Equations over sets of natural numbers

Alexander Okhotin

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January 3, 2008
Equations over sets of numbers

\[
\begin{cases}
X_1 = \varphi_1(X_1, \ldots, X_n) \\
\vdots \\
X_n = \varphi_n(X_1, \ldots, X_n)
\end{cases}
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- \( X_i \): subset of \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \).
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- $\varphi_i$: variables, singleton constants, operations on sets.
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- For $S, T \subseteq \mathbb{N}_0$, 

\begin{align*}
S \cup T &= S \cup T \\
S \cap T &= S \setminus S \\
S + T &= \{x + y \mid x \in S, y \in T\}
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Example

$X = (X^+ + \{2\}) \cup \{0\}$

Unique solution: the even numbers 

Representing sets by unique solutions.
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Unique solution: the even numbers

- Representing sets by unique solutions.
Resolved equations with \( \{ \cup, + \} \)

Equivalent to context-free grammars over the alphabet \( \{a\} \).
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- Fundamental model of syntax in computer science.
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  - Balanced brackets: $S \rightarrow SS \mid aSb \mid \varepsilon$. 

Theorem (Bar-Hillel et al., 1961)

Every context-free language over \{ a \} is ultimately periodic.
Resolved equations with \( \{\cup, +\} \)

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\textbf{Theorem (Bar-Hillel et al., 1961)}

\textit{Every context-free language over } \( \{a\} \text{ is ultimately periodic.} \)
Resolved equations with \{\cup, \cap, +\}

Equivalent to conjunctive grammars over \{a\}.

Problem

Are conjunctive languages over \{a\} always ultimately periodic?
Resolved equations with \{\cup, \cap, +\} are equivalent to conjunctive grammars over \{a\}.

- Extend context-free grammars with conjunction (Okhotin, 2000).

Can represent nontrivial non-context-free formal languages, such as \{wcw | w \in \{a, b\}^*\}.
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**Problem**

Are conjunctive languages over \( \{ a \} \) always ultimately periodic?
Generating nonperiodic sets using \( \{ \cup, \cap, + \} \)

**Example (Jež, DLT 2007)**

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\begin{align*}
X_1 &= (X_2 + X_2 \cap X_1 + X_3) \cup \{1\} \\
X_2 &= (X_6 + X_2 \cap X_1 + X_1) \cup \{2\} \\
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X_6 &= (X_3 + X_3 \cap X_1 + X_2)
\end{align*}
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Generating nonperiodic sets using \{∪, ∩, +\}

Example (Jeż, DLT 2007)

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>((X_2 + X_2 \cap X_1 + X_3) \cup {1})</th>
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<th>({4^n \mid n \geq 0})</th>
</tr>
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<tbody>
<tr>
<td>(X_2)</td>
<td>((X_6 + X_2 \cap X_1 + X_1) \cup {2})</td>
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Let \( k \geq 2 \) and consider:
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Let \(k \geq 2\) and consider:

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Let \( k \geq 2 \) and consider:

1. a language over \( \{0, 1, \ldots, k-1\} \) of base-\( k \) positional notations.
2. a trellis automaton (one-way real-time cellular automaton) recognizing it.
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Then a system defining the corresponding set of numbers can be constructed.
Growth rate of sets defined using \( \{ \cup, \cap, + \} \)

- A large class of sets can be defined.
Growth rate of sets defined using \{\cup, \cap, +\}

- A large class of sets can be defined.
- Any limits of the expressive power of these equations?

Theorem (Je˙z, Okhotin, CSR 2007)

For every recursive enumerable set \(S \subseteq \mathbb{N}\), a faster-growing set can be defined.

1. A Turing machine recognizing \(S = \{n_1, n_2, \ldots\}\).
2. The language VALC of its computation histories: \(\{w_1, w_2, \ldots\}\).
3. Define VALC over the alphabet of digits.
4. VALC defined by trellis automaton = a system for the corresponding set of numbers.
5. Grows faster than \(S\).
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A single equation with \( \{\cup, \cap, +\} \)

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**Example (Okhotin, Rondogiannis, *in preparation*)**

The equation

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has an exponentially growing unique solution.
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*No superexponentially growing sets are representable.*
Resolved equations with \{\sim, +\}

Denote $S^2 = S + S$. 

Example (Leiss, 1994) The equation $X = X^2 + 1$ has a unique solution $L_0 = \{n \mid 2^3 i \leq n < 2^3 i + 2 \text{ for some } i \geq 0\}$.

Lemma (Okhotin, Yakimova, DLT 2006) The symmetric difference $L_0 \triangle \{n \mid n = 2^3 i \text{ or } n = 2^3 i + 1\} \triangle \text{Even}$ is not representable by any system with \{\sim, +\}. 

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Unresolved equations with \{\bigcup, +\}

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- Recursive sets: mathematical notion of “effectively computable”.

Theorem (Jeż, Okhotin, in preparation)

$S \subseteq \mathbb{N}$ is given by unique solution of such a system if and only if $S$ is recursive.

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- cf. Diophantine equations.

Example

1. A Diophantine equation with PRIMES as the range of $x$.
2. An equation over sets of numbers with PRIMES as the unique value of $X$. 

Any number-theoretic methods?

Problem

Construct a set not representable by equations with $\{\cup, \cap, +\}$.
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Let PRIMES be the set of all primes.
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Let PRIMES be the set of all primes.

1. A Diophantine equation with PRIMES as the range of $x$.
2. An equation over sets of numbers with PRIMES as the unique value of $X$.

Any number-theoretic methods?

Problem

*Construction a set not representable by equations with $\{\cup, \cap, +\}$.\*