Profile likelihood interval

This loads the package.

```<<StatisticalInference````

Contents

Profile likelihood region

Profile likelihood based confidence interval

Calculation of profile likelihood based confidence intervals

Profile likelihood region

Most practical problems of parametric inference, it seems to me, fall in the category of "inference in the presence of nuisance parameters," with the acknowledgement that various functions \( g(\theta) \), which might be components of a parameter vector, are simultaneous interest. That is, in practice, we may wish to make inferences about each of many such functions \( g(\theta) \) in turn, keeping in mind that joint inferences are being made, yet requiring separate statements for ease of contemplation. (Kass in Reid 1988, p. 236)

Parameter function of interest

Let \( y_{\text{obs}} \) be the observed response and \( \mathcal{M} = \{ p(y; \omega) : y \in \mathcal{Y}, \omega \in \Omega \} \) its statistical model. In most practical problems only part of the parameter vector or more generally the value of a given function of the parameter vector is of interest. In the former case the rest of the parameter vector consists of the nuisance parameters.

Example/ Fuel consumption

Let us consider the fuel consumption data. Assume that the fuel consumption measurements may be modeled as a sample from some normal distribution. Usually in a situation like this the mean of the normal model is the parameter of interest and the variance is a nuisance parameter.

Example/ Respiratory disorders

Assume that the counts of the favorable cases in the test and placebo groups of the respiratory disorder data may be modeled as observations from binomial models. Now the parameter function of interest might be the odds ratio \( \omega_T / (1-\omega_T) / (1-\omega_P) \), where \( \omega_T \) and \( \omega_P \) are the 'success' probabilities in test and placebo groups, respectively.

Profile likelihood function

Let \( g(\omega) \) be a given interest function with \( q \)-dimensional real vector \( \psi \) as value. Then the function
\[ L_g(\psi; y_{\text{obs}}) = \max_{\omega \in \Omega : g(\omega) = \psi} L(\omega; y_{\text{obs}}) = L(\hat{\omega}_g; y_{\text{obs}}) \]

is called the \textit{profile likelihood function} of the interest function \( g \) induced by the statistical evidence \((y_{\text{obs}}, M)\). The value \( \hat{\omega}_g \) of the parameter vector maximizes the likelihood function in the subset \( \{\omega \in \Omega : g(\omega) = \psi\} \) of the parameter space. The function

\[ l_g(\psi; y_{\text{obs}}) = \max_{\omega \in \Omega : g(\omega) = \psi} \ln(L(\omega; y_{\text{obs}})) = \ln(L(\hat{\omega}_g; y_{\text{obs}})) \]

is called the \textit{logarithmic profile likelihood function} of the interest function \( g \) induced by the statistical evidence \((y_{\text{obs}}, M)\). Furthermore functions

\[ R_g(\psi; y_{\text{obs}}) = \frac{L_g(\psi; y_{\text{obs}})}{L_g(\hat{\omega}_g; y_{\text{obs}})} \]

and

\[ r_g(\psi; y_{\text{obs}}) = L_g(\psi; y_{\text{obs}}) - L_g(\hat{\omega}_g; y_{\text{obs}}) = l_g(\psi; y_{\text{obs}}) - l_g(\hat{\omega}_g; y_{\text{obs}}) \]

are called the \textit{relative profile likelihood function} and the \textit{logarithmic relative profile likelihood function}.

When the interest function \( g \) is real valued, the parameter vectors \( \hat{\omega}_g \) form a curve in the parameter space. This curve is called the \textit{profile curve} of the interest function \( g \). If \( h(\omega) \) is some other real valued function of the parameter vector, the function \( h(\hat{\omega}_g) \) of \( \psi \) is called the \textit{profile trace} of \( h \) with respect to \( g \).

**Example/ Fuel consumption**

Assume that the fuel consumption measurements may be modeled as a sample from some normal distribution and the mean of the normal distribution is the parameter of interest.

```mathematica
dataFile = ToFileName[Directory[], "FuelConsumption.dat"];
TableForm[dataOnFuelConsumption = Import[dataFile], TableAlignments -> Right, TableHeadings -> {None, "Day", "Fuel consumption", "Temperature", "Wind velocity"}]

Day Fuel consumption Temperature Wind velocity
1 14.96 -3.15 15.3
2 14.1 -1.8 16.4
3 23.76 -10.41 21.2
4 13.2 0.7 9.7
5 18.6 -5.1 19.3
6 16.79 -6.3 11.4
7 21.83 -15.5 5.9
8 16.25 -4.2 24.3
9 20.98 -8.8 14.7
10 16.88 -2.3 16.1

fuelConsumption = dataOnFuelConsumption[[All, 2]]; temperature = dataOnFuelConsumption[[All, 3]]; windVelocity = dataOnFuelConsumption[[All, 4]]; n = Length[fuelConsumption];
```

The likelihood function of the statistical model is

\[ M = \text{SamplingModel}[	ext{NormalModel}[\mu, \sigma], \text{Length}[\text{fuelConsumption}]] \]

--- SamplingModel ---
$L = \text{LikelihoodFunction}[\mathcal{M}, \text{fuelConsumption}, \{\mu, \sigma\}]$

$$\frac{1}{32 n^5 \sigma^{10}} \left( e^{-\frac{(11.2 y_0^2 + 16.1 y_1^2 + 16.3 y_2^2 + 16.6 y_3^2 + 16.7 y_4^2 + 16.8 y_5^2 + 16.9 y_6^2 + 17.0 y_7^2 + 17.2 y_8^2 + 17.3 y_9^2 + 17.7 y_{10}^2 + 17.8 y_{11}^2 + 18.2 y_{12}^2 + 18.7 y_{13}^2 + 19.0 y_{14}^2 + 20.0 y_{15}^2 + 20.8 y_{16}^2 + 22.0 y_{17}^2 + 22.8 y_{18}^2 + 23.7 y_{19}^2 + 24.7 y_{20}^2)}{2 \sigma^2} \right)$$

To find the profile likelihood function of the mean $\mu$ we need to maximize the likelihood function with respect to the standard deviation $\sigma$. For example in case the mean has value 15 the plot of the likelihood function as the function of $\sigma$ is

$$\text{Plot[Evaluate[L / . \mu \rightarrow 15], \{\sigma, 0, 10\}, Axes -> False, Frame -> True, PlotRange -> All];}$$

![Plot](image_url)

Now in general if the response $y_{\text{obs}}$ consists of $n$ real numbers and is assumed to be a sample from some normal distribution with unknown mean and variance the likelihood function as the function of the standard deviation $\sigma$ has the form

$$(2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{SS}{2\sigma^2}}$$

where

$$SS = \sum_{j=1}^{n} (y_j - \mu)^2$$

is the sum of squares with respect to given value $\mu$ of the mean. Assuming that not all the observations have same value the sum of squares is positive for all values of $\mu$. Because the value of the standard deviation $\sigma$ maximizing the likelihood function also maximizes the logarithmic likelihood function we can consider the function

$$-\frac{n}{2} \log(2\pi) - n \log(\sigma) - \frac{SS}{2\sigma^2}$$

the derivative of which is

$$-\frac{n}{\sigma} + \frac{SS}{\sigma^2}.$$  

Clearly this derivative takes the value zero at $\sqrt{\frac{SS}{n}}$ and is positive for smaller values of $\sigma$ and negative for larger values of $\sigma$. This means that $\sigma_{\mu} = \sqrt{\frac{SS}{n}}$ and the profile likelihood function is

$$L_{\mu}(\psi; y_{\text{obs}}) = \max_{\sigma > 0} (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{SS}{2\sigma^2}} = \left(2 \pi \frac{\sum_j (y_j - \mu)^2}{n}\right)^{-\frac{n}{2}} e^{-\frac{SS}{2\sigma^2}}$$

The sum of squares
\[ \sum_{j=1}^{n} (y_j - \psi)^2 = \sum_{j=1}^{n} (y_j - \bar{y})^2 + n(\bar{y} - \psi)^2 \]

is minimized when \( \psi \) takes the value \( \bar{y} \) and so the relative profile likelihood function has the following form

\[
R_{\psi}(\psi; y_{\text{obs}}) = \frac{L_{\psi}(y_{\text{obs}})}{L_{\psi}^*(y_{\text{obs}})} = \left( \frac{\sum_{j=1}^{n} (y_j - \psi)^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2} \right)^{-\frac{1}{2}} = \left( 1 + \frac{n(\bar{y} - \psi)^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2} \right)^{-\frac{1}{2}} = \left( 1 + \frac{1}{nT^2} T_{\psi}(y_{\text{obs}})^2 \right)^{-\frac{1}{2}},
\]

where \( T_{\psi} \) is the \( t \)-statistic

\[ T_{\psi}(y) = \frac{\sqrt{n} (\bar{y} - \psi)}{\sqrt{\sum_{j=1}^{n} (y_j - \bar{y})^2}} = \frac{\bar{y} - \psi}{\sqrt{v_n}}. \]

The logarithmic profile likelihood function and the logarithmic relative profile likelihood function are

\[
L_{\psi}(\psi; y_{\text{obs}}) = -\frac{n}{2} \log \left( \frac{\sum_{j=1}^{n} (y_j - \psi)^2}{n} \right) - \frac{n}{2}
\]

\[
r_{\psi}(\psi; y_{\text{obs}}) = -\frac{n}{2} \log \left( 1 + \frac{n(\bar{y} - \psi)^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2} \right) = -\frac{n}{2} \log \left( 1 + \frac{1}{nT^2} T_{\psi}(y_{\text{obs}})^2 \right).
\]

In case of fuel consumption data the plot of the relative profile likelihood function is

\[
SS = \text{Plus @} (\text{fuelConsumption} - \psi)^2;
\]

\[
\bar{y} = \text{Plus @} \text{fuelConsumption} / n;
\]

\[
R = \left( \frac{SS}{\bar{y} \cdot \psi \rightarrow \bar{y}} \right)^{-\frac{1}{2}};
\]

\[
\text{Plot[Evaluate[R], \{\psi, 14, 21\}];}
\]

and that of the logarithmic relative profile likelihood function
In 7.6 functions are given to plot the profile likelihood functions for general real valued interest functions in general parametric models.

Profile likelihood region and it's uncertainty

With help of the relative profile likelihood function one can construct the so called profile likelihood regions. The set

\[ \{ \psi : R_g(\psi; y_{\text{obs}}) \geq \epsilon \} = \{ \psi : r_g(\psi; y_{\text{obs}}) \geq \log(\epsilon) \} \]

of values of the interest function \( g(\omega) \) is the 100 \( c \) % profile likelihood region. The value \( \psi = g(\omega) \) of the parameter function does not belong to the 100 \( c \) % profile likelihood region, if the response is such that

\[ r_g(\psi; y) < \log(\epsilon). \]

The probability of this event calculated at a given value of the parameter vector \( \omega \) is used as a measure of uncertainty of the statement that the unknown value of the interest function belongs to the 100 \( c \) % profile likelihood region provided that this probability has the same value or at least approximately the same value for all values of the parameter vector. One minus this probability is called the confidence level of the profile likelihood region and the region is called the (approximate) \( (1 - \alpha) \)-level profile likelihood based confidence region for the interest function. Because it can be shown that under mild assumptions concerning the interest function \( g(\omega) \) and its statistical model the random variable

\[ -2 r_g(\psi; y) \]

has approximately the \( \chi^2[q] \)-distribution, the set
\[
\{ \psi : r_\psi(\psi) \leq \chi^2_{1 - \alpha}(q) \} = \{ \psi : -2 r_\psi(\psi) \leq \chi^2_{1 - \alpha}(q) \} \approx \{ \psi : 2 r_\psi(\psi) \leq L_y \}
\]
is the approximate \((1 - \alpha)\)-level confidence region for the interest function \(g(\omega)\). In some cases the distribution of \(-2 r_\psi(\psi)\) is exactly the \(\chi^2_{1 - \alpha}(q)\)-distribution and then the set is exact confidence region. Sometimes the random variable \(-2 r_\psi(\psi)\) has some other known distribution, usually the \(F\)-distribution.

Example/ Fuel consumption

Assume that the fuel consumptions in the fuel consumption data can be considered as a sample from some normal distribution with unknown mean and variance.

```plaintext
dataFile = ToFileName[Directory[], "FuelConsumption.dat"];
TableForm[dataOnFuelConsumption = Import[dataFile], TableAlignments -> Right,
          TableHeadings -> {None, {"Day", "Fuel consumption", "Temperature", "Wind velocity"}}]
```

<table>
<thead>
<tr>
<th>Day</th>
<th>Fuel consumption</th>
<th>Temperature</th>
<th>Wind velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.96</td>
<td>-3.0</td>
<td>15.3</td>
</tr>
<tr>
<td>2</td>
<td>14.1</td>
<td>-1.8</td>
<td>16.4</td>
</tr>
<tr>
<td>3</td>
<td>23.76</td>
<td>-10.0</td>
<td>41.2</td>
</tr>
<tr>
<td>4</td>
<td>13.2</td>
<td>0.7</td>
<td>9.7</td>
</tr>
<tr>
<td>5</td>
<td>18.6</td>
<td>-5.1</td>
<td>19.3</td>
</tr>
<tr>
<td>6</td>
<td>16.79</td>
<td>-6.3</td>
<td>11.4</td>
</tr>
<tr>
<td>7</td>
<td>21.83</td>
<td>-15.5</td>
<td>5.9</td>
</tr>
<tr>
<td>8</td>
<td>16.25</td>
<td>-4.2</td>
<td>24.3</td>
</tr>
<tr>
<td>9</td>
<td>20.98</td>
<td>-8.8</td>
<td>14.7</td>
</tr>
<tr>
<td>10</td>
<td>16.88</td>
<td>-2.3</td>
<td>16.1</td>
</tr>
</tbody>
</table>

```plaintext
fuelConsumption = dataOnFuelConsumption[[All, 2]];
temperature = dataOnFuelConsumption[[All, 3]];
windVelocity = dataOnFuelConsumption[[All, 4]];
n = Length[fuelConsumption];
```

Let the mean \(\mu\) be the parameter function of interest.

\[
M = \text{SamplingModel}[\text{NormalModel}[\mu, \sigma], \text{Length}[\text{fuelConsumption}]]
\]

--- SamplingModel ---

The 100\(c\)% profile likelihood region for the mean \(\mu\) is

\[
\{ \psi : r_\psi(\psi, y_{\text{obs}}) \geq \log(c) \} = \left\{ \psi : -2 \frac{\hat{\sigma}}{\sqrt{n}} \log(1 + \frac{1}{n-1} T_\psi(y_{\text{obs}}^2) \geq \log(c) \right\} = \left\{ \psi : T_\psi(y_{\text{obs}}^2) \leq (n - 1) \left( c^{-2} - 1 \right) \right\}.
\]

Because the \(t\)-statistic \(T_\psi\) has the \(\text{Student}\(1 - \alpha\)\,-distribution, when the mean \(\mu\) has the value \(\psi\), the uncertainty of the 100\(c\)% profile likelihood region is \(\alpha\), if \(c\) is chosen such that

\[
(n - 1) \left( c^{-2} - 1 \right) = \left( r_\psi^2 \right)^2.
\]

Thus the interval

\[
\left[ \bar{y}_{\text{obs}} - r_\psi^2 \sqrt{\frac{n - 1}{n}}, \bar{y}_{\text{obs}} + r_\psi^2 \sqrt{\frac{n - 1}{n}} \right]
\]
is the \((1 - \alpha)\)-level confidence interval for the mean \(\mu\).
\[
k = \text{QuantileAtParameter}[	ext{StudentTModel}[n-1], 0.975, \{\}\];
\]
\[
\hat{y} = \text{Plus} @ fuelConsumption / n;
\]
\[
s = \text{Sqrt}[\text{Plus} @ (fuelConsumption - \hat{y})^2 / (n-1) ];
\]
\[
\{\hat{y} - k \frac{s}{\sqrt{n}}, \hat{y} + k \frac{s}{\sqrt{n}}\}
\]
\{15.238, 20.232\}
\[
J = \text{ProfileInterval}[M, \text{fuelConsumption}, \mu, (\mu, \sigma),
\text{ConfidenceQuantile} -> n \text{Log}[1 + \frac{k^2}{n-1}]];\]
\[
\text{ConfidenceLimits}[J]
\]
\{15.238, 20.232\}

**Profile likelihood based confidence interval**

For real valued interest functions the profile likelihood based confidence regions are usually intervals and so those regions are called (approximate) \((1 - \alpha)\)-level profile likelihood based confidence intervals. Thus the set

\[
\{\psi; l_s(\psi; y_{obs}) \geq l_s(\hat{\psi}; y_{obs}) - \frac{1}{2} \text{E}\}\]

forms the (approximate) \((1 - \alpha)\)-level profile likelihood based confidence interval for the interest function \(\psi = g(\omega)\). The statistics \(\hat{\psi}_L\) and \(\hat{\psi}_U\) are called the lower and upper limit of the confidence interval, respectively.

**Example/ Respiratory disorders**

Let us consider the placebo and test groups of the respiratory disorder data.

```math
\text{dataFile} = \text{ToFileName}[\text{Directory}[], "RespiratoryDisorders.dat"];
\text{TableForm}[\text{dataOnRespiratoryDisorders} = \text{Import}[\text{dataFile}, \text{TableAlignments} -> \text{Right},
\text{TableHeadings} -> \{\text{None}, \{"Treatment", "Favorable", "Unfavorable"\}\}]
```

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Favorable</th>
<th>Unfavorable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Placebo</td>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>Test</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

\text{treatment} = \text{dataOnRespiratoryDisorders}[[\text{All}, 1]];\n\text{favorableOutcomes} = \text{dataOnRespiratoryDisorders}[[\text{All}, 2]];\n\text{unfavorableOutcomes} = \text{dataOnRespiratoryDisorders}[[\text{All}, 3]];\n\text{total} = \text{favorableOutcomes} + \text{unfavorableOutcomes};

Assume that the counts of the favorable cases in both groups may be modeled as observations from binomial distributions and that the parameter function of interest is the odds ratio \(\frac{\omega_T}{(1-\omega_T)\omega_P}\), where \(\omega_T\) and \(\omega_P\) are the 'success' probabilities in test and placebo groups, respectively.

\[
M = \text{IndependenceModel}[[\text{BinomialModel}[64, \omega_P], \text{BinomialModel}[60, \omega_T]]]
\]

--- IndependenceModel ---

The \(SIP\)-function \(\text{ProfileInterval}\) calculates the (approximate) 0.95-level confidence interval.
Calculation of profile likelihood based confidence intervals

If the (approximate) \((1 - \alpha)\)-level profile likelihood based confidence region of the real valued interest function \(\psi = g(\omega)\) is an interval, its end points \(\hat{\psi}_L\) and \(\hat{\psi}_U\) satisfy the relations \(\hat{\psi}_L < \psi < \hat{\psi}_U\) and

\[
I_i(\hat{\psi}_L) = I_i(\hat{\psi}_U) = I_i(\hat{\psi}) - \frac{\hat{\chi}^2_{i-1}[1]}{2}.
\]

Thus \(\hat{\psi}_L\) and \(\hat{\psi}_U\) are roots of the equation \(I_i(\psi) = I_i(\hat{\psi}) - \frac{\hat{\chi}^2_{i-1}[1]}{2}\). Consequently most applications of the profile likelihood based intervals have determined \(\hat{\psi}_L\) and \(\hat{\psi}_U\) using some iterative root finding method. This approach involves the solution of an optimisation problem in every trial value and depending on the method also the calculation the derivatives of the logarithmic profile likelihood function at the same trial value.

The approximate \((1 - \alpha)\)-level profile likelihood confidence set for parameter function \(\psi = g(\omega)\) satisfies the relation

\[
\{ \psi : I_i(\psi) > I_i(\hat{\psi}) - \frac{\hat{\chi}^2_{i-1}[1]}{2} \} = \{ \omega \in \mathcal{R}_{1-\alpha}(y_{obs}) \}
\]

where

\[
\mathcal{R}_{1-\alpha}(y_{obs}) = \left\{ \omega : l(\omega) > l(\hat{\omega}) - \frac{\hat{\chi}^2_{i-1}[1]}{2} \right\}
\]

is a likelihood region for the whole parameter vector. This result is true, because the real number \(\psi\) belongs to the profile likelihood region, if and only if there exist a parameter vector \(\omega^*\) such that \(g(\omega^*) = \psi\) and

\[
\sup_{\omega \in \Omega : g(\omega) = \psi} l(\omega) = l(\psi) > l(\psi) > I_i(\hat{\psi}) - \frac{\hat{\chi}^2_{i-1}[1]}{2}.
\]

That is if and only if there exists a parameter vector \(\omega^*\) belonging to the likelihood region \(\mathcal{R}_{1-\alpha}(y_{obs})\) such that it satisfies the equation \(\psi = g(\omega^*)\). So the number \(\psi\) belongs to the profile likelihood region, if and only if it belongs to the set \(\{ \omega \in \mathcal{R}_{1-\alpha}(y_{obs}) \}\).

Assume now that the profile likelihood based confidence region is an interval. Then the end points \(\hat{\psi}_L\) and \(\hat{\psi}_U\) satisfy the following relations

\[
\hat{\psi}_L = \inf_{\omega \in \mathcal{R}_{1-\alpha}(y_{obs})} g(\omega) \quad \text{and} \quad \hat{\psi}_U = \sup_{\omega \in \mathcal{R}_{1-\alpha}(y_{obs})} g(\omega).\]

Let now the likelihood region be a bounded and connected subset of the parameter space. If the log-likelihood and the interest functions are continuous in the likelihood region, the latter with non-vanishing gradient, then the profile likelihood based confidence region is an interval \((\hat{\psi}_L, \hat{\psi}_U)\) with...
\[
\hat{\psi}_L = g(\tilde{\omega}_L) = \inf_{l(\omega) = l(\tilde{\omega}) - \chi^2_{1}/2} g(\omega),
\]

and

\[
\hat{\psi}_U = c(\tilde{\omega}_U) = \sup_{l(\omega) = l(\tilde{\omega}) - \chi^2_{1}/2} g(\omega).
\]

This follows from the assumptions, because they imply, that the set \( R_{1-a}(y_{obs}) \) is open, connected, and bounded subset of the \( p \)-dimensional space. Thus the closure of \( R_{1-a}(y_{obs}) \) is a closed, connected, and bounded set. Form the assumptions concerning \( g \) it follows, that it attains its infimum and supremum on the boundary of \( R_{1-a}(y_{obs}) \), and takes every value between infimum and supremum somewhere in \( R_{1-a}(y_{obs}) \).

Under the above assumptions the solutions of the following constrained minimisation (maximisation) problem

\[
\min (\max) g(\omega) \text{ under } l(\omega; y_{obs}) = l(\hat{\omega}; y_{obs}) - \frac{\chi^2_{1}}{2}.
\]

This problem is rarely explicitly solvable and requires use of some kind of iteration. (Uusipaikka 1996)

### Example/ Respiratory disorders

Let us consider the placebo and test groups of the respiratory disorder data.

```mathematica
dataFile = ToFileName[Directory[], "RespiratoryDisorders.dat"];
TableForm[dataOnRespiratoryDisorders = Import[dataFile], TableAlignments -> Right, 
          TableHeadings -> {None, "Treatment", "Favorable", "Unfavorable"}]

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Favorable</th>
<th>Unfavorable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Placebo</td>
<td>16</td>
<td>48</td>
</tr>
<tr>
<td>Test</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

treatment = dataOnRespiratoryDisorders[[All, 1]]; 
favorableOutcomes = dataOnRespiratoryDisorders[[All, 2]]; 
unfavorableOutcomes = dataOnRespiratoryDisorders[[All, 3]]; 
total = favorableOutcomes + unfavorableOutcomes;
```

Assume that the counts of the favorable cases in both groups may be modeled as observations from binomial distributions and that the parameter function of interest is the ratio \( \frac{\omega_T}{\omega_P} \), where \( \omega_T \) and \( \omega_P \) are the 'success' probabilities in test and placebo groups, respectively.

\[ M = \text{IndependenceModel}[[\text{BinomialModel}[64, \omega_P], \text{BinomialModel}[60, \omega_T]]] \]

The following figure includes a collection of contours of the log-likelihood regions for some confidence levels and the level curves of the interest function. The end points of the corresponding intervals are the values of the interest function calculated at points where the level curves of the log-likelihood and the interest functions are tangential to each other. Iteration steps in the course of calculation of confidence intervals of various confidence levels are also included.
Plot of iteration
Interest function: $\frac{\omega_T}{\omega_P}$