In the following undecidability questions you may use the known undecidability results concerning Wang tiles given on the reverse side of this paper.

1. Prove that it is undecidable if a given two-dimensional CA has any fixed point configurations. Is the question decidable or undecidable for one-dimensional CA?

2. Let $A$ be a fixed CA with quiescent state $q$. The Garden-of-Eden Problem for $A$ asks whether a given $q$-finite configuration is a Garden-of-Eden. Prove that there exists a two-dimensional CA whose Garden-of-Eden Problem is undecidable. Is the question decidable or undecidable for one-dimensional CA?

3. Consider the two-dimensional CA constructed in the proof of Proposition 46 for a given tile set $T$.
   (a) If the tile set $T$ admits a tiling of the plane, is the corresponding CA periodic? (Recall: CA $G$ is periodic if $G^n$ is the identity function for some $n \geq 1$.)
   (a) If the tile set $T$ does not admit a tiling of the plane, is the corresponding CA periodic?
   (c) Is it decidable to determine if a given two-dimensional CA is periodic?

4. Consider the two-dimensional CA constructed in the proof of Proposition 45 for a given tile set $T$ and a blank tile $B \in T$.
   (a) If the tile set $T$ admits a valid, finite, non-trivial tiling of the plane, is the corresponding CA surjective on periodic configurations?
   (a) If the tile set $T$ does not admit a valid, finite, non-trivial tiling of the plane, is the corresponding CA surjective on periodic configurations?
   (c) Is it decidable to determine if a given two-dimensional CA is surjective on periodic configurations?

Recall that CA $G : S^{Z^d} \rightarrow S^{Z^d}$ is nilpotent if there is positive integer $n$ such that $G^n(S^{Z^d})$ contains only one configuration, the quiescent configuration.

5. A CA $G$ with quiescent state $q$ is called nilpotent on finite configurations if every finite configuration evolves into the quiescent configuration, that is, for every finite configuration $c$ there is positive integer $n$ such that all cells in $G^n(c)$ are quiescent.
   (a) Prove that every nilpotent CA is also nilpotent on finite configurations, but the converse is not true.
   (b) Prove that it is undecidable if a given two-dimensional CA is nilpotent on finite configurations.

6. A CA $G$ with quiescent state $q$ is called nilpotent on periodic configurations if every totally periodic configuration evolves into the quiescent configuration.
   (a) Prove that every nilpotent CA is also nilpotent on periodic configurations, but the converse is not true.
   (b) Prove that it is undecidable if a given two-dimensional CA is nilpotent on periodic configurations.

7. Prove that the following questions concerning one-dimensional cellular automata are undecidable:
   (a) Is a given one-dimensional CA eventually periodic?
   (b) Is a given one-dimensional CA nilpotent on periodic configurations? (Hint: recall Problem 7 from last week.)
The following decision problems have been proved undecidable in the *Tilings and Patterns* class. Here we use them without a proof:

**Tiling Problem**
Instance: A finite set $T$ of Wang tiles
Problem: Does $T$ admit a valid tiling?

**Tiling Problem with the Seed Tile**
Instance: A finite set $T$ of Wang tiles and a seed tile $s \in T$
Problem: Does there exist a valid tiling $t$ of the plane such that $t(0,0) = s$?

**Finite Tiling Problem**
Instance: A finite set $T$ of Wang tiles and a blank tile $b \in T$
Problem: Does there exist a valid tiling $t$ such that $\{(i, j) \in \mathbb{Z}^2 \mid t(i, j) \neq b\}$ is finite but non-empty?

**Periodic Tiling Problem**
Instance: A finite set $T$ of Wang tiles
Problem: Does $T$ admit a periodic tiling?

There is a fixed set $T$ of Wang tiles such that the following decision problem is undecidable:

**Completion Problem for Tile Set $T$**
Instance: A finite pattern $p$
Problem: Does $T$ admit a valid tiling that contains a copy of $p$?