The previous short proof did have a gap. January 20

Theorem 1.38 (König). *Let P be a poset satisfying both the finite chain and antichain conditions (FCC and FAC). Then P is finite.*

Proof. Notice first that FCC implies both ACC and DCC. For each $A \subseteq P$, let

$$\min(A) = \{x \mid x \text{ minimal in } A\}\,.$$

The set $\min(A)$ is an antichain and thus it is always finite by FAC. By DCC, for each $y \in A$, there exists an $x \in A$ such that $x \leq_P y$, i.e., $A \subseteq {\uparrow}\min(A)$. (Indeed, for $x <_P y$, consider the half open interval $[x, y]_P \setminus \{x\}$.)

Suppose contrary to the claim that P is infinite, and let

$$S(x) = \min({\uparrow}x \setminus \{x\})\,.$$

Then there exists an element $x_1 \in \min(P)$ such that ${\uparrow}x_1$ is infinite. Inductively, we obtain a sequence $x_1, x_2, \ldots$ such that $x_i \in \min(S(x_{i-1}))$ and ${\uparrow}x_i$ is infinite. However, now $x_0 <_P x_1 <_P x_2 < \ldots$ contradicts the fact that P satisfies ACC. $\square$