The Size of Switching Classes with Skew Gains

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Abstract

Gain graphs are graphs in which each edge has a gain (a label from a group, say Γ, so that reversing the direction inverts the gain). In this paper we take a generalized view of gain graphs in which the gain of an edge is related to the gain of the reverse edge by an anti-involution. These gain graphs will be called skew gain graphs. We define switching by a selector, a generalization of switching (or Seidel switching) of an undirected graph. In this paper we compute the sizes of the resulting equivalence classes of skew gain graphs. This size can be determined by computing the size of an appropriate subgroup of Γ.

We first examine the case that the graph is complete. Then we show how to reduce the general problem to connected graphs and prove that if the graph is connected, but not bipartite, it can be reduced to the complete case. The connected, bipartite case is solved separately.

1 Introduction

Gain graphs are graphs in which each edge has a gain (a label from a group so that reversing the direction, inverts the gain), see Zaslavsky in [11]. In this paper we take a generalized view of gain graphs in which the gain of an edge is related to the gain of the reverse edge by an "anti-involution", i.e., an anti-automorphism of order at most two. These gain graphs will be called skew gain graphs.

In the special case when the group equals $\mathbf{Z}_2$, the skew gain graphs, which are ordinary gain graphs, are known as signed graphs, the study of which was originated by Harary [5]. For further results on signed graphs, we refer to papers by Zaslavsky, [9] and [10].

The present work is mainly motivated by the study of labelled 2-structures of Ehrenfeucht and Rozenberg [2].

Let Γ be a finite group. We shall say that a function $\delta : \Gamma \to \Gamma$ is an *anti-involution*, if it is an anti-automorphism of order at most two, that is, δ is a bijection and for all $x, y \in \Gamma$, $\delta(xy) = \delta(y)\delta(x)$ and $\delta^2(x) = x$. We write (Γ, δ) for a group Γ with a given anti-involution δ.

Let V be a finite set, and denote by $E_2(V) = \{(u, v) \mid u, v \in V, u \neq v\}$ the set of all nonreflexive ordered pairs of V. For a pair $e = (u, v) \in E_2(V)$, we denote by e^{-1} its *reverse pair* (v, u).

We consider graphs $G = (V, E)$ where the set of edges $E \subseteq E_2(V)$ satisfies the condition: if $e \in E$ then also $e^{-1} \in E$. Such graphs can be considered as undirected graphs where the edges have been given a two-way orientation.

Let $G = (V, E)$ be a graph and (Γ, δ) a finite group with anti-involution. A pair (G, g) where g is a mapping $g : E \to (\Gamma, \delta)$ into the group Γ is called a (Γ, δ)-*gain graph (on G)* (or a *graph with skew gains*), if g satisfies the condition

$$g(e^{-1}) = \delta(g(e)) \tag{1}$$

for all $e \in E$. In the future we will refer to a skew gain graph (G, g) simply by g unless confusion arises. The class of (Γ, δ)-gain graphs on G will be denoted by $\mathbf{L}_G(\Gamma, \delta)$ or simply by $\mathbf{L}_G$. A gain graph is a $(\Gamma, {}^{-1})$-gain graph; in topological graph theory these are called voltage graphs (see Gross and Tucker [4]).

For each function $\sigma : V \to \Gamma$, called a *selector*, we associate with g a new (Γ, δ)-gain graph g^σ on $G = (V, E)$ by letting, for each $(u, v) \in E$,

$$g^\sigma(u, v) = \sigma(u) \cdot g(u, v) \cdot \delta(\sigma(v)) \ .$$

We use $\mathbf{S}(V, \Gamma)$ or simply $\mathbf{S}$, to denote the set of the selectors from V into Γ. The set V will be referred to both as the *domain* of G as well as the domain of g.

The relation $g \sim h$, which holds if there exists a selector σ such that $h = g^\sigma$, is an equivalence relation on the (Γ, δ)-gain graphs, and we denote by

$$[g] = \{g^\sigma \mid \sigma : V \to \Gamma\}$$

the equivalence class, called the *switching class*, of g determined by this relation.

Further definitions and basic results on (Γ, δ)-gain graphs are given in Preliminaries. We shall now give a short account of the results of later sections.

The main part of this paper is devoted to an investigation of the sizes of the switching classes $\mathcal{G}$ of graphs with skew gains. We shall first study this problem for complete graphs $G = (V, E)$, i.e., the case where $E = E_2(V)$, so $g : E_2(V) \to (\Gamma, \delta)$. In the last section we reduce the general case where G is not bipartite to the complete case – the bipartite case is solved differently.

If G is complete, it turns out that, when (Γ, δ) is abelian, the sizes are the same for each switching class $\mathcal{G}$ with a given domain V, while in the nonabelian case the size of each $\mathcal{G}$ depends (only) on the "content number" $c(\mathcal{G})$, which is determined using the set of elements of Γ that appear in an $h \in \mathcal{G}$ for which $h(u, v) = 1_\Gamma$ for all $v \in V - \{u\}$ (see page 6) Such an h exists, and $c(\mathcal{G})$ turns out be independent of the choice of h and u. The formula will then read

$$|\mathcal{G}| = k^{n-1} \cdot k / c(\mathcal{G}) \ ,$$

where k is the order of the group Γ and n is the number of elements of the domain V. The content number $c(\mathcal{G})$ always divides the order k of the group Γ.

We conclude this section with some comments on the origin of the problem area. In [3] switching classes are called dynamic labelled 2-structures. This name comes from an interpretation of a group-labelled 2-structure as a network, and a switching class $\mathcal{G}$ as an evolutionary history of this network, when the evolution is carried out by selectors.

In the case where $\Gamma = \mathbf{Z}_2$, the cyclic group of order 2, g can be interpreted as an ordinary undirected simple graph as follows: $g(u,v) = 1$ means that (u,v) is an edge of g, and $g(u,v) = 0$ means that (u,v) is not an edge of g. Note that the identity mapping is the only anti-involution of $\mathbf{Z}_2$. In this case, the class $[g]$ is called the switching class of the graph g. We refer to [7] and [8] for connections to two-graphs and other combinatorial objects. In switching, a selector $\sigma : V \to \mathbf{Z}_2$ can be identified with the subset $S = \{u \mid u \in V, \sigma(u) = 1\}$, and therefore g^σ is obtained from the graph g by omitting the edges that enter S, and adding all nonedges that enter S, and leaving the edges and nonedges inside S and $V - S$ intact.

2 Preliminaries

In this paper V denotes always a finite set, Γ a finite group and δ one of its anti-involutions. The identity of Γ is denoted by 1_Γ.

We note that g^σ is well defined, that is, it satisfies the reversibility condition (1). Indeed,

$$\begin{aligned} g^\sigma(u,v) &= \sigma(u)g(u,v)\delta(\sigma(v)) = \sigma(u)\delta(g(v,u))\delta(\sigma(v)) = \sigma(u)\delta(\sigma(v)g(v,u)) \\ &= \delta(\sigma(v)g(v,u)\delta(\sigma(u))) = \delta(g^\sigma(v,u)) \ . \end{aligned}$$

The set $\mathbf{S}(V,\Gamma)$ of selectors can be made into a group in a natural way by defining the product elementwise: for all σ and τ,

$$(\sigma\tau)(u) = \sigma(u) \cdot \tau(u)$$

for all $u \in V$. Since $(g^\sigma)^\tau = g^{\tau\sigma}$, $\mathbf{S}(V,\Gamma)$ is a group that *acts on* the (Γ,δ)-gain graphs (on the left; that is, by the action $\sigma g = g^\sigma$), and therefore the orbits of this action are exactly the switching classes. From these considerations it follows that each switching class $\mathcal{G}$ is generated by each of its elements: for all σ and g, $[g^\sigma] = [g]$.

In the group $\mathbf{S}(V,\Gamma)$ the *trivial selector*, for which $\sigma_1(u) = 1_\Gamma$ for all $u \in V$, is the group identity; and the inverse σ^{-1} of a selector is found by inverting the selected values in the nodes, that is, $\sigma^{-1}(u) = \sigma(u)^{-1}$.

The following results that were developed earlier (see, e.g., Lemma 3.1 in [10]) will be used extensively in our paper.

A node $u \in V$ is called a *horizon* of a (Γ,δ)-gain graph g on $(V, E_2(V))$, if for all $v \in V - \{u\}$, $g(u,v) = 1_\Gamma$.

Let g be a (Γ,δ)-gain graph on $G = (V, E_2(V))$. For a node $u \in V$ and an element $a \in \Gamma$, define a selector σ_a such that

$$\sigma_a(u) = a \text{ and } \sigma_a(v) = \delta(ag(u,v))^{-1} \text{ for } v \in V - \{u\} \ . \tag{2}$$

It is easy to see that u is a horizon in g^{σ_a} if $(u, v) \in E$ for all $v \in V - \{u\}$.

In fact, a more general result can be shown to hold, see Zaslavsky [9].

Lemma 2.1
For every switching class $\mathcal{G}$ on a graph $G = (V, E)$ and an acyclic graph $F = (V, A)$ such that $A \subseteq E$, there is a (Γ, δ)-gain graph $h \in \mathcal{G}$ such that for all edges $e \in A$, $h(e) = 1_\Gamma$. $\square$

We continue now with some basic results on anti-involutions.

A function δ is an anti-involution of Γ if and only if the function α defined for all $a \in \Gamma$ by $\alpha(a) = \delta(a)^{-1}$ is an automorphism of Γ of order at most two. From this we can conclude that for a group Γ, its inverse function is an anti-involution and also that Γ is abelian if and only if the identity function is an anti-involution of Γ.

The next lemma says that each anti-involution δ, that is not the identity nor the inverse function, possesses both a nontrivial fixed point, $\delta(a) = a$, and a nontrivial 'inverse point', $\delta(b) = b^{-1}$.

Lemma 2.2
Let δ be an anti-involution of a finite group Γ.

i. Either there exists an element $a \neq 1_\Gamma$ such that $\delta(a) = a$, or δ is the inverse function of Γ and Γ has odd order.

ii. Either there exists an element $a \neq 1_\Gamma$ such that $\delta(a) = a^{-1}$, or δ is the identity function and Γ is an abelian group of odd order.

Proof:
For (i), assume that δ has no nontrivial fixed points. We note that for all $a \in \Gamma$, $\delta(a\delta(a)) = a\delta(a)$, hence each $a\delta(a)$ is a fixed point of δ. By the assumption, therefore $a\delta(a) = 1_\Gamma$ for all a; hence $\delta(a) = a^{-1}$, and hence δ is the inverse function. In this case, the order of Γ is odd, because if it were even, then (by Cauchy's theorem) Γ would have an element x of order 2, $x^2 = 1_\Gamma$, and then $\delta(x) = x^{-1} = x$.

For (ii), we deduce that if $a\delta(a) = b\delta(b)$, then $\delta(a^{-1}b) = \delta(b)\delta(a^{-1}) = b^{-1}a = (a^{-1}b)^{-1}$. On the other hand, if $a\delta(a) \neq b\delta(b)$ for all $a \neq b$, then clearly $\Gamma = \{a\delta(a) \mid a \in \Gamma\}$ and thus for each $b \in \Gamma$ there exists an a such that $b = a\delta(a)$, which implies that for all b, $\delta(b) = b$. Thus δ is the identity function. In this case, Γ is abelian, and it has odd order, since $\delta(1_\Gamma) = 1_\Gamma$ and the other elements come in pairs $\{a, a^{-1}\}$. $\square$

Recall that the *center* of a group Γ is the normal subgroup

$$\mathrm{Z}(\Gamma) = \{x \in \Gamma \mid xy = yx \text{ for all } y \in \Gamma\} .$$

Theorem 2.3
The center $\mathrm{Z}(\Gamma)$ of Γ is closed under every anti-involution of Γ. Furthermore, for each anti-involution δ either $\delta(z) = z$ for all $z \in \mathrm{Z}(\Gamma)$ or there exists an element $x \in \mathrm{Z}(\Gamma)$ such that $\delta(x) = x^{-1}$ with $x \neq 1_\Gamma$.

Proof:
For all $x \in \mathrm{Z}(\Gamma)$ and $y \in \Gamma$, $\delta(x)y = \delta(\delta(y)x) = \delta(x\delta(y)) = y\delta(x)$, which shows that also $\delta(x) \in \mathrm{Z}(\Gamma)$. Hence, if δ is an anti-involution of Γ, then δ is an anti-involution of $\mathrm{Z}(\Gamma)$. The second claim follows from Lemma 2.2. $\square$

3 Complete graphs with skew gains

It is technically convenient to assume that the domains of our switching classes contain at least three nodes. We therefore first treat the case in which the size is either one or two. The following simple result holds for all groups.

Lemma 3.1
Let a set V be such that $1 \leq |V| \leq 2$ and let G be any graph on V. For all groups Γ, anti-involutions δ on Γ, and $g \in \mathbf{L}_G(\Gamma, \delta)$, we have $[g] = \mathbf{L}_G(\Gamma, \delta)$. □

In the remainder of this section assume that $|V| \geq 3$.

Recall from group theory, see *e.g.* [6], that since $\mathbf{S}$ acts on $\mathbf{L}_G$, then for all $g \in \mathbf{L}_G$,

$$|\mathbf{S}| = |\mathrm{Stab}(g)| \cdot |[g]| \ , \tag{3}$$

where $\mathrm{Stab}(g) = \{\sigma \in \mathbf{S} \mid g^\sigma = g\}$ is the *stabilizer* of $g \in \mathbf{S}$, and $[g]$ is the orbit containing g.

If $k = |\Gamma|$, equation (3) can be rewritten as

$$k^n = |\{\sigma \mid g = g^\sigma\}| \cdot |[g]| \ . \tag{4}$$

Note that (4) is independent of the chosen generator g of the switching class.

We are interested in determining the size of the stabilizer $\mathrm{Stab}(g)$, because this determines by (4) the remaining unknown, the size of the switching class.

Throughout this section the underlying graph will be complete, meaning that $E = E_2(V)$. (In the last section of this paper we give a general formula for determining the sizes of the switching classes $\mathcal{G}$ of arbitrary (Γ, δ)-gain graphs.) Let $\mathbf{L}_K = \mathbf{L}_K(\Gamma, \delta)$ where K is the complete graph on V.

To determine directly the size of the stabilizer of an arbitrary g seems to be a rather complicated task. Fortunately we can, by Lemma 2.1, reduce this problem to determining the size of $\mathrm{Stab}(g)$ for a generator of the switching class that has a horizon. The horizons allow us also to reduce the problem further to the δ-centralizers of the gains of (Γ, δ)-gain graphs g with a horizon, and in this way the problem is simply to determine the order of a certain subgroup of the group Γ of gains.

We adopt first some notations. Let g be a (Γ, δ)-gain graph on K with a horizon u, We denote by

$$A^u(g) = \{a \in \Gamma \mid g(e) = a \ \text{ for some } \ e \in E_2(V - \{u\})\}$$

the set of gains that occur in the subgraph of g that results when the horizon u is removed.

For any given $a \in \Gamma$, the *δ-centralizer* of a is defined as the set

$$C_a^\delta = \{x \in \Gamma \mid \delta(x)a = ax^{-1}\} \ .$$

It is plain that C_a^δ is a subgroup of Γ and it is also easy to show that for each $a \in \Gamma$, $C_a^\delta = C_{\delta(a)}^\delta$ and $C_{a^{-1}}^\delta = \{\delta(x) \mid x \in C_a^\delta\}$. Further, let

$$C_u^\delta(g) = \{x \in \Gamma \mid \delta(x)a = ax^{-1} \ \text{ for all } a \in A^u(g)\} = \bigcap_{a \in A^u(g)} C_a^\delta \ .$$

This is evidently a subgroup of Γ.

We say that a selector $\sigma : V \to \Gamma$ is *constant below* u, if

$$\sigma(v) = \delta(\sigma(u))^{-1} \qquad \text{for all } v \in V - \{u\} \ . \tag{5}$$

Lemma 3.2
Let g be a (Γ, δ)-gain graph with a horizon u. Then $g = g^\sigma$ if and only if σ is constant below u and $\sigma(u) \in C_u^\delta(g)$.

Proof:
Assume that $g^\sigma = g$ and $a \in A^u(g)$. Let $(v, w) \in E_2(V - \{u\})$ be such that $g(v, w) = a$. We have $g^\sigma(u, v) = \sigma(u)\delta(\sigma(v)) = 1_\Gamma$, hence σ is constant below u. Also, $g(v, w) = g^\sigma(v, w)$, and thus $a = \sigma(v)a\delta(\sigma(w))$, yielding, by the first part of the claim, that $a = \delta(\sigma(u))^{-1}a\sigma(u)^{-1}$. The second claim follows from this.

The converse is easily verified along the same lines as the above, and we omit it. □

Theorem 3.3
Let g be a (Γ, δ)-gain graph with a horizon u. Then $C_u^\delta(g) \cong \mathrm{Stab}(g)$.

Proof:
The function $\alpha : \mathrm{Stab}(g) \to C_u^\delta(g)$ defined by $\alpha(\sigma) = \sigma(u)$ is easily seen to be a homomorphism, and it is bijective by Lemma 3.2. □

It follows from this result that if g and h are two complete (Γ, δ)-gain graphs with horizons u and v respectively, then $[g] = [h]$ implies that $C_u^\delta(g) \cong C_v^\delta(h)$, since the stabilizers $\mathrm{Stab}(g)$ and $\mathrm{Stab}(h)$ are conjugate. Therefore, for a switching class $\mathcal{G}$, the integer

$$c(\mathcal{G}) = |C_u^\delta(g)| \ ,$$

called the *content number* of $\mathcal{G}$, is independent of the choice of $u \in V$ and of $g \in \mathcal{G}$ having a horizon u.

We now have by (4) and Theorem 3.3 that

Theorem 3.4
For each switching class $\mathcal{G}$ of (Γ, δ)-gain graphs with $|\Gamma| = k$ and $|V| = n$,

$$|\mathcal{G}| = k^n / c(\mathcal{G})$$

and $c(\mathcal{G})$ divides k. □

In particular, the size of every switching class is a multiple of the order of the group.

Also, the content number $c(\mathcal{G})$ is partly independent of $\mathcal{G}$ in the sense that it is determined by the *set* $A^u(g)$ of gains (where $g \in \mathcal{G}$ has a horizon u). Moreover, since $c(\mathcal{G})$ is independent of the choice of g and u, we get the following result.

Corollary 3.5
Let $\mathcal{G}$ and $\mathcal{H}$ be switching classes of (Γ, δ)-gain graphs (possibly having domains of different sizes) and let $\mathcal{G} = [g]$ and $\mathcal{H} = [h]$, where g and h have horizons u and v, respectively. If for g and h, $A^u(g) = A^v(h)$, then $c(\mathcal{G}) = c(\mathcal{H})$. □

We conclude this section with some details of the switching classes.

For $u \in V$, the u-*subclass* generated by a (Γ, δ)-gain graph h is $\langle h \rangle_u = \{h^\sigma \mid \sigma(u) = 1_\Gamma\}$.

Theorem 3.6
Let $g \in \mathbf{L}_K$ have a horizon u. Let $\mathcal{T}$ be a transversal of the left cosets of $C_u^\delta(g)$. Then $[g] = \bigcup_{a \in \mathcal{T}} \langle g^{\sigma_a} \rangle_u$. Moreover, all $\langle g^{\sigma_a} \rangle_u$ are different and all g^{σ_a} have horizon u.
Proof:
Indeed, $\langle g^{\sigma_a} \rangle_u = \langle g^{\sigma_b} \rangle_u$ if and only if $(g^{\sigma_a})^{\sigma_{b^{-1}}} = g$ if and only if $\sigma(u) = b^{-1}a \in C_u^\delta(g)$, by Lemma 3.2. Hence, it follows that all $\langle g^{\sigma_a} \rangle_u$ are different for $a \in \mathcal{T}$. By (2) it is clear that u is a horizon of each g^{σ_a}.

For $h = g^\tau$, let $b = \tau(u)$. Now $\tau = (\tau {\sigma_b}^{-1})\sigma_b$, where, again by (2), $\tau {\sigma_b}^{-1}(u) = 1_\Gamma$. This shows that $h \in \langle g^{\sigma_b} \rangle_u$. The claim follows, since there exists an $a \in \mathcal{T}$ such that $g^{\sigma_a} = g^{\sigma_b}$ by the first part of the proof. □

Note that the selectors whose value in a fixed node u is 1_Γ form a subgroup of $\mathbf{S}$. Hence, they induce a partition of the switching classes into u-subclasses. The selectors σ_a, as defined in the previous theorem, then take us from one u-subclass to another.

Lemma 3.7
For $a \in \Gamma$, C_a^δ is trivial if and only if Γ is an abelian group of odd order and δ is its identity function.
Proof:
In order to show the claim, let $\alpha : \Gamma \to \Gamma$ be the function defined by

$$\alpha(x) = \delta(x)ax \ .$$

If $\alpha(x) = \alpha(y)$ for some $x, y \in \Gamma$, then $\delta(x)ax = \delta(y)ay$ and so $\delta(y)^{-1}\delta(x)a = a(xy^{-1})^{-1}$. It follows that $xy^{-1} \in C_a^\delta$. Therefore C_a^δ is trivial if and only if α is an injective function, that is, a bijection on Γ. However,

$$\alpha(a^{-1}) = \delta(a^{-1}) = \alpha(\delta(a^{-1})) \ ,$$

and thus, if α is a bijection, then $\delta(a) = a$, which, in turn, implies that for all $x \in \Gamma$,

$$\delta(\alpha(x)) = \delta(\delta(x)ax) = \delta(x)\delta(a)x = \delta(x)ax = \alpha(x) \ .$$

Therefore, if α is a bijection, then $\delta(y) = y$ for all $y \in \Gamma$, and Γ is abelian and of odd order by Lemma 2.2. □

Let g^{-1} be the complete (Γ, δ)-gain graph defined by $g^{-1}(u, v) = g(u, v)^{-1}$ for all $(u, v) \in E_2(V)$. Note that it can be true that $[g] \neq [g^{-1}]$. However, we show next that the switching classes generated by g and g^{-1} are of the same size.

Theorem 3.8
For any (Γ, δ)-gain graph g with a horizon u, $C_u^\delta(g) = \delta(C_u^\delta(g^{-1}))$. Consequently, $|[g]| = |[g^{-1}]|$.
Proof:
Because $C_a^\delta = \delta(C_{a^{-1}}^\delta)$, and δ is a bijection, the claim follows from the definition of $C_u^\delta(g)$ and the facts that g^{-1} has u as its horizon and $A^u(g^{-1}) = \{a^{-1} \mid a \in A^u(g)\}$. □

4 Improvements in some special cases

Throughout this section we let Γ be a group of order k and $\mathcal{G} = [g]$ a switching class where g is a (Γ, δ)-gain graph on a complete graph, which has a horizon u and a domain V of size n.

We observe that, by the definition of $C_u^\delta(g)$, if $1_\Gamma \in A^u(g)$, then necessarily $\delta(x) = x^{-1}$ and $xa = ax$ for all $x \in C_u^\delta(g)$ and $a \in A^u(g)$. As the next theorem states, this generalizes to a wider family of skew gain graphs.

Denote the *centralizer* of a subset $A \subseteq \Gamma$ by

$$C(A) = \{x \in \Gamma \mid ax = xa \ \text{ for all } \ a \in A\} \ .$$

Denote also

$$I^\delta(\Gamma) = \{x \in \Gamma \mid \delta(x) = x^{-1}\} \ .$$

We observe that always

$$I^\delta(\Gamma) \cap C(A^u(g)) \subseteq C_u^\delta(g) \ .$$

Theorem 4.1 i. If there exist $a, b, ab \in A^u(g)$, then $C_u^\delta(g) = I^\delta(\Gamma) \cap C(A^u(g))$.

ii. For the (Γ, δ)-gain graph g_0 that has only identity gains, $k^n = |I^\delta(\Gamma)| \cdot |[g_0]|$.

Proof:

For (i), assume that $a, b, ab \in A^u(g)$.

Thus for any $x \in C_u^\delta(g)$, $\delta(x)b = bx^{-1}$ and $\delta(x)ab = abx^{-1}$. Therefore $\delta(x)ab = a\delta(x)b$, whence $\delta(x)a = a\delta(x)$. But also $\delta(x)a = ax^{-1}$, so $\delta(x) = x^{-1}$. The claim follows.

Since g_0 has the identity as its only gain and $C(\{1_\Gamma\}) = \Gamma$, $C_u^\delta(g_0) = I^\delta(\Gamma)$ and thus $k^n = |I^\delta(\Gamma)| \cdot |[g_0]|$ and (ii) has been proven to hold. □

If a (Γ, δ)-gain graph g with a horizon u has a gain $a \in A^u(g)$ such that $a \in Z(\Gamma)$, then $\delta(x) = x^{-1}$ for all $x \in C_u^\delta(g)$; thus

Theorem 4.2

If $Z(\Gamma) \cap A^u(g) \neq \emptyset$, then $C_u^\delta(g) = I^\delta(\Gamma) \cap C(A^u(g))$. □

In particular, for an abelian group Γ, $C_u^\delta(g) = I^\delta(\Gamma)$ and thus the size of the switching class does not depend on the gains of g.

For the case when δ is the inversion of Γ, we can be more specific.

Corollary 4.3

If the anti-involution is the group inversion, then $k^n = |C(A^u(g))| \cdot |[g]|$. If, moreover, $A^u(g) \subseteq \mathrm{Z}(\Gamma)$ then $|[g]| = k^{n-1}$. □

Example 4.4

Let us consider the symmetric group S_3 on a three-element set. This group has the following presentation:

$$S_3 = \langle a^3 = 1, \ b^2 = 1, \ ba = a^{-1}b \rangle \ .$$

It can be considered as the symmetry group of an equilateral triangle, where a corresponds to a rotation of 120 degrees and b corresponds to one of the reflections. Thus S_3 is a 6-element group, $S_3 = \{1, a, a^2, b, ab, a^2b\}$.

(1) Suppose first that the anti-involution of S_3 is the group inversion.

We can determine that

$$\begin{aligned}
&C_1^\delta = S_3, \quad C_a^\delta = \{1, a, a^2\} = C_{a^2}^\delta, \\
&C_b^\delta = \{1, b\}, \ C_{ab}^\delta = \{1, ab\}, \quad C_{a^2b}^\delta = \{1, a^2b\} \ .
\end{aligned}$$

In particular, if the domain V has three elements and g has a horizon u, then

$$|[g]| = \begin{cases} 36, & \text{if } g \text{ is the identity }, \\ 72, & \text{if } A^u(g) \subseteq \{a, a^2\}, \\ 108, & \text{otherwise } \ . \end{cases}$$

(2) Let δ be defined by $\delta(a) = a$ and $\delta(b) = ab$. This gives us an anti-involution of S_3 for which

$$\delta(1) = 1, \ \delta(a) = a, \ \delta(a^2) = a^2, \ \delta(b) = ab, \ \delta(ab) = b, \ \delta(a^2b) = a^2b \ .$$

In particular, $I^\delta = \{1, a^2b\}$. We can now compute

$$\begin{aligned}
&C_1^\delta = \{1, a^2b\}, \quad C_a^\delta = \{1, ab\}, \quad C_{a^2}^\delta = \{1, b\}, \\
&C_b^\delta = \{1, a, a^2\} = C_{ab}^\delta, \quad C_{a^2b}^\delta = S_3 \ .
\end{aligned}$$

Therefore, *e.g.*, if a complete (S_3, δ)-gain graph g with a horizon u yields a nontrivial subgroup $C_u^\delta(g)$, then $A^u(g)$ can contain at most one of the gains $1, a, a^2, b$. (Note that $ab \in A^u(g)$ if and only if $b \in A^u(g)$, since $\delta(b) = ab$.) If $C_u^\delta(g)$ is the trivial subgroup of S_3, then $|[g]| = 6^n$, where $n \geq 3$ is the size of the domain. This can only happen when $n \geq 4$.

□

5 The general case

In this section we consider an arbitrary (Γ, δ)-gain graph g on $G = (V, E)$. As before we denote $n = |V|$.

If g (that is, the graph G) is disconnected and its connected components are $g_1, \ldots, g_c$, then clearly

$$|[g]| = |[g_1]| \cdot |[g_2]| \cdots |[g_c]|.$$

Thus the size of the switching class is reduced to the sizes of switching subclasses generated by the connected components.

Suppose now that g is connected, and let T be one of its spanning trees. By Lemma 2.1 there exists a $g_T \in [g]$ such that $g_T(e) = 1_\Gamma$ for all $e \in T$.

Let $u \in V$ and let $T = (V, A)$ be a tree. We say that a node $v \in V$ is *odd (even)* with respect to u if the distance between u and v is odd (even) in the tree T.

A simple induction on the length of the paths in a tree from a root u proves the following lemma.

Lemma 5.1
Let $g_T \in [g]$ be such that $g_T(e) = 1_\Gamma$ for all $e \in T$, and let $u \in V$. If $g_T^\sigma = g_T$ for a selector σ, then

$$\sigma(v) = \begin{cases} \sigma(u) & \text{if } v \text{ is even w.r.t. } u, \\ \delta(\sigma(u))^{-1} & \text{if } v \text{ is odd w.r.t. } u. \end{cases} \tag{6}$$

□

We divide our further considerations into two parts according to whether g is bipartite or not.

Theorem 5.2
Let g be a (Γ, δ)-gain graph on a connected bipartite graph $G = (V, E)$, and let T be a spanning tree of G. Then $\mathrm{Stab}(g) \cong C(A^u(g_T))$. As a consequence,

$$|[g]| = k^n / |C(A^u(g_T))| \ .$$

Proof:
Let $u \in V$ be a fixed node. The sets of even nodes and of odd nodes with respect to u form the unique bipartition of G, since G is connected.

Let $e = (v, w) \in E$, where v is even and w is odd. For each selector σ of the form in (6), $g_T(e) = g_T^\sigma(e)$ holds if and only if $g_T(e) = \sigma(v) g_T(e) \delta(\sigma(w)) = \sigma(u) g_T(e) \sigma(u)^{-1}$. Therefore, $g_T = g_T^\sigma$ if and only if $\sigma(u) \in C(A^u(g_T))$.

The mapping $\alpha : \mathrm{Stab}(g_T) \to C(A^u(g_T))$ defined by $\alpha(\sigma) = \sigma(u)$ is easily seen to be an isomorphism. This proves the claim, since it is always the case that $\mathrm{Stab}(g) \cong \mathrm{Stab}(g_T)$. □

For the non-bipartite case we prove the following

Theorem 5.3
Let g be a (Γ, δ)-gain graph on a connected graph $G = (V, E)$ that is not bipartite. Then there exists a complete (Γ, δ)-gain graph $\hat{g}$ on domain V such that $\hat{g}|_E \sim g$ and $\mathrm{Stab}(g) \cong \mathrm{Stab}(\hat{g})$; therefore $|[g]| = |[\hat{g}]|$.
Proof:
Let T be a spanning tree of G, and $u \in V$. Since G is not bipartite, there is an edge $e_0 = (u_0, v_0) \in E$ that connects either two odd nodes or two even nodes with respect to u. We can assume that the former case holds; otherwise we change the fixed node u to one of its neighbours in T. Such a change changes the parity of the nodes.

Denote $a = g_T(e_0)$. We define $\hat{g}$ as follows.
For each $e \in E$, put $\hat{g}(e) = g_T(e)$.
For each $e \notin E$, if the endpoints of e are

- of different parity, then put $\hat{g}(e) = 1_\Gamma$,
- both odd, then put $\hat{g}(e) = a$ and $\hat{g}(e^{-1}) = \delta(a)$,
- both even, then put $\hat{g}(e) = a^{-1}$ and $\hat{g}(e^{-1}) = \delta(a)^{-1}$.

(In the last two cases the choice of priority between e and e^{-1} is arbitrary.)

Since g_T is a subgraph of $\hat{g}$ (that is, $G \subseteq \hat{G} = (V, E_2(V))$ and g respects the gains of $\hat{g}$), it follows that $\mathrm{Stab}(\hat{g}) \subseteq \mathrm{Stab}(g_T)$.

In the other direction, assume that $\sigma \in \mathrm{Stab}(g_T)$, and let $e = (v, w) \in E_2(V)$ be any edge of $\hat{g}$. If $e \in E$ then $\hat{g}(e) = g_T(e)$, and therefore also $\hat{g}^\sigma(e) = \hat{g}(e)$. Suppose that $e \notin E$.

If v and w are of opposite parity (with respect to u), say v is even and w is odd, then $\hat{g}(e) = 1_\Gamma$, and $\sigma(v) = \sigma(u)$, $\sigma(w) = \delta(\sigma(u))^{-1}$ by Lemma 5.1. Now also $\hat{g}^\sigma(e) = \sigma(u)\hat{g}(e)\sigma(u)^{-1} = 1_\Gamma$.

If both v and w are odd, then $\hat{g}(e) = a$ or $\hat{g}(e) = \delta(a)$. In this case, $\sigma(v) = \sigma(u_0) = \sigma(v_0) = \sigma(w)$. Now, if $\hat{g}(e) = a$ $(= g_T(e_0))$ then $\hat{g}^\sigma(e) = \sigma(v)\hat{g}(e)\delta(\sigma(w)) = \sigma(u_0)g_T(e_0)\delta(\sigma(v_0)) = g_T^\sigma(e_0) = g_T(e_0) = \hat{g}(e)$. The case that $\hat{g}(e) = \delta(a)$ is treated in the same way.

The case where both v and w are even is proved similarly.

This shows that $\sigma \in \mathrm{Stab}(\hat{g})$. Thus, $\mathrm{Stab}(g_T) \cong \mathrm{Stab}(g)$, and hence the proof is completed. □

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