1. \[ x^5 = x^3 + 1, \quad x^6 = x^4 + x, \quad x^7 = x^5 + x^2 = x^3 + x^2 + 1, \quad x^8 = x^7 + x^3 + x \]

2. \[ \alpha^4 = (x^2 + x + 1)^4 = (x^6 + x^2 + x + 1)^2 = x^8 + x^4 + x^2 + 1 \]
   \[ = x^3 + x^2 + x + 1 \]

Let \( \beta = x \) (lecture notes), then \( \alpha \beta = \alpha \) and \( \alpha^4 \beta^4 = \alpha \beta = \alpha \). 

3. \[ \text{ord}(x) \mid 63 \implies x \in \{1, 3, 7, 9, 21, 63\} \]
   \[ x^1 \neq 1, \quad x^2 \neq 1, \quad x^3 = x(x^3 + 1) = x^4 + x^2 \neq 1, \quad x^7 = x^3(x^3 + 1) = x^6 + x^3 = 1 \]
   \[ \implies \text{ord}(x) = 9 \]

4. \[ 2(n+5) = 2n \mod 5, \text{ so } 5 \text{ is a period, and there are no shorter periods; } \]
   \[ (2n) = 2, 4, 1, 3, 0. \]

5. a) Because \( A \) is finite, \( u_i = u_j \) for some \( i, j \) such that \( i < j \).
   
   Then \( u_{n+i} = f(u_i) = f(u_j) = u_{n+j} \) and, by induction, \( u_{n+i} = u_{n+j} \)
   for all \( k \geq 0 \). Thus \( u_{n+i-j} = u_n \forall n \geq i \).

b) Let \( i \) above be minimal, that is, \( u_0, \ldots, u_{i-1} \) only appear once in the seq.
If \( i > 0 \), then \( f(u_i) = u_i = u_j = f(u_{i+1}) \) and \( u_{i+1} \neq u_{i-1} \)
(by the minimality of \( i \)). But this contradicts injectivity of \( f \).
Thus \( i = 0 \) and the sequence is periodic.

6. Let \( f(x) = x^9 - 1 \in \mathbb{F}_q[x] \). For all \( x \in \mathbb{F}_q^* \), \( f(x) = 0 \).
   Thus \( (x^9 - 1) \mid f(x) \), so \( f(x) = g(x) \prod (x - \alpha) \) for some \( g(x) \in \mathbb{F}_q[x] \).

   But \( g(x) \) must have degree 0 and leading coefficient 1, so \( g(x) = 1 \).

a) The coefficient of \( x^9 \cdot 2 \) in \( f(x) \) is 0 and in \( \prod (x - \alpha) \) it is \( \sum_{x \in \mathbb{F}_q} (-\alpha) \), so \( \sum_{x \in \mathbb{F}_q} \alpha = 0 \).

b) The coefficient of \( x^0 \) in \( f(x) \) is -1 and in \( \prod (x - \alpha) \) it is \( \prod_{x \neq \alpha} (-1)^{\deg} \prod_{x \neq \alpha} \alpha = \prod_{x \neq \alpha} (x - \alpha) \), so \( \prod_{x \neq \alpha} \alpha = -1 \).