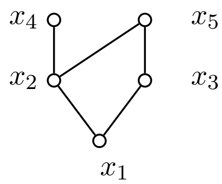
Ordered Sets (2006) / T. Harju

Problem Set 5

1. Find the Möbius function of the given finite poset.



2. Let P and Q be two posets, and consider their direct product $P \times Q$. Show that, for all pairs,

$$\zeta_{P\times Q}((x_1, y_1), (x_2, y_2)) = \zeta_P(x_1, x_2) \cdot \zeta_Q(y_1, y_2)$$

and

$$\delta_{P\times Q}((x_1, y_1), (x_2, y_2)) = \delta_P(x_1, x_2) \cdot \delta_Q(y_1, y_2)\,.$$

3. Prove Theorem 1.68: The Möbius function $\mu_{P\times Q}$ of the direct product $P \times Q$ is the product of the Möbius functions μ_P and μ_Q of P and Q, that is,

$$\mu_{P\times Q}((x_1, y_1), (x_2, y_2)) = \mu_P(x_1, x_2) \cdot \mu_Q(y_1, y_2)\,.$$

4. Compute the Möbius function of the poset $\mathbb{N} \times \mathbb{N}$, where $(x_1, y_1) \le (x_2, y_2)$ if and only if $x_1 \le x_2$ and $x_2 \le y_2$.

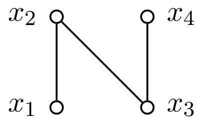
5. Prove Theorem 1.72: Let P be a finite poset. Denote $f^k = f{*}f{*}\cdots{*}f$ (k times) for each $f \in I(P)$.

(1) Show that $\zeta^2(x, y) = |[x, y]_P|$ (the number of elements in the interval $[x, y]$).

(2) Show that $\zeta^k(x, y)$ equals the number of chains of length k (with possible repetitions) from x to y, that is, the number of sequences $(x_0, x_1, \ldots, x_k)$ for which $x = x_0 \le_P x_1 \le_P \cdots \le_P x_{k-1} \le_P x_k = y$.

Here 'with possible repetitions' means that maybe $x_i = x_{i+1}$ in such a chain.

We say that a poset P is **covering** N**-free**, if P does not have four elements such that $x_1 \prec x_2$, $x_3 \prec x_2$, $x_3 \prec x_4$, and otherwise the elements are incomparable, i.e., the Hasse diagram of P does not have a part N as on the right.



6 (Grillet (1969))**.** Let P be a finite poset. Show that P avoids N if and only if every maximal chain of P intersects with every maximal antichain of P.