PARTIALLY ORDERED SETS - POSETS

A poset is a set P with a relation $R \subseteq P \times P$, denoted by

$$R = \leq_P \quad \text{or just} \quad R = \leq$$

such that

- $a \leq_P a$ reflexive
- $a \leq_P b,\ b \leq_P a \implies a = b$ antisymmetric
- $a \leq_P b,\ b \leq_P c \implies a \leq_P c$ transitive

We write $x <_P y$ if $x \leq_P y$ and $x \neq y$.

HASSE DIAGRAMS

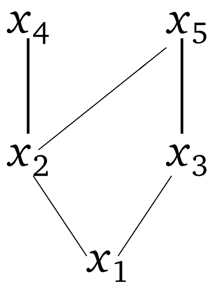
The cover relation in P:

$$x \prec y \iff x \leq_P y \text{ and } x <_P z <_P y \text{ for no } z$$

For a (finite) poset P, itsHasse diagram: there is a line upwards from x to y if $x \prec_P y$.

Hence there is a path upwards from x to y if and only if $x \leq_P y$.

Example. Let $P = \{x_1, \ldots, x_5\}$ with $x_1 \leq_P x_i$ for all $i \in [2, 5]$, $x_2 \leq_P x_4$, $x_2 \leq_P x_5$, $x_3 \leq_P x_5$. (And $x_i \leq_P x_i$ for all i.)



LOCALLY FINITE POSETS

For $x \leq_P y$ the set

$$[x, y]_P = \{z \mid x \leq_P z \leq_P y\}$$

is the interval of x and y.
A poset P is said to be locally finite if all its intervals are finite.

Example.

- All finite posets are locally finite.
- The poset of subsets (powerset) 2^X is a poset w.r.t. $\subseteq$.
  $2^{\mathbb{N}}$ is not locally finite since $[\emptyset, \mathbb{N}]$ is infinite.

SPECIAL POSETS

A poset P is a chain if it is totally ordered: for all $a, b \in P$, either $a \leq_P b$ or $b \leq_P a$.

Example.

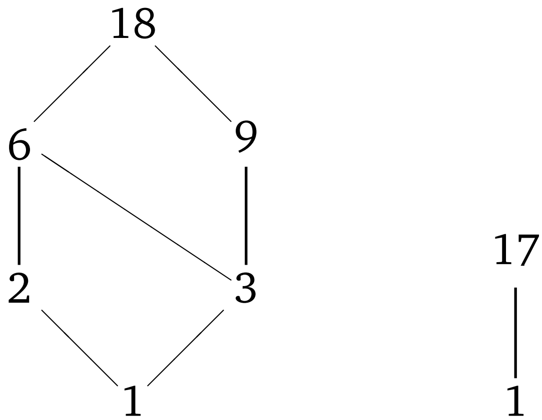
- $(\mathbb{N}, \leq)$ with the usual order is locally finite chain.
- $(\mathbb{Q}, \leq)$ is a chain but not locally finite.

The divisor poset D_n

$(\mathbb{N}_+, |)$ is a locally finite poset of positive integers with the divisor relation.

For each positive integer $n \in \mathbb{N}_+$, the set of divisors of n form a poset

$$D_n = \{k \in [1, n] \ : \ k|n\}.$$



TREES

A rooted tree is a poset $T = (V, E)$, where E satisfies

$$x \leq_T z \text{ and } y \leq_T z \implies x \leq_T y \text{ or } y \leq_T x$$

and T has a smallest element r ,its root, such that

$$r \leq_T x \text{ for all } x.$$

MIN MAX

- $x \in P$ is a minimum element or a zero, if $x \leq_P y$ for all $y \in P$.
- $x \in P$ is the maximum element of P, if $y \leq_P x$ for all $y \in P$.
- These elements may not exist in a poset.

  If they do exist, they are usually denoted by 0 and 1.

Example. The poset $(2^X, \subseteq)$ has minimum element $\emptyset$, and it has maximum element X. It is locally finite only if X is a finite set.

SUBWORD POSET

Consider the set of all words A^* over an alphabet A.
Write $u \leq v$ if

$$v = v_1 u_1 v_2 u_2 \cdots v_n u_n v_{n+1}$$
$$u = u_1 u_2 \cdots u_n,$$

where (some of) u_i and v_i can be the empty word.

Then $(A^*, \leq)$ is a locally finite poset with a zero element (the empty word).

Theorem [Higman] The poset $(A^*, \leq)$ is well-ordered:

> Let $X \subseteq A^*$ be an infinite subset of words.
> Then there are $u, v \in X$ such that $u \leq v$.

In particular, the set of minimal elements if finite for each $X \subseteq A^*$.