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Turku Centre for Computer Science

TUCS Technical Report
No 798, December 2006

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Abstract

We consider relational periods where the relation is a compatibility relation on words induced by a relation on letters. We introduce three types of periods, namely global, external and local relational periods, and we compare their properties by proving variants of the theorem of Fine and Wilf for these periods.

Keywords: period, compatibility relation, partial word, Fine, Wilf

TUCS Laboratory
Discrete Mathematics for Information Technology

1 Introduction

Word relations, i.e., compatibility relations on words induced by relations on letters were introduced as a generalization of *partial words* in [8]. There and in [9, 11] the main focus was on the effect of these relations on coding properties and on the defect theorem of words. In the article [10] we started the study of *periods' interaction properties* with respect to word relations. By the interaction property we mean that if a sufficiently long word has two periods then it also has another nontrivial period depending on the original periods. The theorem of Fine and Wilf is one of the cornerstones in combinatorics on words. In this theorem the derived period is the greatest common divisor of the original periods [7]. Actually, this topic was the starting point of the study of partial words in the seminal paper of J. Berstel and L. Boasson in 1999 [1]. They proved a variant of the theorem of Fine and Wilf for partial words with one hole. Since then several papers on period properties of partial words has been published [2–6, 12]. F. Blanchet-Sadri et al. studied the theorem of Fine and Wilf for partial words with local periods and with arbitrarily many holes. A.M. Shur and Yu.V. Gamzova investigated the case of global periods. We continue the study of this topic by introducing global, external and local relational periods as generalizations of periods of partial words. Using these periods we prove new variants of Fine and Wilf's theorem. Especially, our aim is to compare the interaction properties of different types of periods.

2 Word relations

An *alphabet* $\mathcal{A}$ is a nonempty finite set of symbols, called *letters*, and a *word* over $\mathcal{A}$ is a (finite or infinite) sequence of symbols from $\mathcal{A}$. Denote by $\mathcal{A}^+$ the set of all finite nonempty words over $\mathcal{A}$. The *length* of a word w, denoted by $|w|$, is the total number of (occurrences of) letters in w. For a finite word of length n, we use the notation $w = w_1 w_2 \cdots w_n$, where $w_i \in \mathcal{A}$ is the ith letter of w. If a word $w = w_1 w_2 w_3 \cdots$ is an infinite catenation of a word $x \in \mathcal{A}^+$, we denote $w = x^\omega$.

For a relation $R \subseteq X \times X$, we often write $x\,R\,y$ instead of $(x, y) \in R$. A relation R is a *compatibility relation* on letters if it is both reflexive and symmetric, i.e., (i) $\forall x \in X : \ x\,R\,x$, and (ii) $\forall x, y \in X : \ x\,R\,y \implies y\,R\,x$. For example, both the *identity relation* $\iota_X = \{(x, x) \mid x \in X\}$ and the *universal relation* $\{(x, y) \mid x, y \in X\}$ are compatibility relations on X.

A compatibility relation $R \subseteq \mathcal{A}^+ \times \mathcal{A}^+$ on the set of all words over an alphabet $\mathcal{A}$ will be called a *word relation* if it is induced by its restriction on the letters, i.e.,

$$a_1 \cdots a_m \,R\, b_1 \cdots b_n \iff m = n \text{ and } a_i \,R\, b_i \text{ for all } i = 1, 2, \ldots, m$$

whenever $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}$. Let R be a relation on $\mathcal{A}$. By $\langle R \rangle$ we denote the compatibility relation *generated* by R, i.e., $\langle R \rangle$ is the reflexive and symmetric closure of R. Sometimes we need to consider the restriction of a relation R on a subset X of $\mathcal{A}^+$. We denote $R_X = R \cap (X \times X)$. Words u and v satisfying $u\,R\,v$

are said to be *compatible* or, more precisely, R-*compatible*. If the words are not R-compatible, they are said to be *incompatible*.

Example 1. In the binary alphabet $\{a, b\}$ the only word relation different from the identity relation is the universal relation of all words of equal length. Namely, the relation

$$R = \langle\{(a, b)\}\rangle = \{(a, a), (b, b), (a, b), (b, a)\}$$

makes all words with equal length compatible with each other. On the other hand, in the ternary alphabet $\{a, b, c\}$, where

$$S = \langle\{(a, b)\}\rangle = \{(a, a), (b, b), (a, b), (b, a), (c, c)\},$$

we have $abba\, S\, baab$ but, for instance, the words abc and cac are not compatible.

Example 2. A *partial word* of length n over an alphabet $\mathcal{A}$ is a partial function

$$w\colon \{1, 2, \ldots, n\} \to \mathcal{A}.$$

The domain $D(w)$ of w is the set of positions $p \in \{1, 2, \ldots, n\}$ such that $w(p)$ is defined. The set $H(w) = \{1, 2, \ldots, n\} \setminus D(w)$ is the set of *holes* of w. A partial word can also be seen as a word over the augmented alphabet $\mathcal{A}_\diamondsuit = \mathcal{A} \cup \{\diamondsuit\}$, where $\diamondsuit$ is interpreted as a special "do not know" symbol [1]. In [8] we have shown that using word relations the compatibility relation of partial words can be expressed by

$$R_\uparrow = \langle\{(\diamondsuit, a) \mid a \in \mathcal{A}\}\rangle.$$

3 Types of relational periods

Let $x = x_1 \cdots x_n$ be a word over the alphabet $\mathcal{A}$. An integer $p \geq 1$ is a *(pure) period* of x if, for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$i \equiv j \pmod{p} \implies x_i = x_j.$$

In this case, the word x is called *(purely)* p*-periodic*. The smallest integer which is a period of x is called *the (minimal) period* of x. Here we denote it by $\pi(x)$, or shortly by π, if the word x is clear from the context.

For partial words, two types of periods were defined in [1]: A partial word w has a *(partial) period* p if, for all $i, j \in D(w)$,

$$i \equiv j \pmod{p} \implies w(i) = w(j).$$

A partial word w has a *local (partial) period* p if

$$i, i + p \in D(w) \implies w(i) = w(i + p).$$

For words with compatibility relation on letters, we will now define three types of periods. We call these periods *relational periods*.

Definition 1. Let R be a compatibility relation on an alphabet $\mathcal{A}$. For a word $x = x_1 \cdots x_n \in \mathcal{A}^+$, an integer $p \geq 1$ is

(i) a *global R-period* of x if, for all $i, j \in \{1, 2, \ldots, n\}$, we have

$$i \equiv j \pmod p \implies x_i \, R \, x_j;$$

(ii) an *external R-period* of x if there exists a word $y = y_1 \cdots y_p$ such that, for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, p\}$, we have

$$i \equiv j \pmod p \implies x_i \, R \, y_j.$$

In this case, the word y is called an *external word* of x.

(iii) a *local R-period* of x if, for all $i \in \{1, 2, \ldots, n - p\}$, we have $x_i \, R \, x_{i+p}$.

These definitions generalize naturally to infinite words. For a word x, *the minimal global* (resp. *external, local*) *R-period* is denoted by $\pi_{R,g}(x)$ (resp. $\pi_{R,e}(x)$, $\pi_{R,l}(x)$). In the sequel, we may omit the subscript R or the argument x if the relation R or the word x is clear from the context. Of course, these periods may coincide. Next we give an example where all the above mentioned minimal periods are different.

Example 3. Let $\mathcal{A} = \{a, b, c, d\}$ and define

$$x = babbbcbd.$$

Let $R = \langle \{(a, b), (b, c), (c, d), (d, a)\} \rangle$ be a compatibility relations on the alphabet $\mathcal{A}$. Clearly, the minimal pure period $\pi(x) = 8$. By the definition of R, we see that 2 is a local R-period of x. Since $(x_7, x_8) = (b, d) \notin R$, 1 is not a local period and therefore, we have $\pi_{R,l}(x) = 2$. Neither 1 nor 2 is an external R-period of x, since otherwise the letter y_1 or respectively y_2 in the external word y is related to all the other letters of the alphabet, which is a contradiction. Since $y = bab$ satisfies the conditions of an external word in Def. 1.(ii), we have $\pi_{R,e}(x) = 3$. Furthermore, since $(b, d) \notin R$, we have $\pi_{R,g}(x) > 5$. Indeed, $\pi_{R,g}(x) = 6$, because of the relation $a \, R \, d$. Hence, for a word x, we have

$$\pi = 8 > \pi_g = 6 > \pi_e = 3 > \pi_l = 2.$$

The next theorem shows how different types of periods are related to each other.

Theorem 1. *Every pure period of a word x is a relational period. Every global R-period of x is an external R-period of x and a local period of x. Thus, for a word x, we always have*

$$\pi \geq \pi_g \geq \max(\pi_e, \pi_l).$$

Proof. By reflexivity, $\iota \subseteq R$, and therefore the first statement holds. Note that if $x = x_1 \cdots x_n$ has a period p, then $y = x_1 \cdots x_p$ is an external word of x. Similarly, this choice of y also shows that a global R-period is an external R-period. Clearly, a global period satisfies the definition of a local period. For the minimal periods, these considerations imply the inequalities of the statement. □

Note that every external period is not necessarily a local period and every local period need not be an external period. For example, in Example 3 the minimal local R-period π_l is not an external R-period, and furthermore, π_e is not a local R-period. There we have $\pi_e > \pi_l$. Next we give an example where $\pi_l > \pi_e$.

Example 4. Let $R = \langle\{(a,b),(b,c),(c,d),(d,a)\}\rangle$ and let

$$x = adcbccccbd.$$

Consider first the minimal local R-period of x. Since $(x_9, x_{10}) = (x_4, x_2) = (b,d) \notin R$ and 3 is a local R-period, we have $\pi_l = 3$. Since $x_1 = a$, $x_4 = b$, $x_7 = c$ and $x_{10} = d$, there cannot exist any external word $y = y_1y_2y_3$ of length 3. Otherwise, y_1 would be compatible with all letters of the alphabet $\{a,b,c,d\}$. Hence, 3 is not an external R-period. For the same reason 1 is not an external R-period, but by choosing $y = bc$, we see that $\pi_e = 2$. As noted above, 2 is not a local period. Since $(a,c) \notin R$, the minimal global period satisfies $\pi_g > 7$. Actually, $\pi_g = 8$ since $a\,R\,b$. Clearly, $\pi = 10$. Hence,

$$\pi = 10 > \pi_g = 8 > \pi_l = 3 > \pi_e = 2.$$

If the compatibility relation R is also transitive, all relational periods coincide.

Theorem 2. *If a word relation R is transitive, and thus an equivalence relation, then $P_g(x) = P_e(x) = P_l(x)$, where $P_g(x)$ (resp. $P_e(x)$, $P_l(x)$) is the set of all global (resp. external, local) relational R-periods of a word x. Moreover,*

$$\pi_g(x) = \pi_e(x) = \pi_l(x).$$

Proof. Let $x = x_1 \cdots x_n$ be a word and let R be an equivalence relation. By Theorem 1, we have $P_g(x) \subseteq P_e(x)$ and $P_g(x) \subseteq P_l(x)$. Consider an external R-period p with an external word $y = y_1 \cdots y_p$. Let $i \equiv j \pmod{p}$, where $i,j \in \{1,2,\ldots,n\}$. Then there exists $k \in \{1,2,\ldots,p\}$ such that $i \equiv k \pmod{p}$ and $j \equiv k \pmod{p}$. Now $x_i\,R\,y_k$ and $x_j\,R\,y_k$ by the definition of an external word. Since R is transitive and symmetric, we have $x_i\,R\,x_j$. Hence, p is a global R-period, and we conclude that $P_e \subseteq P_g$.

Consider then a local R-period q of x. Let $i \equiv j \pmod{q}$, where $i,j \in \{1,2,\ldots,n\}$. We may suppose that $j = i + kq$, where k is a nonnegative integer. We have

$$x_i\,R\,x_{i+q}\,R\,x_{i+2q}\,R\,\cdots\,R\,x_{i+(k-1)q}\,R\,x_{i+kq} = x_j.$$

Since R is transitive, we have $x_i\,R\,x_j$. Thus, q is a global period and we conclude that $P_l(x) \subseteq P_g(x)$. Hence, we have shown that $P_g(x) = P_e(x) = P_l(x)$ and $\pi_g(x) = \pi_e(x) = \pi_l(x)$. □

If R is not transitive, local R-periods differ from global and relational periods by the following property.

Lemma 1. *If p is a global R-period or an external R-period, then any multiple of p is a global R-period or an external R-period, respectively. This need not be the case for local R-periods.*

Proof. Suppose that p is a global R-period of x and let $i \equiv j \pmod{kp}$, where k is a nonnegative integer. Then clearly $i \equiv j \pmod{p}$ and, by the assumption, $x_i \, R \, x_j$. Hence kp is a global R-period. The proof is similar for external R-periods. Consider then a word $x = abc$ and a relation $R = \langle \{(a, b), (b, c)\} \rangle$. The word x has 1 as a local R-period, but 2 is not a local R-period. Thus multiples of local R-periods are not necessarily local R-periods. □

Finally we note that external periods are not very meaningful with partial words. Namely, any integer $p \geq 1$ is an external $R_\uparrow$-period of any partial word. Indeed, we may choose $y = (\diamondsuit)^p$ for an external word. Consequently, for partial words, we always have $\pi_e = 1$.

4 Variants of the theorem of Fine and Wilf

The theorem of Fine and Wilf [7] is well-known in combinatorics on words:

Theorem 3. *If a word x has periods p and q, and has length at least $p + q - \gcd(p, q)$, then x has also a period $\gcd(p, q)$.*

J. Berstel and L. Boasson gave a variant of this theorem for partial words with one hole in [1].

Theorem 4. *Let w be a partial word of length n and suppose that it has local $R_\uparrow$-periods p and q. If $H(w)$ is a singleton and if $n \geq p + q$, then w is purely $\gcd(p, q)$-periodic.*

Furthermore, they showed that this bound on the length is sharp. Generalizations for several holes were considered, for example, by F. Blanchet-Sadri in [3] and F. Blanchet-Sadri and R.A. Hegstrom in [6], where it was shown that local partial periods p and q force a sufficiently long word to have a (global) partial period $\gcd(p, q)$ when certain unavoidable cases (*special words*) are excluded. The bound on the length depends on the number of holes in the word. On the other hand, A.M. Shur and Yu.V. Gamzova found bounds for the length of a word with k holes such that (global) partial periods p and q imply a (global) partial period $\gcd(p, q)$ [12]. These results of partial words with several holes show that finding good formulations for periods' interaction in the case of arbitrary relational periods is not possible except for equivalence relations. Namely, any non-transitive compatibility relation R must have letter relations $(x_1, x_2), (x_2, x_3) \in R$, but $(x_1, x_3) \notin R$ for some letters x_1, x_2, x_3. Then the role of the letter x_2 in R is exactly the same as the role of $\diamondsuit$ in $R_\uparrow$ and all binary counter examples of Fine and Wilf's theorem for partial words apply to words with compatibility relation R over the alphabet $\{x_1, x_2, x_3\}$. However, some periods' interaction results can be obtained.

If the relation R is an equivalence relation, we do not have to specify the type of an R-period, since the definitions of the relational periods coincide by Theorem 2. We have the following theorem proved in [10].

Theorem 5. *Let R be an equivalence relation. If a word has R-periods p and q and the length of the word is at least $p + q - \gcd(p, q)$, then the word has also an R-period $\gcd(p, q)$. The bound on the length is strict.*

As was mentioned above, the theorem of Fine and Wilf cannot be generalized for relational periods of a non-transitive compatibility relation unless some restrictions on the number of relations (holes) and exclusions of some special cases are given. Despite this fact, it might be possible to get some new interesting variations of the theorem, for example, by assuming that one of the periods is pure and only the other one is relational by the relation $R \neq \iota$. Unfortunately, this restriction seems to be insufficient in that extent that sometimes no bound on the length of the word can be obtained for periods' interaction. For example, there exists an infinite word with a pure period q and a local R-period p such that it does not have a local R-period $\gcd(p, q)$.

Example 5. Let $R = \langle\{(a, b), (b, c)\}\rangle$. Note that every non-transitive compatibility relation must have a subrelation similar to this one such that a and c are not compatible. Consider an infinite word

$$x = (bbcab)^\omega.$$

Clearly, w has a pure period $q = 5$. It also has a local R-period $p = 3$, since the distance of the letters a and c in x cannot be 3. Since $(x_3, x_4) = (a, c) \notin R$, $\gcd(p, q) = 1$ is not a local period and neither a global period by Theorem 1.

In the previous example the notion of a local relational period is too weak for the desired periods' interaction result. However, depending on the type of the relational period p we get diverse results as will be shown in the sequel. One variant of Fine and Wilf's theorem was already considered in [10]. The following theorem was obtained.

Theorem 6. *Let P and Q be positive integers with $\gcd(P, Q) = d$. Denote $P = pd$ and $Q = qd$. Suppose that a word w has a (pure) period Q and a global R-period P. Let $B_g = B_g(p, q)$ be defined by Table 1. If $|w| \geq B_g d$, then also $\gcd(P, Q) = d$ is a global R-period of the word w. This bound on the length is sharp.*

Hence, one global period with one pure period is strong enough to imply another nontrivial global period. Moreover, according to Theorem 1, one global

$B_g(p,q)$	$p < q$	$p > q$
p, q odd	$\frac{p+1}{2}q$	$q + \frac{q-1}{2}p$
p odd, q even	$\frac{p+1}{2}q$	$\frac{p+1}{2}q$
p even, q odd	$q + \frac{q-1}{2}p$	$q + \frac{q-1}{2}p$

Table 1: Table of bounds $B_g(p,q)$

period must also imply an external and a global relational period. However, the optimal bound on the length of the word can be different in these cases. The bound B_g in Theorem 6 is just one example of interaction bounds defined more precisely in the following.

Definition 2. Let $P \geq 2$ and $Q \geq 3$ be positive integers with $\gcd(P,Q) = d$ and let t_1 and t_2 be types of relational periods. A positive integer $B = B(P,Q)$ is called the *bound of* t_1-t_2 *interaction for* P *and* Q, if it satisfies the following conditions:

(i) The bound B is *sufficient*, i.e., for any word relation R and for any word w with length $|w| \geq B$ having a (pure) period Q and a t_1-type R-period P, the number $\gcd(P,Q) = d$ is a t_2-type R-period of w.

(ii) The bound is *strict*, i.e., there exist a word relation R and a word w with length $|w| = B - 1$ having a (pure) period Q and a t_1-type R-period P such that $\gcd(P,Q) = d$ is **not** a t_2-type R-period of w.

Note that in the definition we exclude trivial cases by assuming that $P \geq 2$ and $Q \geq 3$. Namely, if $Q \leq 2$, then the word contains at most two letters. This is the case of Theorem 5, since there are no non-transitive compatibility relations on a binary alphabet.

The following lemma shows that it is sufficient to consider the case where $\gcd(P,Q) = 1$. In the proof we use a standard approach which was also used in the proof of Theorem 5 in [10].

Lemma 2. *Let* P *and* Q *be positive integers with* $\gcd(P,Q) = d > 1$. *Denote* $P = pd$ *and* $Q = qd$. *If* B *is the bound of* t_1-t_2*-interaction for* p *and* q*, then* Bd *is the bound of* t_1-t_2*-interaction for* P *and* Q.

Proof. Suppose that a word w has a pure period Q and a relational t_1-type period P. We may assume that $|w| = Bd$. Namely, if $|w| > Bd$, then the theorem holds for any factor of w, and therefore also for w itself. Let us now consider the words

$$w^{(i)} = w_i w_{i+d} \cdots w_{i+(B-1)d}$$

for $i = 1, 2, \ldots, d$. Clearly, each of the words $w^{(i)}$ has a pure period q and a t_1-type relational R-period p. Since $|w^{(i)}| = B$ for every $i = 1, 2, \ldots, d$, then 1 is a t_2-type relational R-period for all the words $w^{(i)}$. Consequently, d is a t_2-type relational R-period of w.

In order to prove that the bound Bd is strict, we give an example of a word u of length $Bd - 1$ such that it has a period Q and an R-period P but no R-period d. Suppose that $v = v_1 v_2 \cdots v_{B-1}$ is a word such that it has a pure period q and a t_1-type period p, but $\gcd(p, q) = 1$ is not a t_2-type relational period of v. By the definition of B, such a word exists. Let a be some letter in the alphabet $\mathcal{A}$ and define the word u by the following formula:

$$u = a^{d-1} v_1 a^{d-1} v_2 \cdots a^{d-1} v_{B-1} a^{d-1}.$$

Now u has a pure period $Q = qd$ and a t_1-type period $P = pd$, but by the properties of v, $\gcd(P, Q) = d$ cannot be a t_2-type R-period of u. □

Hence, using our new notation and the previous lemma we may state the result of Theorem 6 in the following way.

Theorem 6'. *Let p and q be positive integers with $\gcd(p, q) = 1$. The bound of global-global interaction for p and q is $B_g(p, q)$ given by Table 1.*

5 Global-local interaction

Instead of attaining a global period $\gcd(p, q)$ we loosen our requirements and consider the case where the greatest common divisor becomes a local relational period.

Theorem 7. *Let p and q be positive integers with $\gcd(p, q) = 1$. Let k be the smallest integer satisfying $kp \equiv \pm 1 \pmod q$. The bound of global-local interaction for p and q is*

$$B_l(p, q) = \begin{cases} q + kp - 1 & \textit{if } 1 \equiv q - 1 \pmod p \textit{ and } kp \equiv +1 \pmod q, \\ q + kp & \textit{otherwise.} \end{cases}$$

We divide the proof into two parts. In the sequel, we use the notation $[n]_q$ for the least positive residue of an integer $n \pmod q$, i.e., $[n]_q$ is the positive integer m satisfying $1 \leq m \leq q$ and $m \equiv n \pmod q$.

Lemma 3. *The bound $B_l(p, q)$ defined in Theorem 7 is sufficient.*

Proof. Denote $B_l = B_l(p, q)$. Assume that a word w has a pure period q and a global R-period p. We show that 1 is a local R-period of w if $|w| \geq B_l$. By the assumption, the word w is a rational power of a word of length q. Thus in w there are at most q different letters. Hence, the word w has a local R-period 1 if and only if, for all $n = 1, 2, \ldots, q$, we have

$$w_{[n]_q} \, R \, w_{[n+1]_q}. \tag{1}$$

We show that, for each $n \in \{1, 2, \dots, q\}$, there exist integers $i_n, j_n \in \mathbb{N}$ such that

$$[n]_q + i_n q \equiv [n+1]_q + j_n q \pmod p \tag{2}$$

and both sides of the congruence belong to the set $\{1, 2, \dots, B_l\}$. This implies together with the global period p of w that Eq. (1) must be satisfied if $|w| \geq B_l$.

Case 1. Assume first that $kp \equiv 1 \pmod q$. For $n \in \{1, 2, \dots, q-1\}$, choose $j_n = \frac{kp-1}{q}$ and $i_n = 0$. Note that j is an integer by the definition of k. Then

$$(n+1) + j_n q = n + 1 + kp - 1 = n + kp \equiv n \pmod p.$$

Clearly, both sides of the congruence belong to $\{1, 2, \dots, B_l\}$. Furthermore, let $j_q = \frac{kp-1}{q} + 1$ and $i_q = 0$. Now

$$1 + j_q q = 1 + kp - 1 + q = q + kp \equiv q \pmod p.$$

The left hand side is less than or equal to B_l only if $1 \not\equiv q-1 \pmod p$. However, in the special case $1 \equiv q-1 \pmod q$, we can choose $i_q = \frac{kp-1}{q}$ and $j_q = 0$ so that

$$q + iq = q + kp - 1 \equiv q - 1 \equiv 1 \pmod p.$$

Now the left hand side is exactly B_l.

Case 2. Assume that $kp \equiv -1 \pmod q$ and, for $n \in \{1, 2, \dots, q-1\}$, let $i_n = \frac{kp+1}{q}$ and $j_n = 0$. Hence,

$$n + i_n q = n + kp + 1 \equiv n + 1 \pmod p.$$

Choose furthermore $i_q = \frac{kp+1}{q} - 1$ and $j_q = 0$. Then

$$q + i_q q = q + kp + 1 - q \equiv 1 \pmod p.$$

Note that both sides of both congruences belong to the set $\{1, 2, \dots, B_l\}$. Hence, we have shown that Eq. (1) is satisfied for all $n = 1, 2, \dots, q$ if $|w| \geq B_l$. Therefore w must have $\gcd(p,q) = 1$ as a local relational period. □

Lemma 4. *The bound $B_l(p,q)$ defined in Theorem 7 is strict.*

Proof. We prove that there exists a word w of length $B_l - 1$ such that it has a global period p and a pure period q but no local period $\gcd(p,q) = 1$. We show that, at least for one index $n \in \{1, 2, \dots, q\}$, there is no solution i_n, j_n of Eq. (2) such that both sides of the equation belong to the set $\{1, 2, \dots, B_l - 1\}$. Without contradicting the assumption that p is a global period of w we may then assume that $(w_{[n]_q}, w_{[n+1]_q}) \notin R$ and therefore $\gcd(p,q) = 1$ is not a local R-period of w.

Case A. Let us first assume that $kp \equiv 1 \pmod q$ and $1 \not\equiv q-1 \pmod p$. Consider the equation

$$q + iq \equiv 1 + jq \pmod p.$$

Note that in the solution $j = j_q = \frac{kp-1}{q} + 1$, $i = i_q = 0$, we have $1 + j_q q = q + kp = B_l$. We prove that there is no smaller solution, i.e., there are no integers i and j such that $\max(q+iq, 1+jq) < B_l$. Note that if such a solution exists, then we may assume that either $i = 0$ or $j = 0$. Namely, if $i > j$ for some solution, then $q + (i-j)q \equiv 1 \pmod p$ is a smaller solution. Similarly, if $j > i$, then $q \equiv 1 + (j-i)q \pmod p$ is a smaller solution. Thus, assume first that, for some $j \in \mathbb{N}$, we have

$$q \equiv 1 + jq \pmod p$$

and $\max(q, 1+jq) < q + kp$. Now $j > 0$. Otherwise, $1 + lp = q$ for some $l \in \mathbb{N}$. This means that $lp \equiv -1 \pmod q$. By the definition of k, we have $l > k$ so that $1 \equiv kp \pmod q$ and $1 < kp < lp = q - 1$. This is a contradiction. Hence, $j \neq 0$ and $\max(q, 1+jq) = 1 + jq$. Since $q \equiv 1 + jq \pmod p$, there exists $s \in \mathbb{N}$ such that $1 + jq - q = sp$. This means that $sp \equiv 1 \pmod q$ and therefore $s \geq k$. Thus, we have

$$\max(q, 1+jq) = 1 + jq = sp + q \geq kp + q.$$

Again, we have a contradiction.

Assume next that, for some $i \in \mathbb{N}$, we have

$$q + iq \equiv 1 \pmod p$$

and $\max(q + iq, 1) = q + iq < q + kp$. Hence, there exists $s \in \mathbb{N}$ such that $q + iq - 1 = sp$ and consequently $sp \equiv -1 \pmod q$. By the definition of k, we again have $s > k$. Now $q > q + iq - kp = sp + 1 - kp > 1$. On the other hand, $q + iq - kp \equiv -1 \pmod q$. We conclude that $q + iq - kp = q - 1$. Hence,

$$1 \equiv q + iq \equiv q + iq - kp = q - 1 \pmod p$$

and we end up in a contradiction. Thus, let us define a word

$$w = (ac^{q-2}b)^{(B_l - 1)/q}$$

in ternary alphabet $\{a, b, c\}$ with length $B_l - 1$. By the above considerations, if $a \, R \, c$ and $b \, R \, c$ and $\gcd(p, q) = 1$, then the word w has a period q and a local R-period p. However, 1 is not a local period of w if a and b are unrelated by the compatibility relation R.

Case B. Assume next that $kp \equiv 1 \pmod q$ and $1 \equiv q - 1 \pmod p$. Consider the congruence

$$(q-1) + iq \equiv q + jq \pmod p.$$

Note that in the solution $i = i_{q-1} = 0$, $j = j_{q-1} = \frac{kp-1}{q}$ we have $q + j_q q = q + kp - 1 = B_l$. Assume then that there is a smaller solution. Again, we may assume that either $i = 0$ or $j = 0$. Suppose that, for some $j \in \mathbb{N}$, we have

$$q - 1 \equiv q + jq \pmod p$$

and $\max(q-1, q+jq) < B_l$. Now $jq+1 = sp$ for some $s \in \mathbb{N}$. As before, we have $sp \equiv 1 \pmod{q}$. Thus we must have $s \geq k$. Hence

$$\max(q-1, q+jq) = q+jq = q+sp-1 \geq q+kp-1 = B_l;$$

a contradiction. Suppose then that, for some $i \in \mathbb{N}$,

$$q-1+iq \equiv q \pmod{p}$$

and $\max(q-1+iq, q) < B_l$. Note that $i > 0$. Now there exists $s \in \mathbb{N}$ such that $iq-1 = sp$. Hence $sp \equiv -1 \pmod{q}$ and $s > k$. Thus,

$$\max(q-1+iq, q) = q-1+iq = (q-1)+(sp+1) > q+kp > B_l.$$

Again we end up in a contradiction. In this case, the word

$$w = (c^{q-2}ab)^{(B_l-1)/q}$$

and the relation $R = \langle\{(a,c),(b,c)\}\rangle$ together with the above calculations show the necessity of our bound B_l like in the previous case.

Case C. Finally assume that $kp \equiv -1 \pmod{q}$. Consider the same congruence as in Case B. However, note that now $B_l = q+kp$. Similarly as above, we see that, for any $i \in \mathbb{N}$ satisfying

$$q-1+iq \equiv q \pmod{p},$$

we must have $\max(q-1+iq, q) \geq kp+q = B_l$. If $j \in \mathbb{N}$ satisfies

$$q-1 \equiv q+jq \pmod{p},$$

then $j > 0$ and $q+jq-q+1 = sp$ for some positive integer s. We have $sp \equiv 1 \pmod{q}$ and therefore $s > k$. It follows that

$$\max(q-1, q+jq) = sp+q-1 \geq (k+1)p+q-1 = (kp+q)+p-1 > B_l.$$

Hence, the word

$$w = (c^{q-2}ab)^{(B_l-1)/q}$$

and $R = \langle\{(a,c),(b,c)\}\rangle$ show the necessity of our bound B_l also in this case. □

Theorem 7 follows now directly from Lemma 3 and Lemma 4. Note that the value of k can be calculated easily using an elementary theorem by Fermat and Euler. Namely, the smallest solution k' of the equation $k'p \equiv 1 \pmod{q}$ is called the reciprocal of p modulo q and, by the theorem,

$$k' = [p^{\varphi(q)-1}]_q,$$

where φ is the Euler's totient function. Thus, we have $k = \min(k', q-k')$, since $(q-k')p \equiv -1 \pmod{q}$.

6 Global-external interaction

Under the same assumptions as in the previous section but replacing the local relational periodicity by external periodicity we obtain the next interaction theorem. Like before, $[n]_q$ is the least positive residue of an integer $n \pmod q$.

Theorem 8. *Let p and q be positive integers with $\gcd(p, q) = 1$. Denote $h = 1 + \left\lfloor \frac{q}{2} \right\rfloor p$. The bound of global-external interaction for p and q is*

$$B_e(p,q) = \begin{cases} \min(h + [h]_q - 1, h + (q - [h]_q) + 1) & \text{if } q \text{ is odd,} \\ \max(h, h + [h]_q - (p+1)) & \text{if } q \text{ is even.} \end{cases}$$

The proof of the theorem is divided into two lemmata like in the previous section.

Lemma 5. *The bound $B_e(p, q)$ defined in Theorem 8 is sufficient.*

Proof. Assume that a word w has a pure period q and a global R-period p. Like in Lemma 3, the word w is a rational power of a word of length q and therefore contains at most q different letters. If one of the letters, say a, is R-compatible with all the other letters, then the word w has also an external relational period 1. Namely, $y = a$ is an external word of w. On the other hand, if this is not the case and the considered alphabet $\mathcal{A}$ does not contain any letters not occurring in w, then 1 is not an external R-period. Hence, the existence of such a letter a is crucial for the bound of global-external interaction.

We use the following notation. For an integer $n \in \{1, 2, \ldots, q\}$, we define $\tau(n) = \max\{m \mid 1 \leq m \leq |w|,\ m \equiv n \pmod q\}$. Note that if the word w has q different letters, then $\tau(n)$ is the last occurrence of the letter w_n in w. Since w has the global relational period p, it follows that w_n must be related to all letters in the positions

$$S(n) = \{n + ip \mid i = 0, 1, \ldots, \left\lfloor \frac{|w| - n}{p} \right\rfloor\}$$

and

$$T(n) = \{\tau(n) - ip \mid i = 1, 2, \ldots, \left\lfloor \frac{\tau(n) - 1}{p} \right\rfloor\}.$$

Next we prove that if $|w| \geq B_l$, then the union $S(n) \cup T(n)$ contains at least q numbers, i.e.,

$$|S(n) \cup T(n)| = 1 + \left\lfloor \frac{|w| - n}{p} \right\rfloor + \left\lfloor \frac{\tau(n) - 1}{p} \right\rfloor \geq q. \tag{3}$$

Since $\tau(n) \equiv n \pmod q$, this means that these numbers form a complete residue system (mod q). Hence, w_n must be R-compatible with all letters w_i, for $i = 1, 2, \ldots, q$, by the q periodicity of w and therefore 1 is an external R-period of w.

Consider the case where q is odd. Suppose first that $|w| \geq B_e = h + [h]_q - 1$, where $h = 1 + \frac{q-1}{2}p$. Then the letter $w_h = w_{[h]_q}$ occurring in the positions h and $[h]_q$ is related to all the other letters. Namely, by the definition of B_e, we have $\tau([h]_q) \geq h$ and

$$1 + \left\lfloor \frac{|w| - [h]_q}{p} \right\rfloor + \left\lfloor \frac{\tau([h]_q) - 1}{p} \right\rfloor \geq 1 + \frac{q-1}{2} + \frac{q-1}{2} = q.$$

Hence, Eq. (3) is satisfied for $n = [h]_q$. Suppose next that $|w| \geq B_e = h + (q - [h]_q) + 1$. Now the letter in position 1 is related to all other letters. Namely, we have $\tau(1) \geq B_e$ and

$$\left\lfloor \frac{|w| - 1}{p} \right\rfloor \geq \left\lfloor \frac{\tau(1) - 1}{p} \right\rfloor \geq \frac{q-1}{2}.$$

Hence, $|S(1) \cup T(1)| \geq 1 + \frac{q-1}{2} + \frac{q-1}{2} = q$ like above.

Let us then assume that q is even. Hence $h = 1 + \frac{q}{2}p$. We note first that $\max(h, h + [h]_q - (p+1)) = h$ if and only if $[h]_q \leq p$. If this is the case, we have

$$\left\lfloor \frac{|w| - [h]_q}{p} \right\rfloor \geq \left\lfloor \frac{\frac{q}{2}p + 1 - [h]_q}{p} \right\rfloor \geq \frac{q}{2} - 1.$$

On the other hand, if $[h]_q > p$, we have

$$\left\lfloor \frac{|w| - [h]_q}{p} \right\rfloor \geq \left\lfloor \frac{\frac{q}{2}p + 1 + [h]_q - (p+1) - [h]_q}{p} \right\rfloor = \frac{q}{2} - 1.$$

Furthermore, $\tau([h]_q) \geq h$ in both cases and

$$\left\lfloor \frac{\tau([h]_q) - 1}{p} \right\rfloor \geq \left\lfloor \frac{\frac{q}{2}p + 1 - 1}{p} \right\rfloor = \frac{q}{2}.$$

Thus, Eq. (3) is satisfied for $n = [h]_q$. □

Lemma 6. *The bound $B_e(p, q)$ defined in Theorem 8 is strict.*

Proof. In order to prove that our bound is strict, we show that, for some suitable R, there exists a word w of length $B_e - 1$ with a period q and with a global period p such that none of its letters is related to all other letters. We use the notation of Lemma 5. It suffices to prove that, for every integer $n \in \{1, 2, \ldots, q\}$, the set $S(n) \cup T(n)$ does not contain a complete residue system (mod q), i.e., Eq. (3) is not satisfied if $|w| = B_e - 1$. Namely then we may define the relation R in the alphabet $\mathcal{A} = \{a_1, a_2, \ldots, a_q\}$ in such a way that w_n is R-compatible only with the letters in the positions $S(n) \cup T(n)$ and hence none of the q different letters is related to all other letters. Then the word

$$w = (a_1 a_2 \cdots a_q)^{\frac{B_e - 1}{q}}$$

has a pure period q and a global period p, but it does not have $\gcd(p,q)=1$ as an external R-period. We consider four cases:

Case 1. Let q be odd and $B_e = h + [h]_q - 1 = \frac{q-1}{2}p + [h]_q$. Assume that $|w| = B_e - 1 = \frac{q-1}{2}p + [h]_q - 1$. Let $1 \le n \le q$ and suppose furthermore that $n = [h]_q + ip + j$, where $i \in \mathbb{Z}$ and $0 \le j < p-1$. Now

$$\left\lfloor \frac{|w|-n}{p} \right\rfloor = \left\lfloor \frac{\frac{q-1}{2}p + [h]_q - 1 - ([h]_q + ip + j)}{p} \right\rfloor = \frac{q-1}{2} - i - 1.$$

Since $\gcd(p,q) = 1$, we have $[h]_q \neq 1$. Thus, $B_e = h + [h]_q - 1 < h + (q - [h]_q) + 1$ by the definition of the bound. This implies that, for any number $l \in \{h, h+1, \ldots, B_e\}$, we have $[l]_q \ge [h]_q$. Therefore,

$$\tau(n) = \begin{cases} h + ip + j & \text{if } n \in \{1, 2, \ldots, [B_e]_q - 1\}, \\ h - q + ip + j & \text{if } n \in \{[B_e]_q, [B_e]_q + 1, \ldots, q\} \end{cases}$$

and moreover,

$$\left\lfloor \frac{\tau(n)-1}{p} \right\rfloor \le \left\lfloor \frac{1 + \frac{q-1}{2}p + ip + j - 1}{p} \right\rfloor = \frac{q-1}{2} + i.$$

We conclude that the set $S(n) \cup T(n)$ contains at most $(\frac{q-1}{2} - i) + (\frac{q-1}{2} + i) = q-1$ elements. Hence, it does not form a complete residue system (mod q).

Case 2. Let q be odd and $B_e = h + (q - [h]_q) + 1) = \frac{q-1}{2}p + q + 2 - [h]_q$. Then $|w| = B_e - 1 = \frac{q-1}{2}p + q + 1 - [h]_q$. Like above, denote $n = [h]_q + ip + j$, where $i \in \mathbb{Z}$ and $0 \le j < p-1$. By the assumption, $h + [h]_q - 1 \ge h + q - [h]_q + 1$ and therefore $2[h]_q \ge q + 2$. Thus, we have

$$\begin{aligned} \left\lfloor \frac{|w|-n}{p} \right\rfloor &= \left\lfloor \frac{\frac{q-1}{2}p + q + 1 - [h]_q - ([h]_q + ip + j)}{p} \right\rfloor \\ &\le \left\lfloor \frac{\frac{q-1}{2}p - 1 - ip - j}{p} \right\rfloor = \frac{q-1}{2} - i - 1. \end{aligned}$$

By the same reasoning as in Case 1, we have $\tau(n) \le h + ip + j$ and

$$\left\lfloor \frac{\tau(n)-1}{p} \right\rfloor \le \frac{q-1}{2} + i.$$

This means that Eq. (3) is not satisfied for any n.

Case 3. Let q be even and $|w| = B_e - 1 = h - 1 = \frac{q}{2}p$. For any $n \in \{1, 2, \ldots, B\}$, we have

$$\left\lfloor \frac{|w|-n}{p} \right\rfloor \le \frac{q}{2} - 1 \quad \text{and} \quad \left\lfloor \frac{\tau(n)-1}{p} \right\rfloor \le \frac{q}{2} - 1.$$

Thus, again Eq. (3) is not satisfied.

Case 4. Let q be even and $|w| = B_e - 1 = h + [h]_q - (p+1) - 1 = \frac{q}{2}p + [h]_q - p - 1$. Like in the previous cases, denote $n = [h]_q + ip + j$, where $i \in \mathbb{Z}$ and $0 \leq j < p - 1$. We have

$$\left\lfloor \frac{|w| - n}{p} \right\rfloor = \left\lfloor \frac{\frac{q}{2}p + [h]_q - p - 1 - ([h]_q + ip - j)}{p} \right\rfloor = \frac{q}{2} - i - 2.$$

Next we prove that, for each $l \in \{h, h+1, \ldots, B_e - 1\}$, we have $[l]_q \geq [h]_q$. Let us assume the contrary. Then, for some $l \in \{h, h+1, \ldots, B_e - 1\}$, we have $[l]_q = 1$. Consider now the number $l - \frac{q}{2}p$. On one hand,

$$l - \frac{q}{2}p \equiv l - \frac{q}{2}p + qp \equiv 1 + \frac{q}{2}p \equiv [h]_q \pmod{q},$$

and on the other hand,

$$l - \frac{q}{2}p \leq \frac{q}{2}p + [h]_q - p - 1 - \frac{q}{2}p < [h]_q.$$

This is a contradiction. Hence, $[l]_q \geq [h]_q$ and therefore $\tau(n) \leq h + ip + j$, and

$$\left\lfloor \frac{\tau(n) - 1}{p} \right\rfloor \leq \left\lfloor \frac{\frac{q}{2}p + 1 + ip + j - 1}{p} \right\rfloor = \frac{q}{2} + i.$$

Thus, $|S(n) \cup T(n)| \leq 1 + (\frac{q}{2} - i - 2) + (\frac{q}{2} + i) = q - 1$. □

7 External interactions

In the last two sections we found interaction bounds for one pure period and one global relational period. On the other hand, Example 5 shows that if we replace the global period by a local period such bounds do not necessarily exist. Does this hold also if the global period is replaced by an external period?

Let us assume that a word w has a pure period q and an external period p. Let $y = y_1 \cdots y_p$ be an external word of w, i.e., for every $j \in \{1, 2, \ldots, p\}$, $y_j \, R \, w_i$ if $i \equiv j \pmod{p}$. Denote by $\mathrm{Alph}(w)$ the set of the letters occurring in w. The succeeding easy example shows that some conditions on the letters of the external word are needed for *external-global* and *external-local* interactions.

Example 6. Consider a three letter alphabet $\mathcal{A} = \{a, b, c\}$ and let

$$R = \langle \{(a, c), (b, c)\} \rangle.$$

Consider the infinite word $w = (a^{q-1}b)^\omega$ for any integer $q \geq 2$ and choose p such that $\gcd(p, q) = 1$. Clearly any p is an external R-period of w, since c is related to both a and b. However, 1 is not a global nor a local R-period of w.

Hence, the example implies the following.

Theorem 9. *The bound of external-global interaction and the bound of external-local interaction do not exist.*

Because of this, in the formulation of the next theorem we consider only external periods satisfying a special condition. Namely, we require that, for an external R-period p of a word w, there exists an external word $y = y_1 \cdots y_p$ of w satisfying

$$|\text{Alph}(w) \setminus \text{Alph}(y)| \leq 1. \tag{4}$$

By considering only these *restricted external periods* it is possible to find a bound of interaction.

Theorem 10. *Let p and q be positive integers with $\gcd(p,q) = 1$. The bound of external-global interaction C_g for a restricted external period p and a pure period q is pq. Similarly, the bound of external-local interaction C_l for a restricted external period p and a pure period q is pq.*

Proof. Suppose that w is of length pq and it has a pure period q and a restricted external period p. Let $y = y_1 \cdots y_p$ be an external word of w satisfying Eq. (4). Consider a letter w_n in position $n \in \{1, 2, \ldots, q\}$. Since the set $\{n + iq \mid i = 0, 1, \ldots, p-1\}$ is a complete residue system (mod p), it means that w_n is related to all letters in Alph(y). By the assumption, there may exist only one letter which does not occur in y. If this letter is w_n, then it is trivially related to itself and therefore to all letters in Alph(w). On the other hand, if $w_n \in$ Alph(y), then there exists a position k such that $y_k = w_n$. Now $\{k + jp \mid j = 0, 1, \ldots, q-1\}$ is a complete residue system (mod q). Hence $y_k = w_n$ is related to all letters of w. This shows that all letters are compatible with all other letters. Hence, 1 is a global and therefore also a local period of w.

Modification of the previous example shows that the bound $C_g = C_l = pq$ is strict. Assume that R is like in Example 6 and

$$w = (a^{q-1}b)^{p-1}a^{q-1}.$$

We may choose $y = c^{p-1}a$. Namely, $y_p = a$ must be only related to letters in positions $p + ip$ for $i = 0, 1, \ldots, q-2$, which are all a's. Hence, w has an external word which satisfies Eq. (4), but 1 is not a local neither a global R-period of w. □

For the external-external interaction no additional conditions are needed.

Theorem 11. *Let p and q be positive integers with $\gcd(p,q) = 1$. The bound of external-external interaction for p and q is $C = 1 + (q-1)p$.*

Proof. Assume that $y = y_1 \cdots y_p$ is an external word of w. Clearly, if $|w| \geq C$, then y_1 is related to all letters in Alph(w). Namely, the set $\{1 + ip \mid i = 0, 1, \ldots, q-1\}$ forms a complete residue system (mod q).

In order to prove that this bound is strict, consider the word

$$w = (a_1 \cdots a_q)^{\frac{C-1}{q}}$$

with q different letters $a_1, \dots, a_q$. Furthermore, let us assume that the alphabet $\mathcal{A}$ under consideration has p extra letters not occurring in w. Suppose that these letters are $y_1, \dots, y_p$. We may now define that y_k, where $k \in \{1, 2, \dots, p\}$, is not related to the letter $a_{[k+(q-1)p]_q}$, but it is related to all letters w_{k+ip} for $i = 0, 1, \dots, q-2$. Note that the length of w and the assumption that w has q different letters ensures that this is well defined. Hence, $y = y_1 \cdots y_p$ is an external word of w. Furthermore, we may assume that two different letters in Alph(w) are not compatible with each other. Hence, no letter in the alphabet $\mathcal{A}$ is related to all letters in Alph(w). Therefore, the word w does not have 1 as an external R-period. □

On the other hand, it might be more interesting to restrict our considerations to the case where there are no extra letters in the alphabet, i.e., the external word consists only of letters occurring in Alph(w).

Theorem 12. *Let p and q be positive integers with $\gcd(p, q) = 1$ and assume that the alphabet is $\mathcal{A} =$ Alph(w). Then the bound of external-external interaction for p and q is*

$$\overline{C}(p, q) = \begin{cases} (q-2)p + (q-1) & \text{if } q \text{ is odd and either } q < p \text{ or } q = p+1, \\ (q-1)p + 1 & \text{otherwise.} \end{cases}$$

This theorem is a direct consequence of the following two lemmata.

Lemma 7. *The bound $\overline{C}(p, q)$ defined in Theorem 12 is sufficient.*

Proof. First of all, Theorem 11 implies that the bound $\overline{C} = 1 + (q-1)p$ is sufficient. Hence, let us consider those cases where $\overline{C} = (q+2)p + (q-1)$, i.e., q is odd and either $q < p$ or $q = p+1$. Assume that a word w of length $|w| \geq \overline{C}$ has a pure period q and an external period p. First we fix some notation. We introduce a parameter t which is the maximal number of different letters the word w can contain. This parameter t will be used also in the proof of the next theorem. In this proof, we always have $t = q$. In order to simplify the notation, we also set $U = \{1, 2, \dots, t-1\}$. For $1 \leq k \leq p$, set $\mathcal{W}_k = \{w_j \mid j \equiv k \pmod{p}\}$. For each k, denote $k' = [(q-1)p + k]_q$. Furthermore, $a \, R \, \mathcal{Y}$ means that the letter a is compatible with all the letters in $\mathcal{Y}$. For example, the kth letter y_k of the external word y is, by the definition, compatible with all the letters in $\mathcal{W}_k$, i.e.,

$$y_k \, R \, \mathcal{W}_k. \tag{5}$$

Note that if $k \in U$, then $|w| - k \geq (q-2)p$ and the set $\mathcal{W}_k$ contains at least $q-1$ different letters. In other words,

$$\mathcal{W}_k = \left\{ w_{k+ip} \;\middle|\; i = 0, 1, \dots, \left\lfloor \frac{|w| - k}{p} \right\rfloor \right\} \supseteq \mathcal{A} \setminus \{w_{k'}.\} \tag{6}$$

Next we make a couple of important observations, which will be needed throughout the proof.

(i) If $k \in U$ and $y_k = w_{k'}$, then $y_k \, R \, \mathcal{A}$.

(ii) If there exist $k, l \in U$ $(k \neq l)$ such that $y_k = y_l = a$, then $y_k \, R \, \mathcal{A}$.

(iii) If there exist $k, l \in U$ $(k \neq l)$ such that $y_l = w_{k'}$ but $y_k \neq w_{l'}$, then $y_k \, R \, \mathcal{A}$.

Let us prove these statements in brief. First, consider (i). It follows from Eq. (5) and Eq. (6) that $y_k \, R \, \mathcal{A} \setminus \{w_{k'}\}$. Since the word relation R is reflexive and $w_{k'} = y_k$, it follows that $y_k \, R \, \mathcal{A}$. Next, consider (ii). Like in (i), we have $y_k \, R \, (\mathcal{A} \setminus \{w_{k'}\})$ and $y_l \, R \, (\mathcal{A} \setminus \{w_{l'}\})$. Now $k' \neq l'$, since $k, l \in \{1, 2, \ldots, q-1\}$. Hence, $y_k \, R \, w_{l'}$ and $y_l \, R \, w_{k'}$. Since $y_k = y_l = a$, we have $y_k \, R \, \mathcal{A}$. Finally, consider (iii). Again, $y_k \, R \, (\mathcal{A} \setminus \{w_{k'}\})$ and $y_l \, R \, (\mathcal{A} \setminus \{w_{l'}\})$. Since $y_k \neq w_{l'}$, we have $y_k \in (\mathcal{A} \setminus \{w_{l'}\})$. Therefore $y_l = w_{k'} \, R \, y_k$, which implies that $y_k \, R \, \mathcal{A}$. Indeed, if $k, l \in U$ $(k \neq l)$, $y_l = w_{k'}$ and $y_k = w_{l'}$, then relations $y_k \, R \, \mathcal{W}_k$ and $y_l \, R \, \mathcal{W}_l$ do not imply that $w_{k'} \, R \, w_{l'}$. In other words, it is possible that $(w_{k'}, w_{l'}) \notin R$. Such a pair (k, l) is called a *match* and the set of indices belonging to some match is denoted by M in the sequel.

If any of the assumptions of $(i) - (iii)$ is satisfied, then the word w has necessarily an external period 1. Namely, $y = y_k$ is an external word of w. Thus, let us assume that none of them is satisfied. Assume first that, at least for one index $k \in U$, the letter in the position k' occurs also in another position $1 \leq n \leq q$. Denote $w_{k'} = w_n = a$. Since $\mathcal{W}_k$ must contain a letter which is in a position congruent to n, we have $a \in \mathcal{W}_k$ and $\mathcal{W}_k = \mathcal{A}$. Thus, $y_k \, R \, \mathcal{A}$ and 1 is an external period of w.

Finally, assume that, for each $k \in U$, the letter $w_{k'}$ occurs only in positions congruent to k' (mod q). Then all letters in positions $\{i' \mid i \in U\}$ are different. Moreover, by (ii), all letters in Alph$(y_1 \cdots y_{t-1})$ are different. Set $m = [(q-1)p + t]_q$. Hence, m does not belonging to the set $\{i' \mid i \in U\}$. By Eq. (6), $w_m \in \mathcal{W}_i$ for every $i \in U$. Thus, by Eq. (5), $y_i \, R \, w_m$ for $1 \leq i \leq t-1$. Suppose that w_m does not occur in the $t-1$ different letters of Alph$(y_1 \cdots y_{t-1})$. Then $w_m \, R \, (\mathcal{A} \setminus \{w_m\})$, since t is the maximal number of letters occurring in t. By reflexivity of R, we have $w_m \, R \, \mathcal{A}$ and $y = w_m$ is an external word of w.

Furthermore, the case where $y_k = w_m$ for any $k \in U$ is impossible. This is based on the fact that $t - 1 = q - 1$ is even. Consider a position $l \in U \setminus \{k\}$. Since none of the assumptions of the observations is satisfied, we have

$$(i)\; y_l \neq w_{l'}, \quad (ii)\; y_l \neq y_k = w_m \quad (iii)\; y_l \neq w_{k'}.$$

Moreover, since there are at most t letters in w, we have $\{w_{i'} \mid i \in U\} =$ Alph$(w) \setminus \{w_m\}$, and there must exists a unique match $s \in U \setminus \{k, l\}$ such that $y_l = w_{s'}$ and $y_s = w_{l'}$. Since the set $U \setminus \{k\}$ has odd number $t-2$ elements, there cannot be a unique match for each l. Thus, this case is impossible, and we have showed that if $|w| \geq \overline{C}$, then $\gcd(p, q) = 1$ is an external period of w. $\square$

Lemma 8. *The bound $\overline{C}(p,q)$ defined in Theorem 12 is strict.*

Proof. We construct a word w of length $\overline{C}(p,q)-1$ and a relation R such that w has a pure period q and an external R-period p, but 1 is not an external period. In other words, using the notation of Lemma 7, we want to show that we can define R and $y = y_1 \cdots y_p$ in such a way that no letter is R-compatible with all other letters of the alphabet and $y_i \, R \, \mathcal{W}_i$ for all $i = 1, 2, \ldots, p$. First, consider the bound $\overline{C} = (q-1)p + 1$.

(a) Assume that $q < p$ and q is even. Set $\mathcal{A} = \{a_1, \ldots, a_q\}$ and $w = (a_1 \cdots a_q)^{(\overline{C}-1)/q}$. Since q is even, we can make a partition P of the set $\{i' \mid i = 1, 2, \ldots, q\} = \{1, 2, \ldots, q\}$ into pairs, i.e., subsets of cardinality two. If m and n belong to the same subset in P, we denote $(m, n) \in P$ and define

$$(a_m, a_n) \notin R. \tag{7}$$

Define furthermore that these are the only R-incompatible pairs. Hence, each letter is incompatible with exactly one other letter. Taking benefit of this partition P, we define for every $i, j \in \{1, 2, \ldots, q\}$ satisfying $(i', j') \in P$ that

$$y_i = a_{j'} \quad \text{and} \quad y_j = a_{i'}. \tag{8}$$

Then $y_i = a_{j'} \, R \, \mathcal{W}_i = \mathcal{A} \setminus \{a_{i'}\}$ for $i \in \{1, 2, \ldots, q\}$. Furthermore, set $y_i = y_{[i]_q}$ for $i = q+1, q+2, \ldots, p$. Note that $\mathcal{W}_i \subseteq \mathcal{W}_{[i]_q}$. Namely, if $i = [i]_q + tq \leq p$, then by the q-periodicity of w,

$$\mathcal{W}_i = \{w_j \mid j \equiv [i]_q + tq \pmod{p}\} \subseteq \{w_{j-tq} \mid j - tq \equiv [i]_q \pmod{p}\} = \mathcal{W}_{[i]_q}.$$

Hence, $y_i \, R \, \mathcal{W}_i$ for all $i = 1, 2, \ldots, p$.

(b) Assume that $q > p+1$ and p is even. Consider the word

$$w = (ab^{q-p-1} a_{q-p+1} a_{q-p+2} \cdots a_q)^{(\overline{C}-1)/q}$$

in the alphabet $\mathcal{A} = \{a, b, a_{q-p+1}, a_{q-p+2}, \ldots, a_q\}$. We make a partition P of the set $\{q-p+1, q-p+2, \ldots, q\}$ into pairs like in (a). This is possible since the set has p elements and p is even. Define R-incompatible pairs by Eq. (7). Since this concerns only letters $\{a_{q-p+1}, a_{q-p+2}, \ldots, a_q\}$, we also set $(a, b) \notin R$. Let these be the only R-incompatible pairs. It is clear that no letter in $\mathcal{A}$ is compatible with all other letters. We use Eq. (8) to define the external word y. This is possible, since $i' = q - p + i$ for all $i = 1, 2, \ldots, p$. We conclude that $y_i = a_{j'} \, R \, \mathcal{W}_i = \mathcal{A} \setminus \{a_{i'}\}$.

(c) Assume that $q > p$ and p is odd. Set $\mathcal{A} = \{a, a_{q-p+1}, a_{q-p+2}, \ldots, a_q\}$ and

$$w = (a^{q-p} a_{q-p+1} a_{q-p+2} \cdots a_q)^{(\overline{C}-1)/q}.$$

Since p is now odd, we partition only the set $\{q-p+1, q-p+2, \ldots, q-1\}$ and make $(p-1)/2$ incompatible pairs using Eq. (7). Additionally, set $(a_q, a) \notin R$.

Assume moreover that these are the only R-incompatible pairs. Again, each letter is incompatible with exactly one other letter. Define $y_1 \cdots y_{p-1}$ using Eq. (8) and set $y_p = a$. Now $y_i \, R \, \mathcal{W}_i$ for $i = 1, 2, \ldots, p-1$ like in the previous cases and $y_p = a \, R \, \mathcal{W}_p = \mathcal{A} \setminus \{a_q\}$.

Next assume that $\overline{C} = (q-2)p + (q-1)$. In the following four subcases the alphabet is $\mathcal{A} = \{a_1, \ldots, a_q\}$ and the word is $w = (a_1 \cdots a_q)^{(\overline{C}-1)/q}$.

(d) Assume that $q = p+1$ and q is odd, i.e., $p = q-1$ is even. Like in (b), we form P and R-incompatible pairs (7) by partitioning the set $\{q-p+1, q-p+2, \ldots, q\}$. Define also the external word as in (b). In this case, there is no letter b like in (b) and the definition (7) does not concern a_1. In order to forbid a_1 to be related to all the other letters, we define $(a_1, y_p) \notin R$. Note that since $\overline{C} = (q-2)p + (q-1) = (q-2)p + p$, $\mathcal{W}_p = \mathcal{A} \setminus \{a_q, a_1\}$. Hence, $y_i \, R \, \mathcal{W}_i$ for all i, especially for $i = p$.

(e) Assume that $q < p$, q is odd and neither $p+1$ nor $p-1$ is divisible by q. Denote

$$\begin{aligned} a &= a_{[(q-2)p+(q-1)]_q}, & b &= a_{[(q-2)p+q]_q}, \\ c &= a_{[(q-1)p+(q-1)]_q}, & d &= a_{[(q-1)p+q]_q}. \end{aligned}$$

Note that by the above divisibility properties all these four letters are different. Now $\{a_{i'} \mid i = 1, 2, \ldots, q-2\} = \mathcal{A} \setminus \{c, d\}$. Hence, there exist numbers $k, l \in \{1, 2, \ldots, q-2\}$ such that $a_{k'} = a$ and $a_{l'} = b$. We make a partition P of the set $\{1, 2, \ldots, q-2\} \setminus \{l\}$ into pairs. This is possible since the set contains an even number $q-3$ of elements. We use Eq. (7) to define R-incompatible pairs of P and furthermore, define $(b, c) \notin R$ and $(b, d) \notin R$. Let these be the only incompatible pairs. Hence, except for b, all other letters are R-incompatible with exactly one other letter. Now consider an external word $y = y_1 \cdots y_p$. For indices in the partitioned set, use Eq. (8) like before. In addition, set $y_l = c$, $y_{q-1} = y_k$ and $y_q = d$. Furthermore, like in (a), set $y_i = y_{[i]_q}$ for $i = q+1, q+2, \ldots, p$. Now

$$\begin{aligned} y_l = c \quad R \quad \mathcal{W}_l \quad &= \quad \mathcal{A} \setminus \{b\}, \\ y_{q-1} = y_k \quad R \quad \mathcal{W}_{q-1} \quad &= \quad \mathcal{A} \setminus \{a, c\}, \\ y_q = d \quad R \quad \mathcal{W}_q \quad &= \quad \mathcal{A} \setminus \{b, d\}, \end{aligned}$$

and $y_i \, R \, \mathcal{W}_i$ by Eq. (8) for all the other indices $i \in \{1, 2, \ldots, q-2\} \setminus \{l\}$.

(f) Assume that $q < p$, q is odd and $p + 1 \equiv 0 \pmod{q}$. We use the same notation as in (e). Since $p+1 \equiv 0 \pmod{q}$, $a = d$. Clearly $b \notin \{c, a\}$. Now $\{a_{i'} \mid i = 1, 2, \ldots, q-2\} = \mathcal{A} \setminus \{c, a\}$. Thus, there does not exist $k \in \{1, 2, \ldots, q-2\}$ such that $a_{k'} = a$, but we have l like in (e). Define the relation R and the external word y as in (e) except that now $y_{q-1} = b$. Hence, no letter is related to all the other letters and y is well defined. Namely,

$$y_{q-1} = b \quad R \quad \mathcal{W}_{q-1} \quad = \quad \mathcal{A} \setminus \{a, c\}.$$

(g) Assume that $q < p$, q is odd and $p - 1 \equiv 0 \pmod{q}$. Using the notation of (e), we conclude that $b = c$. Clearly $a \notin \{b, d\}$. Now we have $\{a_{i'} \mid i = 1, 2, \ldots, q-2\} = \mathcal{A} \setminus \{b, d\}$. Hence, using the notation of (e) there exists k but no l in $\{1, 2, \ldots, q-2\}$. This time we make a partition P of the set $\{1, 2, \ldots, q-2\} \setminus \{k\}$ into subsets of cardinality two. Set Eq. (7) and define furthermore that $(a, b) \notin R$ and $(a, d) \notin R$. Assume again that these are the only R-incompatible pairs. In addition to Eq. (8) set $y_k = b$, $y_{q-1} = b$ and $y_q = a$. Again no letter is compatible with all the other letters and y is well defined, since

$$\begin{array}{rcl} y_k = b \quad R \quad \mathcal{W}_k & = & \mathcal{A} \setminus \{a\}, \\ y_{q-1} = b \quad R \quad \mathcal{W}_{q-1} & = & \mathcal{A} \setminus \{a, b\}, \\ y_q = a \quad R \quad \mathcal{W}_q & = & \mathcal{A} \setminus \{b, d\}. \end{array}$$

Hence, we have showed that in all cases there exists a word of length $\overline{C}$ such it has a pure period q and an external word $y = y_1 \cdots y_p$ but 1 is not an external R-period. Moreover, the external word y satisfies Eq. (4) in all the cases. $\square$

We may also consider restricted external periods and have a bound for restricted external-external interaction like in Theorem 10.

Theorem 13. *Let p and q be positive integers with $\gcd(p, q) = 1$. Then the bound of external-external interaction for a restricted external period p and pure period q is*

$$C_e(p, q) = \begin{cases} (q-2)p + (q-1) & \text{if } q \text{ is odd and } q < p, \\ (q-1)p & \text{if } p \text{ is even and } q \geq p+1, \\ (q-1)p + 1 & \text{otherwise.} \end{cases}$$

Proof. It suffices to consider only the case where p is even and $q \geq p + 1$. Lemma 7 and Lemma 8 cover the other cases. For the restricted external period, the maximal number of different letters in w is $p+1$. Hence, in order to show that the bound $C_e = (q-1)p$ is sufficient, set $t = p+1$ in Lemma 7. Finally, consider the word

$$w = (a^{q-p} a_{q-p+1} a_{q-p+2} \cdots a_q)^{(C_e - 1)/q}$$

in the alphabet $\mathcal{A} = \{a, a_{q-p+1}, a_{q-p+2}, \ldots, a_q\}$ with length $C_e - 1$. We make a partition P of the set $\{q-p+1, q-p+2, \ldots, q\}$ into pairs. Define R-incompatible pairs by Eq. (7) and use Eq. (8) to define the external word y like in Lemma 8.(b). Note that $\mathrm{Alph}(y) = \{a_{q-p+1}, a_{q-p+2}, \ldots, a_q\}$. In order to forbid a to be related to all the other letters, we define $(a, y_p) \notin R$.Hence, no letter is related to all other letters. Note also that since $\overline{C} - 1 = (q-2)p - 1$, $\mathcal{W}_p = \mathcal{A} \setminus \{a_q, a\}$. Hence, $y_i \, R \, \mathcal{W}_i$ for all i, especially for $i = p$. This means that w has a pure period q and a restricted external period p, but 1 is not an external period of w. $\square$

8 Local interactions

Despite the negative result in Example 5 there exist interaction bounds for some integers p and q also in the case where p is local. If no bound B exists, i.e., there is an infinite word w such that $\gcd(p, q)$ is not a t_2-type period of w, we set $B = \infty$.

Theorem 14. *Let p and q be positive integers with $\gcd(p,q)=1$. Then the bound of local-local interaction for p and q is*

$$D_l = \begin{cases} p+q & \text{if } p-1 \equiv 0 \pmod q \text{ or } p+1 \equiv 0 \pmod q, \\ \infty & otherwise. \end{cases}$$

Proof. Let w be a word of length D_l with a pure period q and a local period p. Suppose that $\gcd(p,q)=1$. Assume first that $p+1 \equiv 0 \pmod q$. By the periodicity assumption, we then have

$$w_i \, R \, w_{i+p} = w_{i-1}$$

for all $i = 2, 3, \dots, q$ and furthermore $w_1 \, R \, w_{1+p} = w_q$. Since q is a period of w, 1 is a local R-period of w. On the other hand, if we set $R = \langle\{(a,c),(b,c)\}\rangle$, the word

$$w = (c^{q-2}ab)^{(p+q-1)/q}$$

has a pure period q and a local R-period p. However, $\gcd(p,q)=1$ is not a local R-period of w, since $(w_{q-1}, w_q) \notin R$. Note that in order to check that w has a local period p, it suffices to ensure that the distance from any occurrence of a to any occurrence of b is not p. By the length of w this holds. Namely, we have $a = w_{q-1} \, R \, w_{q-1+p} = w_{q-2} = c$ and if $q = p+1$, then also $b = w_q \, R \, w_{q-p} = w_1 = c$.

Assume next that $p-1 \equiv 0 \pmod q$. Now $w_i \, R \, w_{i+p} = w_{i+1}$ for all $i = 1, 2, \dots, q$. Like above, this means that w has a local R-period 1. Our bound is strict, since setting again $R = \langle\{(a,c),(b,c)\}\rangle$, the word

$$w = (ac^{q-2}b)^{(p+q-1)/q}$$

has a pure period q and a relational R-period p. However, $(w_q, w_{q+1}) \notin R$ and 1 is not a local R-period. Again the length of w ensures that a and b do not have to be related. We only check that $a = w_1 \, R \, w_{1+p} = w_2 = c$, which is satisfied.

Finally, assume that q does not divide $p-1$ nor $p+1$. Then $w_{i+p} \neq w_{i+1}$ and $w_{i+p} \neq w_{i-1}$ if $\text{Alph}(w) = q$. Thus, if $R = \langle\{(a,c),(b,c)\}\rangle$, then the infinite word

$$w = (abc^{q-2})^{\omega}$$

has a pure period q and a local R-period p, but clearly 1 is not a local R-period of w. □

Local periods are really weak when considering other interactions.

Theorem 15. *Let p and q be positive integers with $\gcd(p,q)=1$. The bounds D_e of local-external interaction and D_g of local-global interaction do not exist, except for $p=2$ and $q=3$, in which case $D_e = D_g = 5$.*

Proof. Suppose first that w is a word with a pure period 3 and a local period 2. If $|w| \geq 5$, we must have $w_i \, R \, \text{Alph}(w)$ for $i = 1, 2, 3$. Hence, 1 is a global and external R-period. Otherwise, consider a four letter alphabet $\{a,b,c,d\}$ and define $R = \langle\{(a,b),(b,c),(c,d),(d,a)\}\rangle$. By Lemma 2, we may assume that $\gcd(p,q) = 1$. Define an infinite word $w = (w_1 \cdots w_q)^\omega$ in the following way. Set

$$w_1 = a,\ w_{[1+p]_q} = b,\ w_{[1+2p]_q} = c \text{ and } w_{[1+ip]_q} = d$$

for $i = 3,4,\ldots,q-1$. Now, by the definition of R, $w_i \, R \, w_{i+p}$ for all $i = 1,2,\ldots,q$. Hence, p is a local R-period of w. However, 1 is not an external neither a global R-period, since no letter is compatible with all the other letters. Hence, $D_e = \infty$. □

9 Summary of bounds

In order to get a clearer picture of all the different variants of Fine and Wilf's theorem represented in the previous sections, we summarize the bounds in Table 2.

By Theorem 1, a global period is a stronger attribute than the other periods, and therefore

$$B_g \geq B_e \text{ and } B_g \geq B_l,$$

for every p and q. Observe also that B-bounds (B_g, B_e and B_l) are in many cases smaller than the other bounds.

On the other hand, if we compare the bounds of global-external and global-local interaction we see, for example, that

$$\begin{array}{rcccccl} B_e(5,9) & = & 23 & > & 19 & = & B_l(5,9), \\ B_e(4,7) & = & 15 & = & 15 & = & B_l(4,7), \\ B_e(3,5) & = & 8 & < & 10 & = & B_l(3,5). \end{array}$$

This indicates the incomparability of external relational period and local relational period, which was already seen in Examples 3 and 4 with respect to minimal periods. However, in some sense the local period seems to be the weakest. In the case where p is an external period, we get interaction bounds, at least, if we assume an extra condition. In the case of a local period p, bounds usually do not even exist. Furthermore, note that

$$C_e < C_g = C_l \text{ and } D_l \leq D_g = D_e.$$

Hence, in these cases the interaction bound is lower if the type of the period $\gcd(p,q)$ is same as the type of the period p. As a final example, we give a complete table of interaction bounds for $p = 6$ and $q = 7$.

interaction type	bound
global-global	$B_g = \begin{cases} \frac{p+1}{2}q & \text{if } (p < q \text{ and } p \text{ is odd}) \\ & \text{or } (p > q \text{ and } q \text{ is even}), \\ q + \frac{q-1}{2}p & \text{otherwise.} \end{cases}$
global-external	$B_e = \begin{cases} \min(h + [h]_q - 1, h + (q - [h]_q) + 1) & \text{if } q \text{ is odd}, \\ \max(h, h + [h]_q - (p+1)) & \text{if } q \text{ is even.} \end{cases}$
global-local	$B_l = \begin{cases} q + kp - 1 & \text{if } 1 \equiv q - 1 \pmod{p} \\ & \text{and } kp \equiv +1 \pmod{q}, \\ q + kp & \text{otherwise.} \end{cases}$
restricted ext.-global	$C_q = pq$
restricted ext.-ext.	$C_e = \begin{cases} (q-2)p + (q-1) & \text{if } q \text{ is odd and } q < p, \\ (q-1)p & \text{if } p \text{ is even and } q \geq p+1, \\ (q-1)p + 1 & \text{otherwise.} \end{cases}$
restricted ext.-local	$C_l = pq$
external-global	∞
external-external	$C = 1 + (q-1)p$
external-local	∞
local-global	$D_g = \begin{cases} 5 & \text{if } p = 2 \text{ and } q = 3 \\ \infty & \text{otherwise} \end{cases}$
local-external	$D_e = \begin{cases} 5 & \text{if } p = 2 \text{ and } q = 3 \\ \infty & \text{otherwise} \end{cases}$
local-local	$D_l = \begin{cases} p + q & \text{if } p - 1 \equiv 0 \pmod{q} \\ & \text{or } p + 1 \equiv 0 \pmod{q}, \\ \infty & \text{otherwise.} \end{cases}$

Table 2: Interaction bounds for p and q, where $\gcd(p, q) = 1$, $h = 1 + \lfloor q/2 \rfloor p$ and k is the smallest integer such that $kp \equiv \pm 1 \pmod{q}$.

t_1 \ t_2	global	external	local
global	25	22	13
restricted external	42	36	42
external	∞	37	∞
local	∞	13	∞

Table 3: Interaction bounds for $p = 6$ and $q = 7$.

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ISBN 952-12-1839-8
ISSN 1239-1891