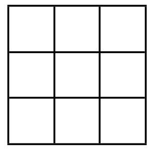
Combinatorial Enumeration – Fall 2011

Problem Set 6 (October 14, 2011)

1. Apply Cauchy-Frobenius to the following 3×3 board, where each square can be coloured by 1 or 2. Here plane isometries determine which boards are the similar, that is, the isometries of the (big) square.



(i) What is the answer if there are no restrictions?

(ii) What is the answer if exactly two of the squares must get a colour 1?

2. Consider a cube with colourings of its vertices. Find the polynomial P_G of Theorem 8.7, and apply it to the case, where we have 2 colours. Similarities are according to rotations in the 3-dimensional space; see Example 7.8.

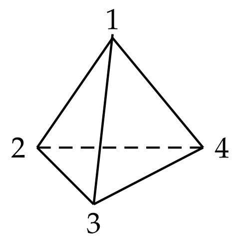
3. Continue the previous exercise with 2 colours, black and white. Count, using Pólya's theorem, the number of dissimilar cubes that have exactly 3 black vertices and 5 white vertices.

4. Consider a wheel that has 24 sectors that rotates around its axis. Count the number of ways the sectors can be coloured with two colours.

5. Consider an 8×8 checker board. In a **configuration** of the board each square is either left empty or a checker is placed on it. Find the polynomial P_G when similarities are defined by the rotations of the board. Find the total number of configurations of the board.

6. Continue the previous exercise. How many different configurations are there on a 8×8 board, when 16 checkers are placed on its squares?

7. Consider a (regular) tetrahedron where the faces are equilateral triangles. In how many ways can the vertices be coloured with three colours? (The symmetries are the rotations in the 3-dimensional space.)



8. Show that the cycle index polynomial of the cyclic group C_n is

$$P_{C_n}(z_1, z_2, \ldots, z_n) = \frac{1}{n} \sum_{d|n} \phi(d) z_d^{n/d} .$$