January 30

Clarified and corrected examples:

Example 1.59(Monotone sequences) Consider the set $N = \{1, 2, \ldots, n^2 + 1\}$ of integers, and let $\sigma = (k_1, k_2, \ldots, k_{n^2+1})$ be a linear order on N. We show that this sequence contains a monotonic subsequence of length $n + 1$.

Consider the set $P = \{(i, k_i) \mid i = 1, 2, \ldots, n^2 + 1\}$ together with the partial order

$$(i, k_i) \leq_P (j, k_j) \iff i \leq j \text{ and } k_i \leq k_j .$$

A subsequence σ' of σ is ascending if and only if σ' corresponds to a chain in the poset P. By Theorem 1.57, P can be partitioned into $w(P)$ chains, and thus σ can be partitioned into $w(P)$ ascending subsequences. Thus σ has an ascending subsequence of length at least $|N|/w(P)$.

Assume that $(i_1, k_{i_1}), (i_2, k_{i_2}), \ldots, (i_t, k_{i_t})$ is an antichain in P ordered so that $i_1 < i_2 < \cdots < i_t$. Then $k_{i_1} > k_{i_2} > \cdots > k_{i_t}$ is a descending subsequence. Now, if $|N|/w(P) \leq n$ (i.e., σ does not have an ascending chain of length $n + 1$), then $w(P) \geq (n^2 + 1)/n \geq n + 1/n$, and so $w(P) \geq n + 1$, as required. □

Example 1.61 (Hall's Marriage Theorem) Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a family of subsets of a finite set X such that $X = \bigcup_{i=1}^{n} S_i$. (The sets S_i can intersect with each other.) A function $\sigma\colon \{1, 2, \ldots, n\} \to X$ is called a **disjoin representative function** for $\mathcal{S}$ if it is injective and, for all $1 \leq i \leq n$, there exists an integer j such that $\sigma(j) \in S_i$. Here $\sigma(i)$ **represents** S_i and only S_i. For an index set $I \subseteq \{1, 2, \ldots, n\}$, denote

$$S(I) = \bigcup_{i \in I} S_i .$$

We show that the family $\mathcal{S}$ has a distinct representative function if and only if the following **Hall's condition** holds

$$|S(I)| \geq |I|$$

for all index sets $I \subseteq \{1, 2, \ldots, n\}$.

If $|I| < |S(I)|$ then clearly no distinct representative function can exist. To prove the sufficiency of Hall's condition, consider the relation $\leq$ on the set $P = \{1, 2, \ldots, n\} \cup X$ defined by

$$x \leq i \iff x \in S_i .$$

Then P is is a partial order (with height $h(P) = 2$).

Let A be an antichain of P with $|A| = w(P)$. Let $A = I \cup A'$, where $I \subseteq \{1, 2, \ldots, n\}$ and $A' \subseteq X$. Now $S(I) \subseteq X \setminus A'$ by the definition of P, and hence $|X| - |A'| \geq |S(I)| \geq |I|$ (by Hall's condition), i.e., $|A| \leq |X|$. Hence $w(P) = |X|$, since X is an antichain.

By Dilworth's theorem, P can be partitioned into $w(P)$ chains. This gives a matching of the elements in $\{1, 2, \ldots, n\}$ with those of X. This proves the sufficiency of Hall's condition. □