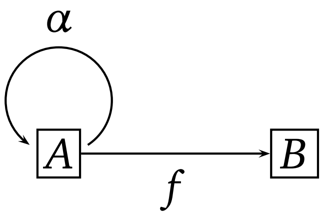
12-FOLD WAY: $g \sim_D f$

Functions $g, f : A \to B$ are domain indistinguishable if

$$\exists \alpha \in S_A : \; g = f\alpha.$$

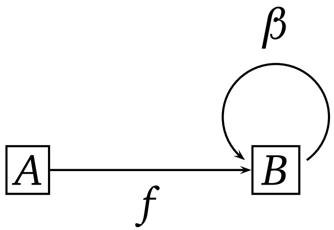


Hence $g \sim_D f \iff |g^{-1}(x)| = |f^{-1}(x)|$ for all $x \in B$, since the sets $g^{-1}(x)$ form a partition of A.

$$g \sim_R f$$

f and g are range indistinguishable if

$$\exists \beta \in S_B : \; g = \beta f.$$

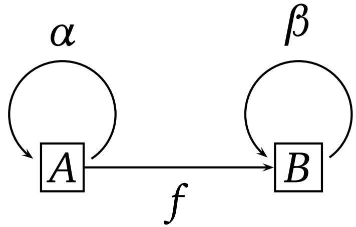


Hence $g \sim_R f \iff \{g^{-1}(x) \mid x \in B\} = \{f^{-1}(x) \mid x \in B\}$.

$$g \sim_{DR} f$$

f and g are domain and range indistinguishable if

$$\exists \alpha \in S_A \; \exists \beta \in S_B : \; g = \beta f \alpha.$$



LEMMA 4.15

Let A and B be given. Let X be D, R or DR.

- $\sim_X$ is an equivalence relations.
- If $f \sim_X g$ and f is injective (respectively, surjective), then so is g.

Proof is an easy exercise.

TWELVE FOLD WAY for $f : A \to B$

Count the equivalence classes with $|A| = n$ and $|B| = k$.
'+D' means 'distinguishable' and '-D' means 'indistinguishable'.

Domain	Range	Any f	Injective f	Surjective f
+D	+D	k^n	$[k]_n$	$k!S(n,k)$
–D	+D	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
+D	–D	$\sum_{i=1}^{k} S(n,i)$	$\begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$	$S(n,k)$
–D	–D	$\sum_{i=1}^{k} p_i(n)$	$\begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$	$p_k(n)$

STIRLING

- $S(n,k) =$ **Stirling number of the second kind:**

  The number of partitions of an n-set into k blocks (partition classes). By convention, $S(0,0) = 1$.

- $p_n(k) =$ the number of **(integer) partitions** of k to exactly n parts: the number of ways to write

$$
\begin{aligned}
k = & k_1 + k_2 + \cdots + k_n \\
& \text{where } 1 \le k_1 \le k_2 \le \cdots \le k_n \le k.
\end{aligned}
$$