Defect theorem

- A folklore result. Skordev and Sendov (1961)
- Linear algebra: notion of dimension. Do sets of words possess dimension properties of some kind?
  - If a set of $n$ words satisfies a nontrivial relation, then these words can be expressed as products of at most $n-1$ words.
- Partly generalizes to infinite words. (A recent topic.)

We need a result, which is interesting on its own.

**Theorem 1.** If $S_i (i \in I)$ are free semigroups of $A^+$, then

$$S = \bigcap_{i \in I} S_i$$

is free or it is empty.

**Proof.** Suppose $S \neq \emptyset$.

Then $S$ is closed:

- If $u, v \in S$, then there is an index $i$ such that $u, v \in S_i$. Hence also $uv \in S_i$ and $uv \in S$.

But

$$u, v, uv, wv \in S \implies \forall i \in I : u, v, uv, wv \in S_i$$

By Criterion, $w \in S_i$ for all $i \in I$, that is, $w \in S$. The claim follows by our criterion. □
By Theorem 1, for any $X \subseteq A^*$,

$$\hat{X} = \bigcap \{S \mid X \subseteq S, \text{ } S \text{ is a free semigroup} \}$$

is a free semigroup.

**Theorem 2.** Every set $X \subseteq A^+$ of words has a unique smallest semigroup $\hat{X}$ such that $X^+ \subseteq \hat{X}$

The above definition of $\hat{X}$ as an intersection tells nothing of the structure of the free hull. For instance,

- If $X$ is finite, then $\hat{X}$ is finitely generated.

- If a word $w \in S$ can be written as

  $$w = uxv \text{ where } u, ux \in S \text{ and } xv, v \in S \quad (x \neq \lambda)$$

  then $x$ is called an **overflow** of $S$.

**Theorem 3.** A semigroup $S \subseteq A^+$ is free iff every overflow of $S$ is in $S$. 
Example

Let $A = \{a, b\}$ and

$$X = \{bb, bbaba, abaa, baaba, baa\}.$$ 

Consider any free $S$ with $X^+ \subseteq S$.

$X^+$ is not free, since $bbaba \cdot abaa = bb \cdot abaa \cdot baa$.

- $bbaba|a.baa$
- $bb|aba,a|baa$

The words $aba$ and $a$ are overflows in $S$, and hence $aba$, $a \in S$ (as well as the elements of $X$).

Also, $baaba \cdot abaa = baa \cdot baa \cdot baa$, and hence $ba$ is an overflow in $S$. So also $ba \in S$.

Hence $\{a, ba, bb\} \subseteq S$, and since $\{a, ba, bb\}$ is a code with $X \subseteq \{a, ba, bb\}^*$,

$$\hat{X} = \{a, ba, bb\}^+, \quad \text{base}(\hat{X}) = \{a, ba, bb\}.$$ 

Note that

$$\text{rank}(X) = |X| = 5 > 3 = \text{rank}(\hat{X}).$$

Defect theorem

**Theorem 4.** [Defect theorem]

Let $X \subseteq A^+$ be finite. If $X$ is not a code, then

$$\text{rank}(\hat{X}) \leq |X| - 1.$$ 

**Theorem 5.**

Let $X \subseteq A^+$ be finite. If $X^+$ is not free, then

$$\text{rank}(\hat{X}) \leq \text{rank}(X) - 1.$$ 

Hence if a set of words $X$ satisfies a nontrivial identity, then $X$ can be expressed with fewer words (than $|X|$).
**Proof.** Define $\alpha: X \to \text{base}(\hat{X})$:

\[
\alpha(u) = v \quad \text{if} \quad u \in v \cdot w \quad \text{for} \quad v \in \text{base}(\hat{X}), \ w \in \hat{X} \cup \{\lambda\}.
\]

Thus $\alpha$ picks the first factor out of $u \in X$.

**Claim 1.** $\alpha$ is well defined.

**Claim 2.** $\alpha$ is not injective.

**Claim 3.** $\alpha$ is surjective.

Conclusion:

\[
\text{rank}(\hat{X}) = |\text{base}(\hat{X})| = |\alpha(X)| < |X|.
\]

□

**Algorithm**

Define for all $Y$:

\[
C(Y) = \{(u, v) \in Y \times \hat{X} \mid u \neq v, \ uY^* \cap vY^* \neq \emptyset\}.
\]

Let $X \subset A^+$ be finite.

Initialize: Set $X_0 = X$, $j = 0$.

1. find $(u, v) \in C(X_j)$ if exists; else stop and return $\hat{X} = X_j$

2. let $w$ be such that $u = vw$

3. if $w \in X_j$, set $X_{j+1} = X_j - \{u\}$; if $w \notin X_j$, set $X_{j+1} = (X_j - \{u\}) \cup \{w\}$; $j := j + 1$

goto (1)
The algorithm works: it returns the free hull of every finite set.

For this consider the sizes

\[ s(X_j) = \sum_{u \in X_j} |u| \]

of the sets \( X_j \).

They decrease strictly when \( j \) grows.

Also, \( \widehat{X} = \widehat{X}_0 \) and
\( \widehat{X}_j = \widehat{X}_{j+1} \) for all \( j \geq 0 \).

Some corollaries

A word \( w \in A^+ \) is primitive, if it is not a proper power of another word, that is,

\[ w = u^k \implies k = 1 \text{ and } u = w. \]

**Corollary 1.** Each word \( w \in A^+ \) is a power of a unique primitive word.

**Proof.** Let \( w = u^n = v^m \) for some \( u \neq v \in A^+ \) and \( n, m \geq 1 \).

Then \( X = \{u, v\} \) is not a code, since \( w \) has two different factorizations over \( \{u, v\} \).

Defect Thm \( \implies 1 = \text{rank}(\widehat{X}) < |X| = 2. \)

Hence base(\( \widehat{X} \)) = \{z\} for some \( z \in A^+ \).

But now \( X \subseteq z^* \). Thus \( u, v \) are powers of \( z \). \( \square \)
Corollary 2. Two words \( u, v \in A^* \) commute, \( uv = vu \) iff they are powers of a common word.

Proof. \( uv = vu \implies X = \{u, v\} \) is not a code.

Defect thm \( \implies \) \( \text{rank}(\hat{X}) < |X| = 2 \).

So \( u \) and \( v \) are powers of a common word. \( \square \)

Example. Let \( X = \{u, v, w\} \subseteq \{a, b\}^+ \).

Suppose that

\[
\begin{cases}
  uvw = vwu \\
  uuv = wvu
\end{cases}
\]

Now \( X \) is periodic:

\[
\text{rank}(X^+) = 1
\]

That is, \( u, v, w \) are all powers of a common word.

Graph lemma

The defect theorem has a generalization in terms of graphs (TH and Karhumäki 1986).

Let \( X \subseteq A^+ \) be finite. Define a \( G_X = (X, E) \) on the set \( X \) of nodes, called the dependency graph of \( X \):

\[
(u, v) \in E \iff \exists s, t \in X^*: \ u s = v t.
\]

Theorem 6. [Graph Lemma] Let \( X \subseteq A^+ \) be a finite set that is not a code. Then

\[
\text{rank}(\hat{X}) \leq c(X) < |X|,
\]

where \( c(X) \) is the number of the connected components of \( G_X \).
Example. Let \( X = \{ab, aba, abb, bab\} \). Then the graph \( G_X \) is given below:

\[
aba \cdot bab = ab \cdot ab \cdot ab \\
abb \cdot ab = ab \cdot bab.
\]

Problem. Which conditions force \( X \) to have rank at most \( |X| - k \) for some \( k \geq 2 \)?

Tilings in the plane

We consider two examples of 2-dimensional words in connection to the defect theorem.

A figure is a partial function \( \tau : \mathbb{Z} \times \mathbb{Z} \to A \) consisting of a finite number of unit squares of integer points labelled by letters of an alphabet.

We consider each point \((i, j)\) as a square of unit length centred at \((i, j)\).

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

There are some changes in the two dimensional case (if you want):

- A letter \( \leftrightarrow \) coloured square
- A word \( \leftrightarrow \) coloured figure
- Factorizations \( \leftrightarrow \) coloured tilings of figures
Tiles

Let \( X = \{T_1, T_2, \ldots, T_n\} \) be a set of figures (called tiles). We say that a figure \( F \) has a tiling from \( X \), if

- \( F \) can covered with (copies of) the tiles without overlapping. The tiles can be translated but not rotated or reflected.

There are similar problems for tilings where one allows rotations and/or reflections.

Problem

A set of tiles \( X \) is decomposable, if there exists a set \( Y \) of tiles such that

(1) \( |Y| < |X| \),

(2) every \( T \in X \) has a tiling from \( Y \).

**Problem.** Does the defect effect hold for set of tiles? That is, given a set \( X \) of \( n \) tiles:

- Is it true that if there exists a figure \( F \) that has two different tilings from \( X \), then \( X \) is decomposable?

The answer to this general questions is **No**:

Figures in the plane do **not** satisfy the defect theorem even in some very simple cases.
Example

The three tiles are:

The below figure has two different tilings, but the tiles cannot be expressed by less than three tiles.

Open problem

**Problem.** Does the defect effect hold for sets of two tiles?

That is,

let \( X = \{T_1, T_2\} \) be a set of two tiles.

Is it true that if there is a figure \( F \) with two different tilings in \( X \) then there exists a tile \( T \) such that both \( T_1 \) and \( T_2 \) can be tiled with \( T \).
Vectors and rectangular figures

- Consider vector tiles (that resemble words!) of the form $1 \times m$ and $m \times 1$.

- The defect effect does hold for sets of two vector tiles: If a (any!) figure has two different tilings from a set of vector tiles, then the tile set is decomposable.

- The defect effect does not hold for sets of four vector tiles:
  
  Let $X = \{ab, cd, (ac)^T, (bd)^T\}$. Then there is a figure $F$, namely the square with rows $ab$ and $cd$, that has two different factorizations over $X$, but the defect effect does not apply to $X$.

Three tile problem

**Problem.** Does the defect effect concern sets $X = \{T_1, T_2, T_3\}$ of three vector tiles?

Squares

- A square is a tile (figure) of the form $n \times n$.

- Easy: the defect effect holds for two squares, $X = \{T_1, T_2\}$.

  If a figure $F$ can be tiled in two different ways in $X$, then there is a square $T$ such that both $T_1$ and $T_2$ can be tiled with $T$. 

Three squares

The above does not hold for three squares.

The squares are:

\[
\begin{array}{ccc}
\text{a} & \text{a} & \text{a} \\
\text{b} & \text{c} & \text{b} \\
\text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} \\
\end{array}
\quad
\begin{array}{ccc}
\text{a} & \text{a} & \text{a} \\
\text{b} & \text{c} & \text{b} \\
\text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} \\
\end{array}
\quad
\begin{array}{ccc}
\text{a} & \text{a} & \text{a} \\
\text{b} & \text{c} & \text{b} \\
\text{a} & \text{a} & \text{a} \\
\text{a} & \text{a} & \text{a} \\
\end{array}
\]