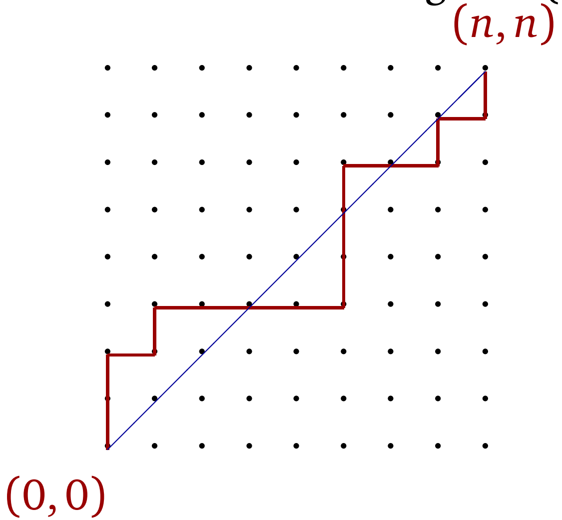
CATALAN NUMBERS: good and bad paths

Consider lattice paths $(0,0) \to (n,n)$ using moves

up u (or $\uparrow$) and right r (or $\to$).

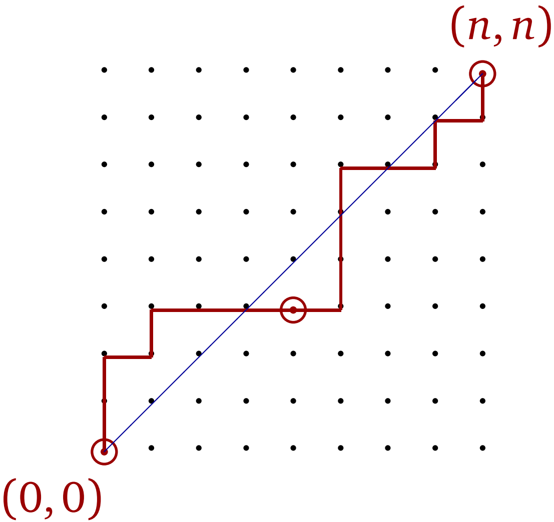
A good path never visits below the diagonal $(0,0) - (n,n)$.



BAD PATH ENTERING FORBIDDEN $(i, i-1)$

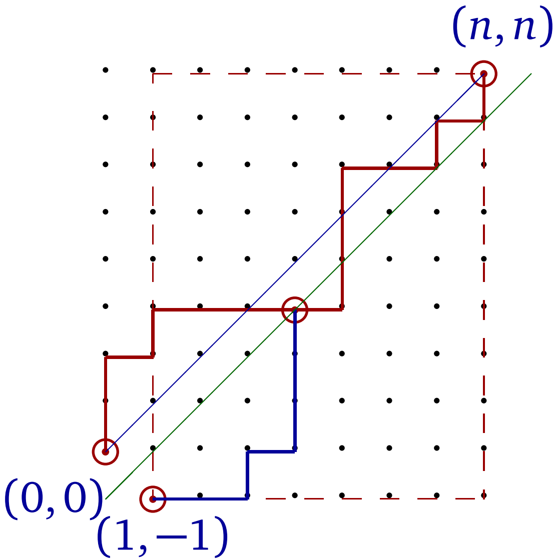
We first count the bad paths.

Assume w is a bad path. Let i be the first place where w enters below the diagonal: it will hit $(i, i-1)$.



REFLECT THE PORTION $(0,0)-(i,i-1)$

w.r.t. the lower diagonal $(0,-1)-(n+1,n)$



Every path $(1,-1) \to (n,n)$ is a reflection of a unique bad path (for some instance i):

So a bijective correspondence.

NOW COUNT

#bad paths $=$ #all paths $(1,-1) \to (n,n)$
$=$ #all paths $(0,0) \to (n-1, n+1)$ (lift the bad paths $\nwarrow$)
$=$

$$\binom{2n}{n-1} = \binom{2n}{n+1}$$

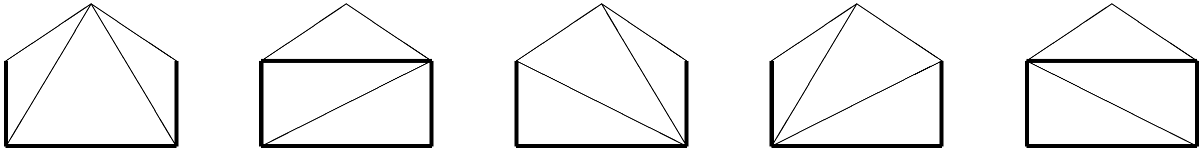
Good paths: All paths $(0,0) \to (n,n)$ - bad paths

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

Catalan numbers are solutions to dozens of combinatorial problems; Stanley, “Combinatorial Enumeration, Vol. II”

TRIANGULATIONS OF A CONVEX $(n+2)$-GON

Divide the polygon with nonintersecting diagonals.
(There are $(n-1)$ such diagonals.)



PARENTHESIZED STRINGS OF $(n+1)$ LETTERS

For $n = 3$, these are

$$(((xx)x)x), (x((xx)x)), ((x(xx))x),$$
$$(x(x(xx))), ((xx)(xx))$$

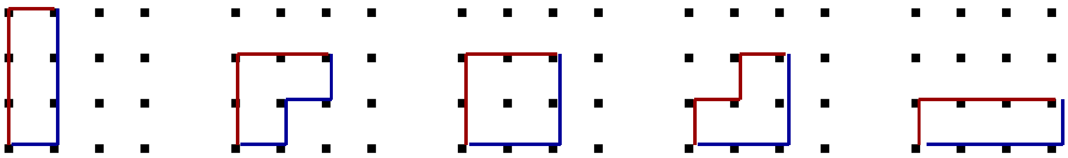
or, without the placeholder x:
For $n = 3$, these are

$$((())), ((())), ((())),$$
$$((())), (()())$$

These are known as the Dyck words in formal language theory.

PAIRS OF LATTICE PATHS w_1, w_2

Using $n+1$ rules u and r, starting from $(0,0)$, ending at the same point, and intersecting only at the endpoints.



SEQUENCES $(a_1, a_2, \ldots, a_{2n})$ OF INTEGERS

$a_i \in \{-1, +1\}$ where $\sum_{i=1}^{k} a_i \geq 0$ for all k and $\sum_{i=1}^{n} a_i = 0$.

$$+1+1+1-1-1-1$$

$$+1+1-1+1-1-1$$

$$+1+1-1-1+1-1$$

$$+1-1+1+1-1-1$$

$$+1-1+1-1+1-1$$

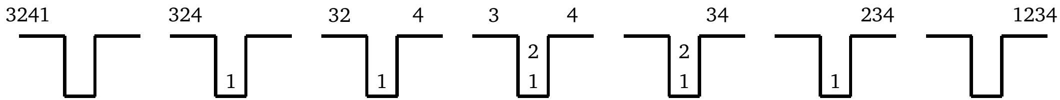
Easily seen to be in bijective correspondence with the Dyck words.

AND OTHER ...

- Sequences $1 \le a_1 \le a_2 \le \cdots \le a_n$ of n integers with $a_i \le i$.

$$(1,1,1)\ (1,1,2)\ (1,1,3)\ (1,2,2)\ (1,2,3)$$

- Permutations in S_n that can be sorted to the identity by one stack.



- The number of ways $2n$ people sitting around a table can shake hands without crossing arms.

SCHRÖDER NUMBERS

The Schröder numbers[1] are defined by

$$s_{n+1} = \frac{1}{2} \sum_{i=0}^{n} \binom{2n-i}{i} c_{n-i} \quad \text{with } s_1 = 1, \tag{1}$$

where c_n is the nth Catalan number.

- This number occurs in many refined enumeration problems.
- s_n equals the number of ways to put parentheses into a word of n letters without unnecessary parenthesis. For instance, there are 11 such words when $n = 4$:

$$1234,\ 12(34),\ 1(234),\ (12)23,\ (123)4,\ (12)(34)$$
$$1(23)4,\ (1(23))4,\ ((12)3)4,\ 1((23)4),\ 1(2(34)).$$

[1] Schröder (1841 - 1902)

- The Schröder numbers occur in plane trees, polygon dissections, Łukasiewicz words, ... The first ten values are:

$$s_1 = 1,\ s_2 = 1,\ s_3 = 3,\ s_4 = 11,\ s_5 = 45,\ s_6 = 197,$$
$$s_7 = 903,\ s_8 = 4279,\ s_9 = 20\,793,\ s_{10} = 103\,049.$$

I am sure you noticed that the value $s_{10} = 103\,049$ was mentioned by Plutarch (50 - 120). In the *Table-Talk*:

> Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparchus, to be sure, refuted this by showing that on the affirmative side there are 103 049 compound statements, and on the negative side 310 952.)

- How the $\eta\epsilon\lambda\lambda$ did Hipparchus count s_{10}? Or did he? Counting the different ways directly is rather impossible, and using (1) is unlikely. It is known that

$$(n+2)s_{n+2} = 3(2n+1)s_{n+1} - (n-1)s_n,$$

but this requires quite a long proof. The first combinatorial proof, due to Foata and Zeilberger, appeared as late as 1997. Maybe Plutarch used the 'easier' reduction

$$s_n = \sum_{i_1+i_2+\cdots+i_k=n} s_{i_1} s_{i_2} \cdots s_{i_k}.$$

- What does "negative side" of 310 952 mean?

- For further information: R. Stanley, Hipparchus, Plutarch, Schröder, and Hough, *Amer. Math. Monthly* **104** (1997), 344 – 350.