1. How many lattice paths \( w \in \{u, r\}^* : (0, 0) \to (n, k) \) are there that visit the point \((x, y)\), where \(0 \leq x \leq n\) and \(0 \leq y \leq k\)?

Solution. There are \( \binom{x+y}{x} \) paths \((0, 0) \to (x, y)\) and \( \binom{n-x+k-y}{n-x} \) paths \((x, y) \to (n, k)\). Hence the answer is

\[
\binom{x+y}{x} \cdot \binom{n-x+k-y}{n-x}.
\]

2. Let \( n \) and \( k \) be fixed positive integers. How many positive integer solutions are there for the inequality

(a) \( x_1 + x_2 + \cdots + x_k = n \)?
(b) \( x_1 + x_2 + \cdots + x_k < n \)?

Solution. (a) Let

\[
A = \{(x_1, x_2, \ldots, x_k) \mid \sum_{i=1}^{k} x_i = n, x_i \geq 1\},
\]

and consider the mapping \( \alpha : A \to 2^{[1..n-1]} \) defined by

\[
\alpha(x_1, x_2, \ldots, x_k) = \{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_{k-1}\}.
\]

Clearly, \( \alpha \) is bijective onto the \((k-1)\)-subsets of \([1..n-1]\) the number of which is \( \binom{n-1}{k-1} \).

(b) The number of solutions is the same as for \( x_1 + x_2 + \cdots + x_{k+1} = n \), i.e., \( \binom{n-1}{k} \).

3. Consider good sequences \( w = a_1 a_2 \cdots a_n \) with \( n \geq 1 \) where each \( a_i \in [0, k-1] \) such that for each \( i = 1, 2, \ldots, n \), we have \( a_{i+1} = a_i \) or \( a_{i+1} = a_i + 1 \ (\text{mod} \ k) \) (where the index \( i+1 \) is modulo \( n \)). For instance, if \( k = 5 \), then \( w = 2234001 \) is a good sequence, but 112 is not, since while closing the sequences, 2 \( \not\equiv 1 \) (mod 5). Show that there are

\[
k \sum_{i \geq 0} \binom{n}{ki}
\]

good sequences of length \( n \). (Here \( \binom{n}{m} = 0 \) if \( m > n \).)

Additional note. For \( k \geq 3 \), further evaluation of the sum is bit complicated. By Guichard’s theorem (1995), \( k \sum_{i \geq 0} \binom{n}{ki} = \sum_{i=1}^{k} (1 + e^{2\pi i/k})^n \), where \( e^{2\pi i/k} \) is a complex primitive \( k \)th root of unity.

Solution. Each good sequence can be represented as \( (a_1; r_1, r_2, \ldots, r_n) \), where \( a_{i+1} = a_i + r_i \) with \( r_i \in \{0, 1\} \). Hence 2234001 is represented by \((2; 0, 1, 1, 0, 1, 1)\). In order for a sequence to be good the number of ones must be divisible by \( k \), i.e., the number of good sequences of length \( n \) is \( k \sum_{i \geq 0} \binom{n}{ki} \), where the coefficient \( k \) tells from which element \( a_1 \) to start.
4. Prove the following form of the **binomial inversion formula**: Let \((b_n)_{n=0}^\infty\) be a sequence of integers, and let

\[ a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i b_i. \]

for all \(n \geq 0\). Then

\[ b_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i. \]

**Solution.** There are several solutions. The easiest one reduces to the original binomial inversion by substituting \((-1)^i b_i\) for \(b_i\).

One can also imitate the proof of the binomial inversion:

\[
\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( \sum_{k=0}^{i} \binom{i}{k} (-1)^k b_k \right) \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{n}{i} \binom{i}{k} (-1)^{i+k} b_k \\
&= \sum_{k=0}^{n} \sum_{i=k}^{n} \binom{n}{k} \binom{n-k}{i-k} (-1)^{i+k} b_k \quad \text{(Example 4.6)} \\
&= \sum_{k=0}^{n} \binom{n}{k} b_k \left( \sum_{i=k}^{n} \binom{n-k}{i-k} (-1)^{i-k} \right) \quad \text{(change order)} \\
&= \sum_{k=0}^{n} \binom{n}{k} b_k \left( \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \right) \quad \text{(substitute } j = i - k) \\
&= \sum_{k=0}^{n} \binom{n}{k} b_k \delta_{nk} \quad \text{(see eq. (4.8))} \\
&= \binom{n}{n} b_n = b_n.
\end{align*}
\]

5. Let \(f(n)\) be the number of integer sequences \(1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n\) such that \(a_i \geq i\) for each \(i \in [1, n]\). Show that \(f(n)\) is a Catalan number.

**Solution.** There is a bijection onto the lattice paths.
6. Show that the number \( a_n \) of permutations \( \alpha \in S_n \), where \( \alpha(i) \neq i + 1 \), for \( i = 1, 2, \ldots, n - 1 \), is

\[
a_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.
\]

**Solution.** Let \( A_i = \{ \alpha \mid \alpha(i) = i + 1 \} \), where \( |A_i| = (n-1)! \), and

\[
|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)! \quad \text{for} \quad i_1 < i_2 < \ldots < i_k.
\]

Then \( a_n = \left| \bigcap_{k=1}^{n} A_{m-k} \right| \). By PIE,

\[
a_n = |S_n| + \sum_{k=1}^{n-1} (-1)^k \sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|
\]

\[
= \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.
\]

7. Let \( P_{n,m} = \left| \{ \alpha \in S_n \mid \alpha(k) = k \text{ for some } k \in A \} \right| \) for a subset \( A \subseteq [1,n] \) of \( m \) elements with \( m \leq n \). Compute \( P_{7,3} \).

**Solution.** By PIE. For each \( k \in A \), let \( A_k = \{ \alpha \mid \alpha(k) = k \} \), and denote \( m = |A| \). Now

\[
\left| \bigcup_{k \in A} A_k \right| = \sum_{k \in A} |A_k| - \sum_{k_1 < k_2 \in A} |A_{k_1} \cap A_{k_2}| + \cdots \\
+ (-1)^{i+1} \sum_{k_1 < \cdots < k_i \in A} |A_{k_1} \cap \cdots \cap A_{k_i}| + \cdots \\
+ (-1)^{m+1} |A_1 \cap A_2 \cap \cdots \cap A_m|.
\]

Here \( |A_k| = (n-1)! \), and, in general, for \( k_1 < \cdots < k_i \) (from \( A \)),

\[
|A_{k_1} \cap \cdots \cap A_{k_i}| = (n-i)!.
\]

Therefore

\[
\left| \bigcup_{k \in A} A_k \right| = \sum_{k \in A} (n-1)! - \sum_{k_1 < k_2 \in A} (n-2)! + \cdots \\
+ (-1)^{i+1} \sum_{k_1 < \cdots < k_i \in A} (n-i)! + \cdots \\
+ (-1)^{m+1} (n-m)!.
\]

Notice that this depends only on the size of \( A \), not on its elements.

For \( P_{7,3} \), we can choose \( A = \{1,2,3\} \), and we have

\[
\left| \bigcup_{k \in A} A_k \right| = \sum_{k \in A} (7-1)! - \sum_{k_1 < k_2 \in A} (7-2)! + \sum_{k_1 < k_2 < k_3 \in A} (7-3)!
\]

\[
= 3 \cdot 6! - \binom{3}{2} 5! + 4! = 2160 - 360 + 24 = 1824.
\]
8. Use PIE to count the number of words \( w \in \{a, b, c\}^n \) of length \( n \) that contain each letter \( a, b \) and \( c \) at least once?

**Solution.** Use PIE. The number of all words of length \( n \) is \( N = 3^n \).

- Missing \( a \) (or \( b \), or \( c \), respectively): \( 2^n \).
- Missing \( a \) and \( b \) (or \( a, c \), or \( b, c \), respectively): \( 1^n = 1 \).
- Missing all: 0 (unless \( n = 0 \)).

\[
\left| X \setminus \bigcup_{i=1}^{3} A_i \right| = N - N_1 + N_2 - N_3 = 3^n - 3 \cdot 2^n + 3.
\]