Finite Metrics in Switching Classes

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Abstract

Let $g\colon D \times D \to \mathbb{R}$ be a symmetric function on a finite set D satisfying $g(x,x) = 0$ for all $x \in D$. A switch g^σ of g w.r.t. a local valuation $\sigma\colon D \to \mathbb{R}$ is defined by $g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)$ for $x \neq y$ and $g^\sigma(x,x) = 0$ for all x. We show that every symmetric function g has a unique minimal semimetric switch, and, moreover, there is a switch of g that is isometric to a finite Manhattan metric. Also, for each metric on D, we associate an extension metric on the set of all nonempty subsets of D, and we show that this extended metric inherits the switching classes on D.

1 Introduction

Finite metric spaces are useful in many applications, where one needs to measure distances or dissimilarities of objects that come out from a large storage of objects, see, e.g., Linial [10]. In some cases, however, the first natural measure to be considered might not be properly a distance function, and it may

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be necessary to distort the measure in order to estimate it by a distance function. We consider this problem with respect to a graph theoretic operation of switching. Our distortions are governed by the local switching operation of the complete undirected graphs, where the edges are weighed by real numbers. Such a graph g on a set D of vertices will be identified with a function $g\colon D\times D \to \mathbb{R}$, called a *symmetric function* (*on* D), that satisfies the following properties, for all $x, y \in D$, $g(x,x) = 0$ and $g(x,y) = g(y,x)$.

Switching of unweighed graphs was introduced by Van Lint and Seidel [11] in connection with a problem in elliptic geometry. For surveys of this topic, see [4,7,8,12,13]. Symmetric functions are special cases of *2-structures* which were introduced in [5] as a framework for decomposition of finite discrete systems. Switching was generalized in [6] to 2-structures under the name of 'dynamic labelled 2-structures', where the dynamic aspect was motivated by the theory of graph transformation systems.

Let g be a symmetric function on a finite domain D. The *switch* of g with respect to a function $\sigma\colon D \to \mathbb{R}$ is the symmetric function g^σ defined by

$$g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)\,,$$

for all $(x,y) \in D\times D$ with $x \neq y$, and $g^\sigma(x,x) = 0$ for all $x \in D$. The *switching class* of g is the set $[g] = \{g^\sigma \mid \sigma\colon D \to \mathbb{R}\}$ of all switches of g.

A symmetric function g can be considered as a generalized distance function allowing negative values, and which need not satisfy the triangle inequality. We shall show that every symmetric function g has a switch that is a metric, and, moreover, each g has a unique minimal semimetric switch. We also show that each symmetric function g has a switch g^σ that is isometric to a finite Manhattan metric. This is interesting also from the point of view of algorithmic complexity, since it is known that the embedding problem of finite metrics to the Manhattan space is NP-complete, see Karzanov [9]. Finally, we consider domains D with weightings $w\colon D \to \mathbb{R}$, where $w(x) > 0$ for all $x \in D$. We extend each metric (symmetric function) g on D to a metric g_w on the set of all nonempty subsets of D. Here $g_w(X,Y)$ corresponds to the weighted mean value of the connections in g between the elements of X and Y. This extension inherits the switching classes on D, i.e., if g is a switch of h then g_w is a switch of h_w for the extensions of g and h.

2 Semimetrics

We shall consider finite semimetric spaces, i.e., pairs (D,d), where D is a finite set of points and $d\colon D\times D \to \mathbb{R}$ is a function, called a *semimetric*, that satisfies the following conditions, for all $x, y, z \in D$,

(i) $d(x,x) = 0,\ \ d(x,y) \geq 0$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq d(x,z) + d(z,y)$.

Hence every semimetric is a symmetric function. Moreover, a semimetric d is a *metric*, if $d(x,y) = 0$ implies $x = y$.

Example 1 Let $G = (D,E)$ be an undirected connected graph, i.e., the domain D is a finite set of vertices and E is a set of edges $\{x,y\}$, $x,y \in D$ with $x \neq y$. Define $d_G \colon D \times D \to \mathbb{R}$ such that $d_G(x,y)$ is the length of a shortest path from x to y in G. Then d_G is a metric on D. $\square$

The functions from a finite set D to $\mathbb{R}$ are provided with the usual operations: $(\sigma + \tau)(x) = \sigma(x) + \tau(x)$ and $(r\sigma)(x) = r \cdot \sigma(x)$, where $r \in \mathbb{R}$ is a constant.

For a symmetric function $g \colon D \times D \to \mathbb{R}$, the switching class $[g]$ is generated by each of its elements, that is, $[g] = [g^\sigma]$ for all $\sigma \colon D \to \mathbb{R}$. This follows from the equality $(g^\sigma)^{-\sigma} = g^{\sigma - \sigma} = g$.

It is clear that if $\sigma(x) = s$ for sufficiently large $s \in \mathbb{R}$, then g^σ is metric. Indeed, for this we can choose any $s > (3/2) \cdot \max\{|g(x,y)| \mid x,y \in D\}$. Therefore all symmetric functions are switches of metrics:

Theorem 2 *Let g be a symmetric function. Then the switching class $[g]$ contains a metric.*

Define a partial order on the symmetric functions on D by $g \leq h$ if and only if $g(x,y) \leq h(x,y)$ for all $x,y \in D$. We shall refer to this ordering as the *natural ordering* of the symmetric functions.

If x,y,z are three different elements of a set D, then the ordered triple (x,y,z) is called a *triangle* in D. For each triangle (x,y,z) and each symmetric function g on D, we let

$$\Delta_g(x,y,z) = g(x,y) + g(x,z) - g(y,z)\,.$$

Now, for each $\sigma \colon D \to \mathbb{R}$, we have $\Delta_{g^\sigma}(x,y,z) = \Delta_g(x,y,z) + 2\sigma(x)$.

Theorem 3 *Let g be a symmetric function. Then the switching class $[g]$ contains a unique minimal semimetric with respect to the natural ordering.*

PROOF. Let g be on the domain D. For $|D| = 1$ the claim is obvious, and if $|D| = 2$, then there exists only one switching class on D, and the minimum semimetric on D is the zero function. Assume then that $|D| \geq 3$, and define

$$\sigma(x) = -(1/2) \cdot \min\{\Delta_g(x,y,z) \mid y,z \in D, (x,y,z) \text{ a triangle}\}\,. \tag{1}$$

We show that g^σ is the unique minimal semimetric in $[g]$. Let $x, y, z \in D$. By (1), $\sigma(x) + \sigma(y) \geq -g(x, y)$, that is, $g^\sigma(x, y) \geq 0$. Similarly, using (1) for $\sigma(z)$, we obtain

$$\begin{aligned} g^\sigma(x,z) + g^\sigma(z,y) &= \sigma(x) + g(x,z) + 2\sigma(z) + g(z,y) + \sigma(y) \\ &\geq \sigma(x) + g(x,y) + \sigma(y) = g^\sigma(x,y)\,, \end{aligned}$$

which shows that g^σ is a semimetric.

For minimality, assume that, for each $x \in D$, the minimum in (1) is obtained in a triangle (x, y_x, z_x) for some $y_x, z_x \in D$. By (1), we have that $\Delta_{g^\sigma}(x, y_x, z_x) = 2\sigma(x) + \Delta_g(x, y_x, z_x) = 0$. Assume then that the switch g^τ is a semimetric for some $\tau\colon D \to \mathbb{R}$. We have $g^\tau = (g^\sigma)^{\tau-\sigma}$, and thus

$$\begin{aligned} 0 \leq \Delta_{g^\tau}(x, y_x, z_x) &= (\tau-\sigma)(x) + g^\sigma(x, y_x) + (\tau-\sigma)(y_x) \\ &+ (\tau-\sigma)(x) + g^\sigma(x, z_x) + (\tau-\sigma)(z_x) - (\tau-\sigma)(y_x) - g^\sigma(y_x, z_x) \\ &- (\tau-\sigma)(z_x) = 2\cdot(\tau-\sigma)(x)\,, \end{aligned}$$

from which $\sigma \leq \tau$ follows. This implies $g^\tau(x, y) = (g^\sigma)^{\tau-\sigma}(x, y) = (\tau-\sigma)(x) + g^\sigma(x, y) + (\tau-\sigma)(y) \geq g^\sigma(x, y)$, where equality holds if and only if $\tau = \sigma$. This proves the claim. □

For each symmetric function g, we denote by $\min(g)$ the *unique minimum semimetric* in $[g]$ provided by Theorem 3.

Example 4 Let $D = \{1, 2, \ldots, n\}$ for $n \geq 4$, and let g be defined by $g(i, j) = (-1)^{i+j}$ for all $i, j \in D$ with $i \neq j$, and $g(i, i) = 0$ for each i. Then g is a symmetric function that is not a semimetric for $n \geq 2$, since g attains negative values. The function σ in the proof of Theorem 3 is constant, $\sigma(i) = 3/2$, since, for each i, one can always choose j and k such that $i+j$ and $i+k$ are odd and $j+k$ is even. Therefore, for $i \neq j$, we have $g^\sigma(i, j) = g(i, j) + 3 = 3 + (-1)^{i+j}$. Since $g^\sigma(i, j) > 0$ for all $i \neq j$, this unique minimum semimetric is also a metric. □

Theorem 5 *Let g be a symmetric function on D with $|D| \geq 3$, and let $\tau\colon D \to \mathbb{R}$. The switch $(\min(g))^\tau$ is a semimetric if and only if τ is non-negative.*

PROOF. Let $h = \min(g)$. For each triangle (x, y, z), we have $\Delta_{h^\tau}(x, y, z) = \Delta_h(x, y, z) + 2\tau(x)$, and hence, if $\tau(x) \geq 0$ for all x, then h^τ is a semimetric, since h is a semimetric. Also, by the proof of Theorem 3, for each x, there exists a triangle (x, y, z) such that $\Delta_h(x, y, z) = 0$. By the above, $\Delta_{h^\tau}(x, y, z) = 2\tau(x)$, and so if h^τ is a semimetric, then $\tau(x) \geq 0$. □

By Theorem 5 we have immediately

Corollary 6 *Let g be a symmetric function. If* $\min(g)$ *is a metric, then so are all semimetrics in* $[g]$.

3 Manhattan geometry

We consider the n-dimensional space $\mathbb{R}^n$ of real vectors. The *Manhattan metric* on $\mathbb{R}^n$ (see, e.g., [2,3]) is defined by

$$d_L(\bar{x}, \bar{y}) = \sum_{i=1}^{n} |x_i - y_i| \tag{2}$$

for all vectors $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$. The metric space $(\mathbb{R}^n, d_L)$ is called an L_1*-space*.

There are finite metrics that can be embedded into the Manhattan space L_1, but not into the Euclidean space with its usual metric. One such metric is defined by $g(x_1, x_i) = 1$ for $i = 2, 3, 4$ and $g(x_i, x_j) = 2$ for $i \neq 1$ and $j \neq 1$.

We say that a symmetric function g is *semi-Manhattan of dimension* n, if there exists a mapping $\alpha \colon D \to \mathbb{R}^n$ such that $g(x, y) = d_L(\alpha(x), \alpha(y))$. If the mapping α is injective, then it is called a *Manhattan isometry for* g, and, in this case, g is a *Manhattan function of dimension* n.

It is well known that all 4-point metrics can be embedded into the Manhattan space; see Remark 3.2.5 of [3]. However, not all finite metrics are Manhattan. Indeed, the distance metric of the complete bipartite graph $K_{2,3}$. is a 5-element metric that is not Manhattan; see [2,3].

It is also interesting to note that the problem whether a finite metric is isometric to a Manhattan metric, is NP-complete, see Karzanov [9].

A semimetric d on D is called a *cut semimetric*, if there is a subset $S \subseteq D$ such that $d(x, y) = 0$ if either $x, y \in S$ or $x, y \notin S$; otherwise $d(x, y) = 1$. The following general result is due to Assouad [1], see also [3].

Theorem 7 *A finite metric can be embedded in* L_1 *if and only if it is a linear combination of cut semimetrics with nonnegative coefficients.*

It follows from this that the sum of two Manhattan functions is also Manhattan.

In the following theorem it is shown that if g is a Manhattan function, then the switching class $[g]$ contains excessively many Manhattan functions.

Theorem 8 *If g is a Manhattan function, so is g^σ for all nonnegative σ.*

PROOF. From Theorem 7 it follows that if g is a Manhattan function, so is g^σ for all nonnegative σ. $\square$

As an immediate corollary to Corollary 6 and Theorem 8, we obtain

Corollary 9 *The minimum semimetric* $\min(g)$ *of the switching class* $[g]$ *is a Manhattan function if and only if all semimetrics in* $[g]$ *are Manhattan.*

We proceed to show that every switching class does have Manhattan functions. To this end, let $A \subseteq D$ and let $g_{(A,a)}$ be defined by

$$g_{(A,a)}(x,y) = \begin{cases} 0 & \text{if } x,y \in A \text{ or } x,y \in D \setminus A\,, \\ a & \text{otherwise}\,. \end{cases}$$

Lemma 10 *Let $A \subseteq D$ and $a \in \mathbb{R}$ be nonnegative. Then $g_{(A,a)}$ is a semi-Manhattan function of dimension* 1.

PROOF. Indeed, consider $\alpha(x) = a$ for $x \in A$ and $\alpha(x) = 0$ for $x \in D \setminus A$. Then $d_L(\alpha(x), \alpha(y)) = g_{(A,a)}(x,y)$ for all $x, y \in D$. $\square$

Theorem 11 *Each switching class contains a Manhattan function.*

PROOF. We can assume that $|D| \geq 4$, since each metric on three elements can be embedded in L_1.

Obviously, each switching class has a negative symmetric function. Assume then that g is negative. For all $u \neq v$, let $g_{uv} = g_{(\{u,v\},-(1/4)g(u,v))}$. Since g is negative, each function g_{uv} is nonnegative. By Lemma 10, each g_{uv}, and hence also the sum $h = \sum_{u\in D}\sum_{v\in D\setminus\{u\}} g_{uv}$ over all ordered pairs $(u,v) \in D \times D$ with $u \neq v$ is semi-Manhattan. Now, $g_{uv} = g_{vu}$ for all u and v, and moreover, $g_{uv}(x,y) = 0$ unless exactly one of u or v is in $\{x,y\}$. Let $A = D \setminus \{x,y\}$. Then we have

$$\begin{aligned} h(x,y) &= 2\sum_{v\in A} g_{xv}(x,y) + 2\sum_{u\in A} g_{uy}(x,y) \qquad (3) \\ &= -(1/2)\sum_{v\neq y} g(x,v) - (1/2)\sum_{u\neq x} g(u,y) \\ &= -(1/2)\sum_{v\in D} g(x,v) - (1/2)\sum_{v\in D} g(y,v) + g(x,y)\,. \end{aligned}$$

Therefore $h = g^\sigma$ for the nonnegative mapping $\sigma(x) = -(1/2)\sum_{v\in D} g(x,v)$. Hence $h \in [g]$. Moreover, if $v \in D \setminus \{x,y\}$, then, by the definition of g_{xv}, we have $g_{xv}(x,y) > 0$. Thus, by (3), $h(x,y) > 0$ whenever $x \neq y$. This proves the claim. □

4 Mean invariance

Denote by $\mathcal{P}_+(D) = \{X \mid X \subseteq D,\ X \neq \emptyset\}$ the set of all nonempty subsets of D. Quotients of 2-structures are defined with respect to partitions of the domain into clans, see, e.g., [4]. Such partitions can be avoided in the present approach of metrics. Indeed, if g is a metric on D, we can define a metric on the set $\mathcal{P}_+(D)$ such that switching classes are inherited through this transformation.

Let D be a finite set. Each function $f\colon D \to \mathbb{R}$ will be extended to the subsets $X \subseteq D$ by setting

$$f(X) = \sum_{x\in X} f(x)\,.$$

By a *weighting* we mean a positive function w on D.

Let g be a symmetric function on D, and let w be a weighting on D. With respect to w, we extend g to $g_w\colon \mathcal{P}_+(D)\times\mathcal{P}_+(D) \to \mathbb{R}$ as follows: $g_w(X,X) = 0$ and

$$g_w(X,Y) = \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)g(x,y)}{w(X)w(Y)} \qquad \text{if } X \neq Y\,. \tag{4}$$

The function g_w is well defined since $w(X) > 0$ for all $X \neq \emptyset$. In the above definition, we do not require that the subsets X and Y are disjoint. Notice that g_w does extend the function d, since for the singleton pairs, we have $g_w(\{x\},\{y\}) = g(x,y)$.

Lemma 12 *Let $g\colon D\times D \to \mathbb{R}$ be a metric, and let w be a weighting on D. Then g_w is a metric on $\mathcal{P}_+(D)$.*

PROOF. Let X, Y and Z be in $\mathcal{P}_+(D)$. We can assume that they are all distinct subsets of D. Now

$$\begin{aligned}
&w(Z) \cdot (g_w(X,Z) + g_w(Z,Y)) = \\
&= \frac{\sum_{x\in X}\sum_{z\in Z} w(x)w(z)g(x,z)}{w(X)} + \frac{\sum_{y\in Y}\sum_{z\in Z} w(y)w(z)g(z,y)}{w(Y)} \\
&= \sum_{z\in Z} w(z) \frac{\sum_{x\in X}\sum_{y\in Y} (w(x)w(y)g(x,z) + w(x)w(y)g(z,y))}{w(X)w(Y)} \\
&\geq \sum_{z\in Z} w(z) \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)g(x,y)}{w(X)w(Y)} = w(Z)g_w(X,Y)\,,
\end{aligned}$$

which shows that $g_w(X,Y) \leq g_w(X,Z) + g_w(Z,Y)$. □

Recall that by Theorem 2, every switching class contains a metrics. In the following theorem the domains of the symmetric functions g_w will be $\mathcal{P}_+(D)$.

Theorem 13 *Let h be a metric on the domain D, and let w be a weighting on D. If $g \in [h]$, then also $g_w \in [h_w]$.*

PROOF. Let $\sigma\colon D \to \mathbb{R}$ be such that $g = h^\sigma$, and define $\bar{\sigma}\colon \mathcal{P}_+(D) \to \mathbb{R}$ such that, for each $X \subseteq D$,

$$\bar{\sigma}(X) = \frac{\sum_{x\in X} w(x)\sigma(x)}{w(X)}\,.$$

For $X \neq Y$, we have

$$\begin{aligned}
&h_w^{\bar{\sigma}}(X,Y) = \bar{\sigma}(X) + h_w(X,Y) + \bar{\sigma}(Y) \\
&= \frac{\sum_{x\in X} w(x)\sigma(x)}{w(X)} + \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)h(x,y)}{w(X)w(Y)} + \frac{\sum_{y\in Y} w(y)\sigma(y)}{w(Y)} \\
&= \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)g(x,y)}{w(X)w(Y)} = g_w(X,Y)\,,
\end{aligned}$$

as required. □

As a special case, for any symmetric function g, consider the constant weighting $w(x) = 1$ on all vertices. In this case,

$$g_w(X,Y) = \sum_{x\in X}\sum_{y\in Y} g(x,y)/|X| \cdot |Y|\,.$$

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