Abstract

We develop a new tool, namely polynomial and linear algebraic methods, for studying systems of word equations. We illustrate its usefulness by giving essentially simpler proofs of several hard problems. At the same time we prove extensions of these results. Finally, we obtain the first nontrivial upper bounds for the fundamental problem of the maximal size of independent systems. These bounds depend quadratically on the size of the shortest equation. No methods of having such bounds have been known before.

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1. Introduction

Combinatorics on words is a part of discrete mathematics. It studies the properties of strings of symbols and has applications in many areas from pure mathematics to computer science. See, e.g., [23] or [3] for a general reference on this subject.

Some of the most fundamental questions in combinatorics on words concern word equations. First such question is the complexity of the satisfiability problem, i.e., the problem of determining whether a given equation with constants has a solution. The satisfiability problem was proved to be decidable by Makanin [24] and proved to be in PSPACE by Plandowski [27], and it has been conjectured to be NP-complete.
A second question is how to represent all solutions of a constant-free equation. Hmelevskii proved that the solutions of an equation on three unknowns can be represented with parametric words, but this does not hold for four unknowns [13]. The original proof has been simplified [19] and used to study a special case of the satisfiability problem [28].

A third fundamental question, which is very important for this article, is the maximal size of an independent system of word equations. It was proved by Albert and Lawrence [1] and independently by Guba [9] that an independent system cannot be infinite. However, it is still not known whether there are unboundedly large independent systems.

One of the basic results in the theory of word equations is that a nontrivial equation causes a defect effect. In other words, if \( n \) words satisfy a nontrivial relation, then they can be represented as products of \( n - 1 \) words. Not much is known about the additional restrictions caused by several independent relations [10].

In fact, even the following simple question, formulated already in [4], is still unanswered: How large can an independent system of word equations on three unknowns be? The largest known examples consist of three equations. This question can be obviously asked also in the case of \( n > 3 \) unknowns. Then there are independent systems of size \( \Theta(n^4) \) [18]. Some results concerning independent systems on three unknowns can be found in [12], [6] and [7], but the open problem seems to be very difficult to approach with current techniques.

There are many variations of the above question: We may study it in the free semigroup, i.e., require that \( h(x) \neq \varepsilon \) for every solution \( h \) and unknown \( x \), or examine only the systems having a solution of rank \( n - 1 \), or study chains of solution sets instead of independent systems. See, e.g., [11], [10], [5] and [20].

In this article we will use polynomials to study some questions related to systems of word equations. Algebraic techniques have been used before, most notably in the proof of Ehrenfeucht’s conjecture, which is based on Hilbert’s basis theorem. However, the way in which we use polynomials is quite different and allows us to apply linear algebra to the problems.

The main contribution of this article is the development of new methods for attacking problems on word equations. This is done in Sections 3 and 5. Other contributions include simplified proofs and generalizations for old results in Sections 4 and 6, and studying maximal sizes of independent systems of equations in Section 6. Thus the connection between word equations and
linear algebra is not only theoretically interesting, but is also shown to be very useful at establishing simple-looking results that have been previously unknown, or that have had only very complicated proofs. In addition to the results of the paper, we believe that the techniques may be useful in further analysis of word equations.

Next we give a brief overview of the paper. First, in Section 2 we define a way to transform words into polynomials and prove some basic results using these polynomials.

In Section 3 we prove that if the lengths of the unknowns are fixed, then there is a connection between the ranks of solutions of a system of equations and the rank of a certain polynomial matrix. This theorem is very important for all the later results.

Section 4 contains small generalizations of two earlier results. These are nice examples of the methods developed in Section 3 and have independent interest, but they are not important for the later sections.

In Section 5 we analyze the results of Section 3 when the lengths of the unknowns are not fixed. For every solution these lengths form an \( n \)-dimensional vector, called the length type of the solution. We prove that the length types of all solutions of rank \( n - 1 \) of a pair of equations are covered by a finite union of \( (n - 1) \)-dimensional subspaces if the equations are not equivalent on solutions of rank \( n - 1 \). This means that the solution sets of pairs of equations are in some sense more structured than the solution sets of single equations. This theorem is the key to proving the remaining results.

We begin Section 6 by proving a theorem about unbalanced equations. This gives a considerably simpler reproof and a generalization of a result in [12]. Finally, we return to the question about sizes of independent systems. There is a trivial bound for the size of a system depending on the length of the longest equation, because there are only exponentially many equations of a fixed length. We prove that if the system is independent even when considering only solutions of rank \( n - 1 \), then there is an upper bound for the size of the system depending quadratically on the length of the shortest equation. Even though it does not give a fixed bound even in the case of three unknowns, it is a first result of its type – hence opening, we hope, a new avenue for future research.
2. Basic Theorems

Let $|w|$ be the length of a word $w$ and $|w|_a$ be the number of occurrences of a letter $a$ in $w$. The set of nonnegative integers is denoted by $\mathbb{N}_0$ and the set of positive integers by $\mathbb{N}_1$. The empty word is denoted by $\varepsilon$.

In this section we give proofs for some well-known results. These serve as examples of the polynomial methods used. Even though the standard proofs of these are simple, we hope that the proofs given here illustrate how properties of words can be formulated and proved in terms of polynomials.

Let $\Sigma \subset \mathbb{N}_1$ be an alphabet of numbers. For a word $w = a_0 \ldots a_{n-1} \in \Sigma^n$ we define a polynomial

$$P_w = a_0 + a_1 X^1 + \cdots + a_{n-1} X^{n-1}$$

and, if $n = |w| > 0$, a rational function

$$R_w = \frac{P_w}{X^n - 1}.$$ 

The mapping $w \mapsto P_w$ is an injection from words to polynomials. Here the assumption $0 \notin \Sigma$ is needed; if injectivity of $P_w$ would not be needed, then also $0$ could be a letter. If $w_1, \ldots, w_m \in \Sigma^*$, then

$$P_{w_1 \ldots w_m} = P_{w_1} + P_{w_2} X^{|w_1|} + \cdots + P_{w_m} X^{|w_1 \ldots w_m| - 1},$$

and if $w_1, \ldots, w_m \in \Sigma^+$, then

$$P_{w_1 \ldots w_m} = R_{w_1}(X^{|w_1|} - 1) + R_{w_2}(X^{|w_1 w_2|} - X^{|w_1|}) + \cdots + R_{w_m}(X^{|w_1 \ldots w_m|} - X^{|w_1 \ldots w_m| - 1}).$$

If $w \in \Sigma^+$ and $k \in \mathbb{N}_0$, then

$$P_{w^k} = P_w X^{k|w|} - 1 = R_w(X^{k|w|} - 1).$$

The polynomial $P_w$ can be viewed as a characteristic polynomial of the word $w$. A polynomial or formal power series obtained from a sequence in this way is sometimes known as the generating function or z-transform of the sequence. We could also replace $X$ with a suitable number $b$ and get a number whose reverse $b$-ary representation is $w$. Or we could let the coefficients of $P_w$ be from some other commutative ring than $\mathbb{Z}$. Similar ideas have been used to analyze words in many places, see, e.g., [22] and [29].
Example 2.1. If \( w = 1212 \), then \( P_w = 1 + 2X + X^2 + 2X^3 \) and
\[
R_w = \frac{1 + 2X + X^2 + 2X^3}{X^4 - 1} = \frac{(1 + X^2)(1 + 2X)}{(X^2 + 1)(X^2 - 1)} = \frac{1 + 2X^2}{X^2 - 1}.
\]

A word \( w \in \Sigma^+ \) is primitive if it is not of the form \( u^k \) for any \( k > 1 \). If \( w = u^k \) and \( u \) is primitive, then \( u \) is a primitive root of \( w \).

Lemma 2.2. If \( w \) is primitive, then \( P_w \) is not divisible by any polynomial of the form \( (X^{|w|} - 1)/(X^n - 1) \), where \( n < |w| \) is a divisor of \( |w| \).

Proof. If \( P_w \) is divisible by \( (X^{|w|} - 1)/(X^n - 1) \), then there are numbers \( a_0, \ldots, a_{n-1} \) such that
\[
P_w = (a_0 + a_1X^1 + \cdots + a_{n-1}X^{n-1}) \frac{X^{|w|} - 1}{X^n - 1} = (a_0 + a_1X^1 + \cdots + a_{n-1}X^{n-1})(1 + X^n + \cdots + X^{|w| - n}),
\]
so \( w = (a_0 \ldots a_{n-1})^{\frac{|w|}{n}} \). \( \Box \)

The next two theorems are among the most basic and well-known results in combinatorics on words (except for item (4) of Theorem 2.4, which, however, appeared in [17] in a slightly different form).

Theorem 2.3. Every nonempty word has a unique primitive root.

Proof. Let \( u^m = v^n \), where \( u \) and \( v \) are primitive. We need to show that \( u = v \). We have
\[
P_u \frac{X^{m|u|} - 1}{X^{|u|} - 1} = P_v \frac{X^{n|v|} - 1}{X^{|v|} - 1} = P_v \frac{X^{n|v|} - 1}{X^{|v|} - 1}.
\]
Because \( m|u| = n|v| \), we get \( P_u(X^{|v|} - 1) = P_v(X^{|u|} - 1) \). If \( d = \gcd(|u|, |v|) \), then \( \gcd(X^{|u|} - 1, X^{|v|} - 1) = X^d - 1 \). Thus \( P_u \) must be divisible by \( (X^{|v|} - 1)/(X^d - 1) \) and \( P_v \) must be divisible by \( (X^{|u|} - 1)/(X^d - 1) \). By Lemma 2.2, both \( u \) and \( v \) can be primitive only if \( |u| = d = |v| \). \( \Box \)

The primitive root of a word \( w \in \Sigma^+ \) is denoted by \( \rho(w) \).

Theorem 2.4. For \( u, v \in \Sigma^+ \), the following are equivalent:

1. \( \rho(u) = \rho(v) \),
2. if $U, V \in \{u, v\}^*$ and $|U| = |V|$, then $U = V$,

3. $u$ and $v$ satisfy a nontrivial relation,

4. $R_u = R_v$.

Proof. (1) $\Rightarrow$ (2): $U = \rho(u)^{|U|/|\rho(u)|} = \rho(u)^{|V|/|\rho(u)|} = V$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (4): Let $u_1 \ldots u_m = v_1 \ldots v_n$, where $u_i, v_j \in \{u, v\}$. Then

$$0 = P_{u_1 \ldots u_m} - P_{v_1 \ldots v_n}$$

$$= \sum_{i=1}^{m} R_{u_i} (X^{[u_1 \ldots u_i]} - X^{[u_1 \ldots u_{i-1}]}) - \sum_{j=1}^{n} R_{v_j} (X^{[v_1 \ldots v_j]} - X^{[v_1 \ldots v_{j-1}]})$$

$$= -R_{u_1} + \sum_{i=1}^{m-1} (R_{u_i} - R_{u_{i+1}}) X^{[u_1 \ldots u_i]} + R_{u_m} X^{[u_1 \ldots u_m]}$$

$$+ R_{v_1} - \sum_{j=1}^{n-1} (R_{v_j} - R_{v_{j+1}}) X^{[v_1 \ldots v_j]} - R_{v_n} X^{[v_1 \ldots v_n]}$$

$$= (R_u - R_v)p$$

for some polynomial $p$, because $X^{[u_1 \ldots u_m]} = X^{[v_1 \ldots v_n]}$ and

$$-R_{u_1} + R_{v_1}, R_{u_m} - R_{v_n}, R_{u_i} - R_{u_{i+1}}, R_{v_j} - R_{v_{j+1}} \in \{R_u - R_v, R_v - R_u, 0\}$$

for all $i, j$. If $m \neq n$ or $u_i \neq v_i$ for some $i$, then $p \neq 0$, and thus $R_u = R_v$.

(4) $\Rightarrow$ (1): We have $P_{u^{[v]}} = R_u (X^{[u[v]]} - 1) = R_v (X^{[v[v]]} - 1) = P_{v^{[u]}}$, so

$w^{[v]} = v^{[u]}$ and $\rho(u) = \rho(w^{[u]}) = \rho(v^{[u]}) = \rho(v)$. \[\square\]

Similarly, polynomials can be used to give a simple proof for the theorem of Fine and Wilf. In fact, one of the original proofs in [8] uses power series. The proof given here is essentially this original proof formulated in terms of our polynomials. Algebraic techniques have also been used to prove variations of this theorem [25].

**Theorem 2.5** (Fine and Wilf). If $u^i$ and $v^j$ have a common prefix of length $|u| + |v| - \gcd(|u|, |v|)$, then $\rho(u) = \rho(v)$. 

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Proof. Let \( \gcd(|u|, |v|) = d \), \( \text{lcm}(|u|, |v|) = m \), \( m/|u| = r \) and \( m/|v| = s \). If \( \rho(u) \neq \rho(v) \), then \( u^r \neq v^s \), so \( u^r \) and \( v^s \) have a maximal common prefix of length \( k < m \). This means that

\[
P_{u^r} - P_{v^s} = \frac{X^{|u|} - 1}{X^{|u|} - 1} P_u - \frac{X^{|v|} - 1}{X^{|v|} - 1} P_v
\]

is divisible by \( X^k \), but not by \( X^{k+1} \), so also the polynomial

\[
\frac{X^{|v|} - 1}{X^d - 1} P_u - \frac{X^{|u|} - 1}{X^d - 1} P_v
\]

is divisible by \( X^k \), but not by \( X^{k+1} \). Thus \( k \) can be at most the degree of this polynomial, which is at most \( |u| + |v| - d - 1 \). \( \square \)

3. Solutions of Fixed Length

In this section we apply polynomial techniques to word equations. From now on, we will assume that there are \( n \) unknowns, they are ordered as \( x_1, \ldots, x_n \) and \( \Xi \) is the set of these unknowns.

A (coefficient-free) word equation \( u = v \) on \( n \) unknowns consists of two words \( u, v \in \Xi^* \). A solution of this equation is any morphism \( h : \Xi^* \rightarrow \Sigma^* \) such that \( h(u) = h(v) \). The equation is trivial if \( u \) and \( v \) are the same word.

The (combinatorial) rank of a morphism \( h \) is the smallest number \( r \) for which there is a set \( A \) of \( r \) words such that \( h(x) \in A^* \) for every unknown \( x \). A morphism of rank at most one is periodic.

Let \( h : \Xi^* \rightarrow \Sigma^* \) be a morphism. The length type of \( h \) is the vector

\[
L = (|h(x_1)|, \ldots, |h(x_n)|) \in \mathbb{N}_0^n.
\]

This length type \( L \) determines a morphism

\[
\text{len}_L : \Xi^* \rightarrow \mathbb{N}_0, \text{len}_L(w) = |h(w)|.
\]

It is important that \( \text{len}_L \) depends only on \( L \) and not on \( h \).

If \( E \) is a word equation, the set of its solutions is denoted by \( \text{Sol}(E) \), the set of solutions of rank \( r \) by \( \text{Sol}_r(E) \), the set of solutions of length type \( L \) by \( \text{Sol}^L(E) \) and the set of solutions of rank \( r \) and length type \( L \) by \( \text{Sol}_r^L(E) \).
These can be naturally generalized for systems of equations. For example, if $E_1$ and $E_2$ are word equations, then $\text{Sol}(E_1, E_2) = \text{Sol}(E_1) \cap \text{Sol}(E_2)$.

For a word equation $E : y_1 \ldots y_k = z_1 \ldots z_l$ (where $y_i, z_i \in \Xi$), a variable $x \in \Xi$ and a length type $L$, let

$$Q_{E,x,L} = \sum_{y_i = x} X^{\text{len}_L(y_1 \ldots y_{i-1})} - \sum_{z_i = x} X^{\text{len}_L(z_1 \ldots z_{i-1})}.$$ 

Informally, this polynomial encodes the positions of $x$ in the equation $E$.

**Theorem 3.1.** A morphism $h : \Xi^* \rightarrow \Sigma^*$ of length type $L$ is a solution of an equation $E : u = v$ if and only if

$$\sum_{x \in \Xi} Q_{E,x,L} P_h(x) = 0.$$ 

**Proof.** Now $h(u) = h(v)$ if and only if $P_{h(u)} = P_{h(v)}$, and the polynomial $P_{h(u)} - P_{h(v)}$ can be written as $\sum_{x \in \Xi} Q_{E,x,L} P_h(x)$ by (1).

Theorem 3.1 means that if we fix a length type $L$, then we can turn a word equation into a linear equation where the polynomials $Q_{E,x,L}$ are the coefficients. A solution for this linear equation is an $n$-dimensional vector over the field of rational functions, and $h \in \text{Sol}^L(E)$ corresponds to a solution $(P_{h(x_1)}, \ldots, P_{h(x_n)})$ of the linear equation.

**Example 3.2.** Let $\Xi = \{x, y, z\}$, $E : xyz = zyx$ and $L = (1,1,2)$. Then

$$Q_{E,x,L} = 1 - X^2, \quad Q_{E,y,L} = X - X^3, \quad Q_{E,z,L} = X^2 - 1.$$ 

If $h$ is the morphism defined by $h(x) = 1$, $h(y) = 2$ and $h(z) = 12$, then $h$ is a solution of $E$ and

$$Q_{E,x,L} P_{h(x)} + Q_{E,y,L} P_{h(y)} + Q_{E,z,L} P_{h(z)} \\
(1 - X^2) \cdot 1 + (X - X^3) \cdot 2 + (X^2 - 1) (1 + 2X) = 0.$$ 

At this point we start using linear algebra. We will do this over two fields: The field of rational numbers (for the first time in Lemma 3.5) and the field of rational functions (for the first time in Lemma 3.6). We start with an example.
**Example 3.3.** Consider the morphism \( h : \{x_1, x_2, x_3\}^* \to \{1, 2\}^* \) of rank 2 defined by \( h(x_1) = 1, h(x_2) = 2, h(x_3) = 12 \). If \( h \) is a solution of an equation \( E \), then so is \( g \circ h \) for every morphism \( g : \{1, 2\}^* \to \{1, 2\}^* \). The length type of \( g \circ h \) is

\[
(|g(1)|, |g(2)|, |g(12)|) = |g(1)| \cdot (1, 0, 1) + |g(2)| \cdot (0, 1, 1).
\]

Because the vectors \((1, 0, 1)\) and \((0, 1, 1)\) are linearly independent, these length types essentially form a two-dimensional space (of course \(|g(1)|\) and \(|g(2)|\) are nonnegative integers, so the length types don’t form the whole space). This observation is formalized and generalized in Lemma 3.5.

A morphism \( \phi : \Xi^* \to \Xi^* \) is an **elementary transformation** if there are two unknowns \( x, y \in \Xi \) so that \( \phi(y) \in \{xy, x\} \) and \( \phi(z) = z \) for \( z \in \Xi \setminus \{y\} \). If \( \phi(y) = xy \), then \( \phi \) is regular, and if \( \phi(y) = x \), then \( \phi \) is singular. The next lemma follows immediately from results in [23].

**Lemma 3.4.** Every solution \( h \) of an equation \( E : u = v \) has a factorization \( h = \theta \circ \phi \circ \alpha \), where \( \alpha(x) \in \{x, \varepsilon\} \) for all \( x \in \Xi \), \( \phi = \phi_1 \circ \cdots \circ \phi_t \), every \( \phi_i \) is an elementary transformation, \( (\phi \circ \alpha)(u) = (\phi \circ \alpha)(v) \), and \( \theta(x) \neq \varepsilon \) for all \( x \in \Xi \). If \( \alpha(x) = \varepsilon \) for \( s \) unknowns \( x \) and \( t \) of the \( \phi_i \) are singular, then the rank of \( \phi \circ \alpha \) is \( n - s - t \).

**Lemma 3.5.** Let \( E \) be an equation on \( n \) unknowns and let \( h \in \text{Sol}^L_r(E) \). There is an \( r \)-dimensional subspace \( V \) of \( \mathbb{Q}^n \) containing \( L \) such that the set of those length types of morphisms in \( \text{Sol}_r(E) \) that are in \( V \) is not covered by any finite union of \((r - 1)\)-dimensional spaces.

**Proof.** For arbitrary morphisms \( F : \Xi^* \to \Xi^* \) and \( G : \Xi^* \to \Sigma^* \), let \( L_G = ([G(x_1)], \ldots, [G(x_n)])^T \) be the length type of \( G \) as a column vector and let \( A_F = ([F(x_i)]_{ij}) \) be an \( n \times n \) matrix. Then \( L_{G \circ F} = A_F L_G \). More generally, if \( F_1, \ldots, F_m \) are morphisms \( \Xi^* \to \Xi^* \), then

\[
L_{G \circ F_{m \circ \cdots \circ F_1}} = A_{F_1} \cdots A_{F_m} L_G.
\]

Let \( h = \theta \circ \phi_m \circ \cdots \circ \phi_1 \circ \alpha \) as in Lemma 3.4. Let \( f = \phi_m \circ \cdots \circ \phi_1 \circ \alpha \). The rank of \( f \) is \( n - s - t \geq r \) if \( s \) and \( t \) are as in Lemma 3.4. The morphism \( g \circ f \) is a solution of \( E \) for every morphism \( g : \Xi^* \to \Sigma^* \). The length type of \( g \circ f \) is \( L_{g \circ f} = L_{g \circ \phi_m \circ \cdots \circ \phi_1 \circ \alpha} = A L_g \), where \( A = A_{\alpha} A_{\phi_1} \cdots A_{\phi_m} \).

For every \( L' \in \mathbb{N}^n_1 \), there is a morphism \( g : \Xi^* \to \Sigma^* \) such that \( L_g = L' \) and the rank of \( g \circ f \) is \( r \). If the rank of \( A \) is at least \( r \), then from
these morphisms we can select \( g_1, \ldots, g_r \) so that the vectors \( ALg_1^i \) are linearly independent. Let \( V \) be the \( r \)-dimensional space generated by these vectors. Every linear combination of the vectors \( Lg_i \) with positive integer coefficients is in \( \mathbb{N}^n \subseteq \{ Lg \mid g \circ f \in \text{Sol}_r(E) \} \), so every linear combination of the vectors \( ALg_i \) with positive integer coefficients is in \( \{ ALg \mid g \circ f \in \text{Sol}_r(E) \} \) and these linear combinations cannot be covered by a finite union of \((r-1)\)-dimensional spaces. Thus \( V \) satisfies the requirements of the lemma. It remains to be shown that the rank of \( A \) is at least \( r \). This can be done by determining the ranks of the matrices \( A_\alpha \) and \( A_\phi \).

The matrix \( A_\alpha \) is a diagonal matrix and the \( i \)th element on the diagonal is 0 if \( \alpha(x_i) = \varepsilon \) and 1 otherwise. Thus the rank of \( A_\alpha \) is \( n - s \).

If \( \phi \) is the elementary transformation defined by \( \phi(x_1) = x_2x_1 \), then

\[
A_\phi = \begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

is a matrix of rank \( n \) (this is an identity matrix except for the second element on the first row). In general, the rank of \( A_\phi \) is \( n \) for every regular elementary transformation \( \phi \).

If \( \phi \) is the elementary transformation defined by \( \phi(x_1) = x_2 \), then

\[
A_\phi = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

is a matrix of rank \( n - 1 \) (this is an identity matrix except for the first two elements on the first row). In general, the rank of \( A_\phi \) is \( n - 1 \) for every singular elementary transformation \( \phi \).

The rank of \( A_\alpha \) is \( n - s \), \( t \) of the matrices \( A_\phi \) have rank \( n - 1 \) and the rest have rank \( n \). Thus the rank of \( A \) is at least \( n - s - t \), which is at least \( r \).

\[\square\]

**Lemma 3.6.** Let \( E : u = v \) be an equation on \( n \) unknowns and let \( h \in \text{Sol}_r(E) \). There are morphisms \( f : \Xi^* \to \Xi^* \) and \( \theta : \Xi^* \to \Sigma^* \) and polynomials \( p_{ij} \) such that the following conditions hold:
1. $h = \theta \circ f,$
2. $f(u) = f(v),$
3. $\theta(x) \neq \varepsilon$ for every $x \in \Xi,$
4. $P_{(g \circ f)(x_i)} = \sum p_{ij} P_{g(x_j)}$ for all $i, j$ if $g : \Xi^* \to \Sigma^*$ is a morphism of the same length type as $\theta,$
5. $r$ of the vectors $(p_{1j}, \ldots, p_{nj}) \in \mathbb{Q}(X)^n,$ where $j = 1, \ldots, n,$ are linearly independent.

**Proof.** The proof is quite similar to the proof of Lemma 3.5.

For arbitrary morphisms $F : \Xi^* \to \Xi^*$ and $G : \Xi^* \to \Sigma^*$ and length type $L$, define an $n$-dimensional column vector $P_G = (P_G(x_1), \ldots, P_G(x_n))^T$ and an $n \times n$ polynomial matrix $B_{F,L} = (b_{ij})$, where

$$b_{ij} = \sum_{ux_j \leq F(x_i)} X^{\text{len}(u)}.$$

If $L$ is the length type of $G$, then $P_{G \circ F} = B_{F,L} P_G$. More generally, if $F_1, \ldots, F_m$ are morphisms $\Xi^* \to \Xi^*$ and $L_k$ is the length type of $G \circ F_m \circ \cdots \circ F_{k+1}$, then

$$P_{G \circ F_m \circ \cdots \circ F_1} = B_{F_1, L_1} \cdots B_{F_m, L_m} P_G.$$

The matrices $B_{F,L}$ will be used to define the polynomials $p_{ij}$.

Let $h = \theta \circ \phi_m \circ \cdots \circ \phi_1 \circ \alpha$ as in Lemma 3.4. Let $f = \phi_m \circ \cdots \circ \phi_1 \circ \alpha$. The first three conditions are satisfied by $\theta$ and $f$. The rank of $f$ is $n - s - t \geq r$ if $s$ and $t$ are as in Lemma 3.4.

Let $L$ be the length type of $\theta$ and let $g$ be a morphism of length type $L$. Then $P_{g \circ f} = P_{g \circ \phi_m \circ \cdots \circ \phi_1 \circ \alpha} = B P_g$, where $B = B_{\alpha, L_0} B_{\phi_1, L_1} \cdots B_{\phi_m, L_m}$ and $L_k$ is the length type of $g \circ \phi_m \circ \cdots \circ \phi_k \circ \alpha$. Let $B = (p_{ij})$. Then the fourth condition holds, because $P_{g \circ f} = B P_g$.

To prove that the last condition holds, it must be proved that the rank of the matrix $B$ is at least $r$. This can be done by determining the ranks of the matrices $B_{\alpha, L}$ and $B_{\phi_k, L}$.

The matrix $B_{\alpha, L}$ is a diagonal matrix and the $i$th element on the diagonal is 0 if $\alpha(x_i) = \varepsilon$ and 1 otherwise. Thus the rank of $B_{\alpha, L}$ is $n - s$.
If $\phi$ is the elementary transformation defined by $\phi(x_1) = x_2x_1$, then

$$B_{\phi,L} = \begin{pmatrix}
X^{\text{len}_L(x_2)} & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & & & & \ddots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}$$

is a matrix of rank $n$ (this is an identity matrix except for the first two elements on the first row). In general, the rank of $B_{\phi,L}$ is $n$ for every regular elementary transformation $\phi$.

If $\phi$ is the elementary transformation defined by $\phi(x_1) = x_2$, then

$$B_{\phi,L} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & & & & \ddots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}$$

is a matrix of rank $n - 1$ (again, this is an identity matrix except for the first two elements on the first row). In general, the rank of $B_{\phi,L}$ is $n - 1$ for every singular elementary transformation $\phi$.

The rank of $B_{\alpha,L_0}$ is $n - s$, $t$ of the matrices $B_{\phi_k,L_k}$ have rank $n - 1$ and the rest have rank $n$. Thus the rank of $B$ is at least $n - s - t$, which is at least $r$.

With the help of these lemmas, we are going to analyze solutions of some fixed length type. Principal (or fundamental) solutions, which were implicitly present in the previous lemmas (see [23]), have been used in connection with fixed lengths also in [14] and [15].

**Theorem 3.7.** Let $E_1, \ldots, E_m$ be a system of equations on $n$ unknowns and let $L \in \mathbb{N}_0^n$. Let $q_{ij} = Q_{E_i,x_j,L}$. If $\text{Sol}^L_r(E_1, \ldots, E_m) \neq \emptyset$, then the rank of the $m \times n$ polynomial matrix $(q_{ij})$ is at most $n - r$. If the rank of the matrix is $1$, at most one component of $L$ is zero and the equations are nontrivial, then $\text{Sol}^L_r(E_1) = \cdots = \text{Sol}^L_r(E_m)$.

**Proof.** Let $h \in \text{Sol}^L_r(E_1, \ldots, E_m)$. If $r = 1$, the first claim follows from Theorem 3.1, so assume that $r > 1$. Let $E$ be an equation that has the same nonperiodic solutions as the system. Lemma 3.6 will be used for this
equation. Fix \( k \) and let \( g_1 : \Xi^* \to \Sigma^* \) be the morphism determined by 
\( g_1(x_i) = 1^{[\theta(x_i)]} \) for all \( i \) and let \( g_2 : \Xi^* \to \Sigma^* \) be the morphism determined by 
\( g_2(x_k) = 21^{[\theta(x_k)]-1} \) and \( g_2(x_i) = 1^{[\theta(x_i)]} \) for all \( i \neq k \). Then \( g_1 \circ f \) and \( g_2 \circ f \) are solutions of every \( E_l \), so
\[
\sum_{i=1}^{n} Q_{E_l, x_i, L} P_{(g_1 \circ f)(x_i)} = 0 \quad \text{and} \quad \sum_{i=1}^{n} Q_{E_l, x_i, L} P_{(g_2 \circ f)(x_i)} = 0
\]
for all \( l \) by Theorem 3.1. Because also \( P_{(g_1 \circ f)(x_i)} = \sum_{j=1}^{n} p_{ij} P_{g_1(x_j)} \) and 
\( P_{(g_2 \circ f)(x_i)} = \sum_{j=1}^{n} p_{ij} P_{g_2(x_j)} \), we get
\[
0 = \sum_{i=1}^{n} Q_{E_l, x_i, L} (P_{(g_2 \circ f)(x_i)} - P_{(g_1 \circ f)(x_i)})
\]
\[
= \sum_{i=1}^{n} Q_{E_l, x_i, L} \sum_{j=1}^{n} p_{ij} (P_{g_2(x_j)} - P_{g_1(x_j)}) = \sum_{i=1}^{n} Q_{E_l, x_i, L} P_{ik}
\]
for all \( l \). Thus the vectors \((p_{ij}, \ldots, p_{nj})\) are solutions of the linear system of equations determined by the matrix \((g_{ij})\). Because at least \( r \) of these vectors are linearly independent, the rank of the matrix is at most \( n - r \).

To prove the second claim, assume that at most one component of \( L \) is zero and that the equations are nontrivial. First, we show that all rows of the matrix are nonzero. Consider the \( i \)th row. Let \( E_i \) be the equation 
\( uy_1 \ldots y_k = u z_1 \ldots z_l \), where \( u \in \Xi^* \), \( y_j, z_j \in \Xi \) for all \( j \), and \( y_1 \neq z_1 \). It is not possible that \( \text{len}_L(y_1) = \text{len}_L(z_1) = 0 \). By symmetry, we can assume that \( \text{len}_L(z_1) > 0 \). Then
\[
Q_{E_i, y_1, L} = \sum_{y_j = y_1} X^{\text{len}_L(u y_1 \ldots y_{j-1})} - \sum_{z_j = y_1} X^{\text{len}_L(u z_1 \ldots z_{j-1})}.
\]
The first term of the first sum is \( X^{\text{len}_L(u)} \), but every term of the second sum is divisible by \( X^{\text{len}_L(u z_1)} \). Thus \( Q_{E_i, y_1, L} \neq 0 \) and the \( i \)th row of the matrix is nonzero. Because \( i \) was arbitrary, all rows are nonzero. If the rank of the matrix is 1, then all rows are multiples of each other and the second claim follows by Theorem 3.1.

\begin{proof}

4. Applications

Based on Theorem 3.7, the polynomial and linear algebraic methods will be developed further in Section 5. However, Theorem 3.7 is already strong
enough to provide reproofs, generalizations and improvements of some results.

The graph of a system of word equations is the graph where $\Xi$ is the set of vertices and there is an edge between $x$ and $y$ if one of the equations in the system is of the form $x \cdots = y \cdots$. The following well-known theorem can be proved with the help of Theorem 3.7.

**Theorem 4.1 (Graph lemma).** Consider a system of equations whose graph has $r$ connected components. If $h$ is a solution of this system and $h(x_i) \neq \varepsilon$ for all $i$, then the rank of $h$ is at most $r$.

**Proof.** We can assume that the connected components are

$$\{x_1, \ldots, x_{i_2-1}\}, \{x_{i_2}, \ldots, x_{i_3-1}\}, \ldots, \{x_i, \ldots, x_n\}$$

and the equations are

$$x_j \cdots = x_{k_j} \cdots,$$

where $j \in \{1, \ldots, n\} \setminus \{1, i_2, \ldots, i_r\}$ and $k_j < j$. Let $q_{ij}$ be as in Theorem 3.7. If we remove the columns $1, i_2, \ldots, i_r$ from the $(n - r) \times n$ matrix $(q_{ij})$, we obtain a square matrix $M$ where the diagonal elements are not divisible by $X$, but all elements above the diagonal are divisible by $X$. This means that $\det(M)$ is not divisible by $X$, so $\det(M) \neq 0$. Thus the rank of the matrix $(q_{ij})$ is $n - r$ and $h$ has rank at most $r$ by Theorem 3.7.

The next theorem generalizes a result from [6] for more than three unknowns.

**Theorem 4.2.** If a pair of nontrivial equations on $n$ unknowns has a solution $h$ of rank $n-1$ where no two of the unknowns commute, then there is a number $k \geq 1$ such that the equations are of the form $x_1 \cdots = x_2^k x_3 \cdots$.

**Proof.** By Theorem 4.1, the equations must be of the form $x_1 \cdots = x_2 \cdots$. Let them be

$$x_1uy \cdots = x_2vz \cdots \quad \text{and} \quad x_1u'y' \cdots = x_2v'z' \cdots,$$

where $u, v, u', v' \in \{x_1, x_2\}^*$ and $y, z, y', z' \in \{x_3, \ldots, x_n\}$. It can be assumed that $z = x_3$ and

$$|h(x_2v)| \leq |h(x_1u)|, |h(x_1u')|, |h(x_2v')|.$$
If it were \(|h(x_1u)| = |h(x_2v)|\), then \(h(x_1)\) and \(h(x_2)\) would commute, so \(|h(x_1u)| > |h(x_2v)|\). If \(v\) would contain \(x_1\), then \(h(x_1)\) and \(h(x_2)\) would commute by Theorem 2.5, so \(v = x_2^{-k-1}\) for some \(k \geq 1\).

Let \(L\) be the length type of \(h\) and let \(q_{ij}\) be as in Theorem 3.7. By Theorem 3.7, the rank of the matrix \((q_{ij})\) must be 1 and thus \(q_{12}q_{23} - q_{13}q_{22} = 0\). The term of \(q_{13}q_{22}\) of the lowest degree is \(X^{[h(x_2^k)]}\). The same must hold for \(q_{12}q_{23}\), and thus the term of \(q_{23}\) of the lowest degree must be \(-X^{[h(x_2^k)]}\). We know that \(x_2v = x_2^k\) and assumed that \(|h(x_2v)| \leq |h(x_2v')|\). If it were \(|h(x_2v)| < |h(x_2v')|\), then \(h(x_3)\) would start in \(h(x_2v'z'\ldots)\) before the end of \(h(x_2v')\), which is not possible. This means that \(|h(x_2v')| = |h(x_2^k)| \leq |h(x_1u')|\) and \(z' = x_3\). As above, we conclude that \(|h(x_2v')| < |h(x_1u')|\), \(v'\) cannot contain \(x_1\) and \(v' = x_2^{-k-1}\).

It was proved in [21] that if

\[ s_0u_1^i s_1 \cdots u_m^i s_m = t_0v_1^i t_1 \cdots v_n^i t_n \]

holds for \(m + n + 3\) consecutive values of \(i\), then it holds for all \(i\). By using similar ideas as in Theorem 3.7, we improve this bound to \(m + n\) and prove that the values do not need to be consecutive. In [21] it was also stated that the arithmetization and matrix techniques in [31] would give a simpler proof of a weaker result. Similar questions have been studied in [16] and there are relations to independent systems [26].

**Theorem 4.3.** Let \(m, n \geq 1\), \(s_j, t_j \in \Sigma^*\) and \(u_j, v_j \in \Sigma^+\). Let

\[ U_i = s_0u_1^i s_1 \cdots u_m^i s_m \quad \text{and} \quad V_i = t_0v_1^i t_1 \cdots v_n^i t_n. \]

If \(U_i = V_i\) holds for \(m + n\) values of \(i\), then it holds for all \(i\).

**Proof.** The equation \(U_i = V_i\) is equivalent to \(P_{U_i} - P_{V_i} = 0\). Because

\[
P_{U_i} = \sum_{j=1}^{m} \left( P_{s_{j-1}} + P_{u_j} \frac{X^{|s_j|} - 1}{X^{|s_j|}} X^{|s_{j-1}|+1} \right) X^{|s_{j-1}|+|s_0\ldots s_{j-2}|}
+ P_{s_m} X^{|s_1\ldots s_m|+|s_0\ldots s_{m-1}|}
\]

and \(P_{V_i}\) is of a similar form, this equation can be written as

\[
\sum_{j=0}^{m} y_j X^{|s_1\ldots s_j|} + \sum_{k \in K} z_k X^{|v_1\ldots v_k|} = 0, \tag{2}
\]
where \( y_j, z_k \) are some polynomials that do not depend on \( i \) and \( K \) is the set of those \( k \in \{0, \ldots, n\} \) for which \(|v_1 \ldots v_k|\) is not any of the numbers \(|u_1 \ldots u_j|\) \((j = 0, \ldots, m)\). If \( U_{i_1} = V_{i_1} \) and \( U_{i_2} = V_{i_2} \), then

\[
(i_1 - i_2)|u_1 \ldots u_m| = |U_{i_1}| - |U_{i_2}| = |V_{i_1}| - |V_{i_2}| = (i_1 - i_2)|v_1 \ldots v_n|.
\]

Thus \(|u_1 \ldots u_m| = |v_1 \ldots v_n|\) and the size of \( K \) is at most \( n - 1 \). If (2) holds for \( m + 1 + \#K \leq m + n \) values of \( i \), it can be viewed as a system of equations where \( y_j, z_k \) are unknowns. The coefficients of this system form a generalized Vandermonde matrix whose determinant is nonzero, so the system has a unique solution \( y_j = z_k = 0 \) for all \( j, k \). This means that (2) holds for all \( i \) and \( U_i = V_i \) for all \( i \). \( \square \)

5. Sets of Solutions

In this section we analyze how the polynomials \( Q_{E,x,L} \) behave when \( L \) is not fixed. Let

\[
\mathcal{M} = \{a_1X_1 + \cdots + a_nX_n \mid a_1, \ldots, a_n \in \mathbb{N}_0\} \subset \mathbb{Z}[X_1, \ldots, X_n]
\]

be the additive monoid of linear homogeneous polynomials with nonnegative integer coefficients on the variables \( X_1, \ldots, X_n \). The monoid ring of \( \mathcal{M} \) over \( \mathbb{Z} \) is the ring formed by expressions of the form

\[
a_1X^{p_1} + \cdots + a_kX^{p_k},
\]

where \( a_i \in \mathbb{Z} \) and \( p_i \in \mathcal{M} \), and the addition and multiplication of these generalized polynomials is defined in a natural way. This ring is denoted by \( \mathbb{Z}[X; \mathcal{M}] \). If \( L \in \mathbb{Z}^n \), then the value of a polynomial \( p \in \mathcal{M} \) at the point \((X_1, \ldots, X_n) = L\) is denoted by \( p(L) \), and the polynomial we get by making this substitution in \( s \in \mathbb{Z}[X; \mathcal{M}] \) is denoted by \( s(L) \).

The ring \( \mathbb{Z}[X; \mathcal{M}] \) is isomorphic to the ring \( \mathbb{Z}[Y_1, \ldots, Y_n] \) of polynomials on \( n \) variables. The isomorphism is given by \( X^{X_i} \mapsto Y_i \). However, the generalized polynomials where the exponents are in \( \mathcal{M} \) are suitable for our purposes.

If \( a_i \leq b_i \) for \( i = 1, \ldots, n \), then we use the notation

\[
a_1X_1 + \cdots + a_nX_n \preceq b_1X_1 + \cdots + b_nX_n.
\]

If \( p, q \in \mathcal{M} \) and \( p \preceq q \), then \( p(L) \leq q(L) \) for all \( L \in \mathbb{N}_0^n \).
For an equation $E : x_{i_1} \ldots x_{i_r} = x_{j_1} \ldots x_{j_s}$ we define

$$S_{E,x} = \sum_{x_k = \bar{x}} X_1^{x_{i_1} + \ldots + x_{i_k} - 1} - \sum_{x_k = \bar{x}} X_1^{x_{j_1} + \ldots + x_{j_k} - 1} \in \mathbb{Z}[X; \mathcal{M}].$$

Then $S_{E,x}(L) = Q_{E,x,L}$. Theorem 3.1 can be formulated in terms of these generalized polynomials.

**Theorem 5.1.** A morphism $h : \Xi^* \rightarrow \Sigma^*$ of length type $L$ is a solution of an equation $E$ if and only if

$$\sum_{x \in \Xi} S_{E,x}(L)P_{h(x)} = 0.$$

**Example 5.2.** Let $E : x_1x_2x_3 = x_3x_1x_2$. Then

$$S_{E,x_1} = 1 - X^{x_1}, \quad S_{E,x_2} = X^{x_1} - X^{x_1 + x_3}, \quad S_{E,x_3} = X^{x_1 + x_2} - 1.$$

The length of an equation $E : u = v$ is $|E| = |uv|$. The number of occurrences of an unknown $x$ in $E$ is $|E|_x = |uv|_x$.

**Theorem 5.3.** Let $E_1, E_2$ be a pair of nontrivial equations on $n$ unknowns. Let $\text{Sol}_{n-1}(E_1) \neq \text{Sol}_{n-1}(E_2)$. For some unknowns $x_k, x_t$, the set of length types of solutions of the pair of rank $n - 1$ is covered by a union of $(|E_1|_{x_k} + |E_1|_{x_t})^2$ $(n - 1)$-dimensional subspaces of $\mathbb{Q}^n$. If $V_1, \ldots, V_m$ is a minimal such cover and $L \in V_i$ for some $i$, then $\text{Sol}_{n-1}^L(E_1) = \text{Sol}_{n-1}^L(E_2)$.

**Proof.** Let $s_{ij} = S_{E_i,x_j}$ for $i = 1, 2$ and $j = 1, \ldots, n$. If all $2 \times 2$ minors of the $2 \times n$ matrix $(s_{ij})$ are zero, then for all length types $L$ of solutions of rank $n - 1$ the rank of the matrix $(q_{ij})$ in Theorem 3.7 is 1 and $E_1$ and $E_2$ are equivalent, which is a contradiction. Thus there are $k, l$ such that

$$t_{kl} = s_{1k}s_{2l} - s_{1l}s_{2k} \neq 0.$$

The generalized polynomial $t_{kl}$ can be written as

$$t_{kl} = \sum_{i=1}^M X^{p_i} - \sum_{i=1}^N X^{q_i},$$

where $p_i, q_i \in \mathcal{M}$ and $p_i \neq q_j$ for all $i, j$. If $L$ is a length type of a solution of rank $n - 1$, then $M = N$ and $L$ must be a solution of the system of equations

$$p_i = q_{\sigma(i)} \quad (i = 1, \ldots, M) \quad (3)$$

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for some permutation $\sigma$. For every $\sigma$ the equations determine an at most $(n - 1)$-dimensional space.

Let the equations be $E_1 : u_1 = v_1$ and $E_2 : u_2 = v_2$. Let

$$
\begin{align*}
|u_1|_{x_k} &= A, & |v_1|_{x_k} &= A', & |u_2|_{x_k} &= B, & |v_2|_{x_k} &= B', \\
|u_1|_{x_l} &= C, & |v_1|_{x_l} &= C', & |u_2|_{x_l} &= D, & |v_2|_{x_l} &= D'.
\end{align*}
$$

Then $s_{1k}, s_{2l}, s_{1l}, s_{2k}$ can be written as

$$
\begin{align*}
s_{1k} &= \sum_{i=1}^{A} X^{a_i} - \sum_{i=1}^{A'} X^{a'_i}, & s_{2l} &= \sum_{i=1}^{B} X^{b_i} - \sum_{i=1}^{B'} X^{b'_i}, \\
s_{1l} &= \sum_{i=1}^{C} X^{c_i} - \sum_{i=1}^{C'} X^{c'_i}, & s_{2k} &= \sum_{i=1}^{D} X^{d_i} - \sum_{i=1}^{D'} X^{d'_i},
\end{align*}
$$

where $a_i \preceq a_{i+1}, a'_i \preceq a'_{i+1}$, and so on. The polynomials $p_i$ form a subset of the polynomials $a_i + b_j, a'_i + b'_j, c_i + d_j$ and $c'_i + d'_j$ (the reason that they form just a subset is that we assumed $p_i \neq q_j$ for all $i, j$). For any $i$, let $j_i$ be the smallest index $j$ such that $a_i + b_j = p_m$ for some $m$. Then for every $i, j, m$ such that $a_i + b_j = p_m$ we have $a_i + b_j \preceq p_m$. We can do a similar thing for the polynomials $a'_i, b'_i$ and $c_i, d_i$ and $c'_i, d'_i$. In this way we obtain at most $A + A' + C + C'$ polynomials $p_i$ such that for any $L$ the value of one of these polynomials is minimal among the values $p_i(L)$. Similarly we obtain at most $A + A' + C + C'$ “minimal” polynomials $q_i$. If $L$ satisfies one of the systems (3), then the smallest of the values $p_i(L)$ must be the same as the smallest of the values $q_i(L)$. Thus $L$ must satisfy some equation $p_i = q_j$, where $p_i$ and $q_j$ are some of the “minimal” polynomials. There are at most

$$(A + A' + C + C')^2 = (|E_1|_{x_k} + |E_1|_{x_l})^2$$

possible pairs of such polynomials, and each of them determines an $(n - 1)$-dimensional space.

Consider the second claim. Because the cover is minimal, there is a solution of rank $n - 1$ whose length type is in $V_i$, but not in any other $V_j$. By Lemma 3.5, there is an $(n - 1)$-dimensional space $V$ such that $L \in V$ and the set of length types of solutions of rank $n - 1$ in $V$ cannot be covered by a finite union of $(n - 2)$-dimensional spaces. If it were $V \neq V_i$, then they would be covered by the spaces $V \cap V_j$, which is a contradiction, so $V = V_i$. Thus one of the systems (3) must determine the space $V_i$. The same holds
for systems coming from all other nonzero $2 \times 2$ minors of the matrix $(s_{ij})$, so $E_1$ and $E_2$ have the same solutions of rank $n - 1$ and length type $L$ for all $L \in V_i$ by Theorem 3.7.

The following example illustrates the proof of Theorem 5.3. It gives a pair of equations on three unknowns where the required number of subspaces is two. We do not know any example where more spaces would be necessary.

**Example 5.4.** Consider the equations

$E_1 : x_1 x_2 x_3 = x_3 x_1 x_2$ and $E_2 : x_1 x_2 x_3 x_2 x_3 = x_3 x_1 x_3 x_2 x_1 x_2$

and the generalized polynomial

$s = S_{E_1, x_1} S_{E_2, x_3} - S_{E_1, x_3} S_{E_2, x_1}$

$= X^{2x_1 + X_2} + X^{X_1 + 2x_2 + x_3} + X^{X_1 + 2x_3} - X^{X_1 + x_2 + X_3} - X^{X_1 + x_3} - X^{2x_1 + 2x_2} - X^{x_1 + x_2 + 2x_3}.$

If $L$ is a length type of a nontrivial solution of the pair $E_1, E_2$, then $s(L) = 0$. If $s(L) = 0$, then $L$ must satisfy an equation $p = q$, where

$p \in \{2X_1 + X_2, X_1 + 2X_3, X_1 + X_2 + X_3\}$ and $q \in \{X_1 + X_3, 2X_1 + 2X_2\}.$

The possible relations are

$X_3 = 0, \quad X_1 + X_2 = X_3, \quad X_2 = 0, \quad X_1 + 2X_2 = 2X_3.$

If $L$ satisfies one of the first three, then $s(L) = 0$. If $L$ satisfies the last one, then $s(L) \neq 0$, except if $L = 0$. So if $h$ is a nonperiodic solution, then

$|h(x_3)| = 0$ or $|h(x_1 x_2)| = |h(x_3)|$ or $|h(x_2)| = 0.$

There are no nonperiodic solutions with $h(x_2) = \varepsilon$, but every $h$ with $h(x_3) = \varepsilon$ or $h(x_1 x_2) = h(x_3)$ is a solution.

### 6. Independent Systems

A system of word equations $E_1, \ldots, E_m$ is independent if it is not equivalent to any of its proper subsystems.

A sequence of nontrivial equations $E_1, \ldots, E_m$ is a chain if

$\text{Sol}(E_1) \supset \text{Sol}(E_1, E_2) \supset \cdots \supset \text{Sol}(E_1, \ldots, E_m).$
The question of the maximal size of an independent system is open. The only things that are known are that independent systems cannot be infinite [1, 9] and there are systems of size $\Theta(n^4)$, where $n$ is the number of unknowns [18]. The question of the maximal size of a chain is similarly open. For a survey on these topics, see [20].

An equation $u = v$ is balanced if $|u|_x = |v|_x$ for every unknown $x$. Harju and Nowotka proved that if an independent pair of equations on three unknowns has a nonperiodic solution, then the equations must be balanced [12]. The proof is long and it is based on a theorem of Spehner [30] (or alternatively a theorem of Budkina and Markov [2]), which also has only a very complicated proof. However, with the help of Theorem 5.3 we get a significantly simpler proof and a generalization for this result.

**Theorem 6.1.** Let $E_1, E_2$ be a pair of equations on $n$ unknowns having a solution of rank $n - 1$. If $E_1$ is not balanced, then $\text{Sol}_{n-1}(E_1) \subseteq \text{Sol}_{n-1}(E_2)$.

**Proof.** If $E_1$ is the equation $u = v$ and $h$ is a solution of $E_1$, then

$$\sum_{i=1}^{n} |u|_x |h(x_i)| = \sum_{i=1}^{n} |v|_x |h(x_i)|$$

and $|u|_x \neq |v|_x$ for at least one $i$. Thus the set of length types of solutions of $E_1$ is covered by a single $(n - 1)$-dimensional space $V$. Because the pair $E_1, E_2$ has a solution of rank $n - 1$, $V$ is a minimal cover for the length types of the solutions of the pair of rank $n - 1$. By Theorem 5.3, $E_1$ and $E_2$ have the same solutions of length type $L$ and rank $n - 1$ for all $L \in V$.

Another way to think of this result is that if $E_1$ is not balanced but has a solution of rank $n - 1$ that is not a solution of $E_2$, then the pair $E_1, E_2$ causes a larger than minimal defect effect.

If $h : \Xi^* \rightarrow \Sigma^*$ is a morphism, then the entire system generated by $h$ is the set of all equations satisfied by $h$. It is denoted by $K_h$. As a consequence of Theorem 6.1, we get the following result about entire systems. The case of three unknowns was proved in [12].

**Corollary 6.2.** If $g, h : \Xi^* \rightarrow \Sigma^*$ are morphisms of rank $n - 1$ and $K_g \neq K_h$, then $K_g \cap K_h$ contains only balanced equations.

**Proof.** It can be assumed that there is an equation $E_2 \in K_g \setminus K_h$. For any equation $E_1 \in K_g \cap K_h$, $g$ is a solution of the pair $E_1, E_2$ and $h$ is a solution of $E_1$ but not of $E_2$. By Theorem 6.1, $E_1$ must be balanced.
As the main application of the tools developed in this article, the following variation of the question about maximal sizes of chains is studied: How long can a sequence of nontrivial equations $E_1, \ldots, E_m$ be if

$$\text{Sol}_{n-1}(E_1) \supseteq \text{Sol}_{n-1}(E_1, E_2) \supseteq \cdots \supseteq \text{Sol}_{n-1}(E_1, \ldots, E_m)?$$

We prove an upper bound depending quadratically on the length of the first equation. For three unknowns we get a similar bound for the size of independent systems and chains. Previously no bounds like those in Theorem 6.3 and Corollary 6.4 have been known.

**Theorem 6.3.** Let $E_1, \ldots, E_m$ be nontrivial equations on $n$ unknowns and let

$$\text{Sol}_{n-1}(E_1) \supseteq \text{Sol}_{n-1}(E_1, E_2) \supseteq \cdots \supseteq \text{Sol}_{n-1}(E_1, \ldots, E_m) \neq \emptyset.$$  

If the set of length types of solutions of the pair $E_1, E_2$ of rank $n-1$ is covered by a union of $N$ $(n-1)$-dimensional subspaces, then $m \leq N + 1$. There are two unknowns $x, y$ such that $m \leq (|E_1|_x + |E_1|_y)^2 + 1$.

**Proof.** It can be assumed that $E_i$ is equivalent to the system $E_1, \ldots, E_i$ for all $i \in \{1, \ldots, m\}$. Let the set of length types of solutions of $E_2$ of rank $n-1$ be covered by the $(n-1)$-dimensional spaces $V_1, \ldots, V_N$. Some subset of these spaces forms a minimal cover for the length types of solutions of $E_3$ of rank $n-1$. If this minimal cover would be the whole set, then $E_2$ and $E_3$ would have the same solutions of rank $n-1$ by the second part of Theorem 5.3. Thus the set of length types of solutions of $E_3$ of rank $n-1$ is covered by some $N-1$ of these spaces. We conclude inductively that the set of length types of solutions of $E_i$ of rank $n-1$ is covered by some $N - i + 2$ of these spaces for all $i \in \{2, \ldots, m\}$. It must be $N - m + 2 \geq 1$, so $m \leq N + 1$. The second claim follows by Theorem 5.3.

In the case of three unknowns, Theorem 6.3 gives an upper bound depending on the length of the shortest equation for the size of an independent system of equations, or an upper bound depending on the length of the first equation for the size of a chain of equations. A better bound in Theorem 5.3 would immediately give a better bound in the following corollary.

**Corollary 6.4.** If $E_1, \ldots, E_m$ is an independent system on three unknowns having a nonperiodic solution, then $m \leq (|E_1|_x + |E_1|_y)^2 + 1$ for some $x, y \in \Xi$. If $E_1, \ldots, E_m$ is a chain of equations on three unknowns, then $m \leq (|E_1|_x + |E_1|_y)^2 + 5$ for some $x, y \in \Xi$. 

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Corollary 6.4 means that as soon as we take one equation on three unknowns, we get a fixed bound for the size of independent systems containing that equation.

It is worth noting that the bounds in Theorem 6.3 and Corollary 6.4 do not depend on the number of unknowns, only on the length of one equation.

Getting a similar bound for the sizes of independent systems or chains in the case of more than three unknowns remains an open problem. Such a bound would have to depend on the number of unknowns. Indeed, in Theorem 6.3 it is not enough to assume that the equations are independent and have a common solution of rank $n - 1$. If the number of unknowns is not fixed, then there are arbitrarily large such systems where the length of every equation is 10 [18].

References


[21] Juha Kortelainen. On the system of word equations $x_0u_1^i x_1u_2^i x_2 \cdots u_m^i x_m = y_0v_1^i y_1v_2^i y_2 \cdots v_n^i y_n$ ($i = 0, 1, 2, \cdots$) in a free monoid. *J. Autom. Lang. Comb.*, 3(1):43–57, 1998.


