RADIATIVE PROCESSES in
ASTROPHYSICS

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Chapter 1

Introduction

Most of the information we obtain about the Universe reaches us in form of the electromagnetic (EM) radiation. The only important exceptions are the measurements in situ at the surfaces of a few planets and their satellites (e.g. Earth and Moon, Mars, Venus, Saturn’s moon Enceladus), measurements of the particles in the solar system, cosmic rays and neutrinos at the Earth surface. In some near future gravitational waves may also be detected and used to probe processes happening in vicinities of relativistic objects.

The vast majority of what we know about the Universe, we learnt by collecting and studying of various forms of EM radiation. In the ancient times, people could only use light in the optical part of the spectrum, while in the 20th century the astrophysical instruments have been developed to study radiation in the radio, infrared, ultraviolet (UV), X-ray and gamma-ray part of the EM spectrum. The radio astronomy appeared only after the 2nd World War when the radars were build to detect emission in radio band. To observe UV, X-ray and gamma-ray photons one needs to make either balloon flights to the height of 10-20 km, or even better to send an observatory to space. Therefore, very active developments in that field were only possible starting from the 1960s when the first rockets and satellites were built. We have learnt that some of the objects emit a large fraction of their radiation in the energy bands invisible from the ground (i.e. Earth surface). An example of such an object is shown in Fig. 1.1.

To understand the physics of such objects, we need to obtain the data simultaneously in various energy band of the EM spectrum. To interpret the EM radiation we receive, we need to know the microphysics, i.e. the radiative processes that give rise to this radiation as well as the processes that change this radiation on the way to our detectors. Much of the information about stars, interstellar
Figure 1.1: Some sources emit radiation from radio to $\gamma$-rays. In order to understand the physics of such objects, one requires to perform coordinated multifrequency observations. Here examples of the spectral energy distributions of a few active galactic nuclei are shown. The spectra extend from the radio to the TeV energies (teraelectron volt, i.e. $10^{12}$ eV). The energy of the most energetic photons exceeds the energy of photons in the visual band by a factor $10^{12}$. 
medium, and galaxies comes from studying their line spectra. However, many objects do not produce strong spectral lines, and we have to study their continuum (i.e. weakly dependent on wavelength) spectra. The most important radiative processes that produce these continua will be studied during our course.

We first start with the basic description and general concepts of the radiation field. We then introduce the radiative transfer equation that describes how the radiation changes when it passes through the medium in the source as well as on the way to us. Next we consider microscopic description of the radiation field as its relation to the Maxwell equations that describe electromagnetic fields. Further we consider radiation of the moving changes. We then discuss radiative processes giving rise to continuum spectra. All of them involve motion of electrons: bremsstrahlung (or free-free emission) involves interaction between electrons and other changed species (e.g. protons), cyclotron or synchrotron radiation which is produced by electrons in the magnetic field, and Compton scattering which is a result of scattering of photons by the electrons. In the order of appearance we will study:

1. **Bremsstrahlung.** Radiation of an electron in the electric field of a proton. Important work from the turn of the 20th century to about 1930s. It is important in HII regions, planetary nebulae, stars, clusters of galaxies. For an example of the later, see Fig. 1.2.

2. **Synchrotron radiation.** Theory was developed mostly in the 2nd half of the 20th century (Ginzburg, Syrovatsky), while some important papers have already appeared in the 1910s (Shott 1912). Found important applications with the start
Figure 1.2: Clusters of galaxies, the largest bound structures in the Universe, are filled with hot \((T \sim 10^7 \text{ K})\) X-ray emitting gas. Upper panel shows the image of Coma cluster taken in the optical band, where you can see hundreds of galaxies. Lower panel, is the XMM image of the Coma cluster in the X-ray, where the glowing gas filling the whole cluster is seen.
of radioastronomy. The process is important in pulsar nebulae and jets from active galaxies. In Fig. 1.1, the lower energy bump is believed to be produced by synchrotron radiation.

3. Coherent Compton scattering (Thomson scattering). Important developments in the 1930s to 1940s by Chandrasekhar, Ambartsumyan, Sobolev. Important as a source of additional opacity in planetary and stellar atmospheres, e.g. white dwarfs.

4. Non-coherent Compton Scattering. Development in the 1940s-50s because of its importance for the nuclear bomb research (Kompaneets). Important applications in the physics of early Universe (Sunyaev, Zeldovich, Illarionov) and astrophysics of relativistic objects (Sunyaev, Titarchuk, Lightman). The theory was further developed in the 1970s-1990s when appeared the need to interpret the data from X-ray satellites. It is a dominating radiative process shaping the spectra from accreting black holes (both stellar-mass and super-massive), neutron stars, and jet from active galactic nuclei (e.g. high-energy bump in Fig. 1.1).
5. $e^\pm$ pair production. At the extreme environments of compact objects such as pulsars and accreting black holes, high-energy photons interact with low-energy photons producing electron-positron pairs. This process was studied in 1980s (by Svensson, Lightman, Zdziarski) in connection with interpretation of the data from active galaxies. With the development of $\gamma$-ray astronomy (including TeV astronomy) it is becoming even more clear how important it is for the correct interpretation of the data from pulsars, jets from active galactic nuclei, and gamma-ray bursts.
Chapter 2

Basics of radiative transfer

2.1 Definitions

- In vacuum the photon wavelength and frequency are related as $\lambda \nu = c$. Photon energy is related to the frequency through Planck constant: $E = h \nu$.

- Energy flux $F_\nu$ [units: erg cm$^{-2}$ Hz$^{-1}$]. The energy passing through the area $dA$ during the time interval $dt$ in the frequency interval $d\nu$ is:

  $$dE = F_\nu \ dA \ dt \ d\nu.$$  \hspace{1cm} (2.1)

  The energy passing per unit time through the sphere of radius $r$ from the isotropic source in steady-state in the absence of sinks on the way is

  $$L = 4\pi r^2 F(r) = 4\pi r^2 F(r_1).$$ \hspace{1cm} (2.2)

  This is just the energy conservation law. The consequence of that is the so called inverse square law:

  $$F(r) = \frac{L}{4\pi r^2} \propto r^{-2}.$$ \hspace{1cm} (2.3)

- Specific intensity $I_\nu$ [units: erg cm$^{-2}$ Hz$^{-1}$ sr$^{-1}$]. The energy passing through the area $dA$ normal to the area within the solid angle $d\Omega$ during the time interval $dt$ in the frequency interval $d\nu$ is:

  $$dE = I_\nu \ d\Omega \ dA \ dt \ d\nu.$$ \hspace{1cm} (2.4)
The flux can be obtained by integrating the intensity over solid angles and taking into account the projection effect:

\[ F_\nu = \int I_\nu \cos \theta \, d\Omega, \quad (2.5) \]

where \( \theta \) is the angle between the normal to the area and the direction of the ray.

- (Specific) momentum flux (pressure) [units: \( \text{dyn cm}^{-2}\ \text{Hz}^{-1} \)]:

\[ p_\nu = \frac{1}{c} \int I_\nu \cos^2 \theta \, d\Omega \quad (2.6) \]

because the photon momentum is \( h\nu/c \). Another \( \cos \theta \) comes from the projection of the transferred momentum to the normal to the area.

- (Specific) energy density [units: \( \text{erg cm}^{-3}\ \text{Hz}^{-1} \)]:

\[ u_\nu = \frac{1}{c} \int I_\nu \, d\Omega = \frac{4\pi}{c} J_\nu, \quad (2.7) \]

where

\[ J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega \quad (2.8) \]

is the mean intensity. Total energy density [units: \( \text{erg cm}^{-3} \)]:

\[ u = \int u_\nu \, d\nu = \frac{4\pi}{c} \int J_\nu \, d\nu. \quad (2.9) \]

- Radiation pressure for isotropic radiation field.

Consider a reflecting enclosure containing isotropic radiation field. Each photon transfers twice its normal component of the momentum on reflection:

\[ p_\nu = \frac{2}{c} \int I_\nu \cos^2 \theta \, d\Omega. \quad (2.10) \]

which agrees with previous formula (2.6) as here we integrate only over half a sphere, \( 2\pi \) steradians. Assuming isotropy, i.e. \( J_\nu = I_\nu \), we get:

\[ p = \frac{2}{c} \int J_\nu \, dv \int \cos^2 \theta \, d\Omega = \frac{1}{3} u. \quad (2.11) \]
2.2 The Equation of Radiative Transfer

- The intensity is conserved along a ray

\[
\frac{dI_\nu}{ds} = 0 \tag{2.12}
\]

unless there is absorption or emission

\[
\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu, \tag{2.13}
\]

where \(\alpha_\nu [\text{cm}^{-1}]\) is the absorption coefficient and \(j_\nu [\text{erg cm}^{-3} \text{ Hz}^{-1} \text{ str}^{-1}]\) is the emission coefficient. This is the equation of radiative transfer. The absorption coefficient can be represented as the product of number density \(n\) of absorbing particles [cm\(^{-3}\)] and the effective cross-section \(\sigma_\nu [\text{cm}^2]\) of an individual particle:

\[
\alpha_\nu = n\sigma_\nu. \tag{2.14}
\]

Alternatively

\[
\alpha_\nu = \rho\kappa_\nu, \tag{2.15}
\]

where \(\rho\) is the density [g cm\(^{-3}\)] and \(\kappa_\nu [\text{cm}^2 \text{ g}^{-1}]\) is the opacity (or mass absorption coefficient).

- Defining the optical depth [units: dimensionless]

\[
d\tau_\nu = \alpha_\nu \, ds \tag{2.16}
\]

and the source function

\[
S_\nu = \frac{j_\nu}{\alpha_\nu}, \tag{2.17}
\]

we can rewrite the radiative transfer equation

\[
\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu. \tag{2.18}
\]

- This has the formal solution

\[
I_\nu(\tau_\nu) = I_0 e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)}S_\nu(\tau'_\nu) \, d\tau'_\nu. \tag{2.19}
\]
Evaluating this expression is not trivial and can often only be done numerically. The biggest problem is that the source function itself often depends on the intensity of radiation. Thus the equation we have at hand is integro-differential, not differential.

If \( S_\nu \) is known and is independent of location, we get

\[
I_\nu(\tau_\nu) = I_0 e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) = S_\nu + e^{-\tau_\nu} (I_0 - S_\nu). \tag{2.20}
\]

Thus when \( \tau_\nu \ll 1 \), we obtain \( I_\nu \approx I_0 + \tau_\nu S_\nu \), while when \( \tau_\nu \gg 1 \) one gets \( I_\nu \approx S_\nu \).

- **Radiation force.**

  If medium absorbs radiation, then the radiation exerts the force on the medium, because the radiation carries momentum. The vector of radiation flux is

  \[
  \vec{F}_\nu = \int I_\nu \vec{n} d\Omega
  \tag{2.21}
  \]

  where \( \vec{n} \) is the unit vector along the direction of the ray. As the photon has momentum \( h\nu/c \), the momentum transferred per unit area per unit length per unit time is

  \[
  \vec{f} = \frac{1}{c} \int \alpha_\nu \vec{F}_\nu d\nu.
  \tag{2.22}
  \]

  One can understand this relation from the following: the amount of energy passing through area \( dA \) in frequency interval \( d\nu \) in solid angle \( d\Omega \) is \( I_\nu dA \cos \theta d\nu d\Omega \) (where \( \theta \) is the angle the photon momentum makes with the normal of \( dA \)). To get the corresponding momentum along the normal to multiply this by \( \cos \theta \) and divide this by \( c \). The probability that photons are absorbed within the length \( ds \) is \( \alpha_\nu ds/\cos \theta \). Now integrate over solid angle (and remember that the volume is \( dV = dA ds \)), we get equation (2.22) gives the force per unit volume. The force per unit mass is

  \[
  \vec{f} = \frac{1}{c} \int \kappa_\nu \vec{F}_\nu d\nu.
  \tag{2.23}
  \]

  Note that there is no additional \( \cos \theta \) factor here (which we had when computing radiation pressure), because we assume that volume can absorb photons. If we consider a problem of e.g. radiation force on a surface, i.e. solar sail, this factor appears.
2.3 Thermal Emission

- Thermal emission is radiation emitted by material in thermal equilibrium (TE); blackbody radiation (BB) is thermal emission that is in TE itself.

- BB is independent of material, shape, color, direction, flavor, etc. It only depends on temperature and wavelength (or frequency)

\[ I_\nu = f(\nu, T) \equiv B_\nu(T). \quad (2.24) \]

- Kirchhoff’s Law: material emitting thermal radiation has

\[ S_\nu = B_\nu(T) \quad (2.25) \]

and therefore

\[ j_\nu = \alpha_\nu B_\nu(T). \quad (2.26) \]

- Thermal radiation has \( S_\nu = B_\nu(T) \) and blackbody radiation has \( I_\nu = B_\nu(T) \). Thermal radiation becomes blackbody radiation for \( \tau \gg 1 \).

- From thermodynamic arguments follows Stefan-Boltzmann Law: energy density

\[ u(T) = a T^4 \quad (2.27) \]

where \( a = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \). Since \( u = \frac{2\pi}{\epsilon} B \) and the flux \( F = \pi B \), we get

\[ F = \sigma T^4 \quad (2.28) \]

where \( \sigma = \frac{ac}{4} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \text{ s}^{-1} \).

- The Planck spectrum

\[ B_\nu(T) = \frac{2\hbar^3}{c^2 \epsilon^4} \frac{1}{e^{\hbar \nu/kT} - 1}. \quad (2.29) \]

or

\[ B_\lambda(T) = \frac{2\hbar c^2}{\lambda^5} \frac{1}{e^{\hbar c/\lambda kT} - 1}, \quad (2.30) \]

and

\[ B_\lambda(T) \, d\lambda = B_\nu(T) \, d\nu. \quad (2.31) \]
CHAPTER 2. BASICS OF RADIATIVE TRANSFER

Figure 2.1: The Planck function for various temperatures.

- Limiting behavior: Rayleigh-Jeans limit $h\nu \ll kT$:
  \[
  I_{\nu}^{RJ}(T) = \frac{2\nu^2}{c^2}kT.
  \]  
  \[ (2.32) \]

- Limiting behavior: Wien limit $h\nu \gg kT$:
  \[
  I_{\nu}^{W}(T) = \frac{2h\nu^3}{c^2} \exp(-h\nu/kT).
  \]  
  \[ (2.33) \]

- If $T_1 > T_2$, then $B_\nu(T_1) > B_\nu(T_2)$ for all $\nu$.

- Wien displacement law
  \[
  \nu_{\text{max}} = 5.88 \times 10^{10} T,
  \]  
  \[ (2.34) \]
where $\nu$ is in Hz and $T$ in K, or

$$\lambda_{\text{max}} = \frac{0.290}{T} \quad (2.35)$$

with $\lambda$ in cm and and $T$ in K. Note that $\lambda_{\text{max}}\nu_{\text{max}} \neq c$.

• Relation to fundamental constants. Let us integrate the Planck function

$$\int_0^\infty B_\nu(T) \, d\nu = \frac{2h}{c^2} \left( \frac{kT}{h} \right)^4 \int_0^\infty \frac{x^3}{e^x - 1} \, dx. \quad (2.36)$$

The integral over $x$ is $\pi^4/15$ (check that at home). Therefore, we have

$$\int_0^\infty B_\nu(T) \, d\nu = \frac{2\pi^4 k^4}{15 c^2 h^3} T^4 = \frac{\sigma T^4}{\pi}. \quad (2.37)$$

• Characteristic temperatures: brightness temperature $T_b$ – a measure of the source intensity (or brightness). Defined so that at a given $\nu$: $I_\nu = B_\nu(T_b)$. Often used in radioastronomy, where the Plack function is replaced with its Rayleigh-Jean limit $B_\nu \propto T_{\text{RJ}}$.

• Characteristic temperatures: color temperature $T_c$ – a measure of the spectral shape. Defined as the temperature for which a black body spectrum has a maximum at the same frequency or wavelength as the measured peak in the observed spectrum (which may not look like a blackbody curve at all).

• Characteristic temperatures: effective temperature $T_{\text{eff}}$ – a measure of the magnitude of the flux. Defined as the temperature for which a black body would have the same flux as the measured flux $F = \sigma_{\text{SB}} T^4$. The actual measured spectrum does not need to look like a blackbody.
2.4 Einstein Coefficients

Linking Kirchhoff’s Law (macroscopic) with microscopic properties.

- For a two-level system with levels $E_2 > E_1$, $E_2 - E_1 = h\nu_0$, and degeneracies $g_1$ and $g_2$, define
  
  1. probability for spontaneous emission (s$^{-1}$) = $A_{21}$.
  2. probability for absorption = $B_{12}\overline{J}$, where $\overline{J} = \int J_\nu \phi(\nu) d\nu$ and $\phi(\nu)$ is the profile function. It describes the finite width around the frequency $\nu_0$, where absorption can take place. For a slowly varying average intensity $J_\nu$ (like the Planck function), $\phi(\nu)$ can be approximated as a $\delta$-function, and $\overline{J} = J_{\nu_0}$.
  3. probability for stimulated emission = $B_{21}\overline{J}$.

- In thermodynamic equilibrium (TE), the number of transitions per unit time per unit volume from state 1 to state 2 should be exactly balanced by the opposite transitions:

$$n_1 B_{12} \overline{J} = n_2 A_{21} + n_2 B_{21} \overline{J}. \quad (2.38)$$

Find $\overline{J}$:

$$\overline{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)/(B_{12}/B_{21}) - 1}. \quad (2.39)$$

In TE:

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp[-(E + h\nu_0)/kT]} = \frac{g_1}{g_2} \exp(h\nu_0/kT). \quad (2.40)$$

Thus

$$\overline{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}. \quad (2.41)$$

We also know that in TE $J_\nu = B_\nu$, so we get the relation between the Einstein coefficients:

$$A_{21}/B_{21} = \frac{2h\nu_0^3}{c^2}, \quad (2.42)$$

$$g_1 B_{12} = g_2 B_{21}. \quad (2.43)$$

These properties do not depend on the temperature of the gas but are the properties of the atoms only. Thus they are valid not only for the TE, but always.
2.4. EINSTEIN COEFFICIENTS

• The macroscopic emission and absorption coefficients can be written in terms of the microscopic Einstein coefficients as

\[ j_\nu = \frac{h \nu}{4\pi} n_2 A_{21} \phi(\nu) \]  

(2.44)

and

\[ \alpha_\nu = \frac{h \nu}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu) \]  

(2.45)

Here, absorption also includes stimulated emission (as negative absorption).

• The source function is

\[ S_\nu = \frac{2h
\nu^3}{c^2} \left( \frac{n_1 g_2}{n_2 g_1} - 1 \right)^{-1}. \]  

(2.46)

• Thermal emission.

When the matter is thermal equilibrium with itself (but not necessarily with radiation), we have

\[ \frac{n_1}{n_2} = \frac{g_1}{g_2} \exp(h\nu/kT), \]  

(2.47)

and

\[ \frac{n_2 g_1}{n_1 g_2} = \exp(-h\nu/kT) < 1. \]  

(2.48)

The matter is said in local thermodynamic equilibrium (LTE). In this case

\[ \alpha_\nu = \frac{h \nu}{4\pi n_1 B_{12}} \left[ 1 - \exp(-h\nu/kT) \right] \phi(\nu), \]  

(2.49)

\[ S_\nu = B_\nu(T). \]  

(2.50)

• Non-thermal emission. Inverted populations, masers.

In real astrophysical circumstances, the matter does not need to be in LTE, i.e.

\[ \frac{n_1}{n_2} \neq \frac{g_1}{g_2} \exp(h\nu/kT). \]  

(2.51)

The extreme case of such a non-thermal population is an inverted population when

\[ \frac{n_1}{g_1} < \frac{n_2}{g_2}. \]  

(2.52)

In such a case, the absorption coefficient is negative, \( \alpha_\nu < 0 \). The intensity of radiation thus increases exponentially along the ray. This is maser - microwave amplification by stimulated emission of radiation, it is similar to laser - light amplification...
2.5 Scattering

In addition to emission and absorption, the photons can also be scattered in the medium. A volume elements emits photons due to the scattering with the rate completely dependent on the amount of radiation falling on the element. Let us consider a simple case of coherent (or elastic, monochromatic, i.e. with no frequency shift) isotropic (i.e. scattering probability is the same in all directions) scattering. Often the 'absorption' coefficient for scattering is denoted $\sigma$ (do not mix it up with cross-section or Stefan-Boltzmann constant!).

- Pure scattering. The radiation energy 'absorbed' by the volume element $dV = dA \, ds$ ($ds = c \, dt$) in time $dt$ is:

$$\int d\Omega \, I_\nu \, dA \, dt \, \sigma \, ds,$$

where $I_\nu \, d\Omega \, dA \, dt$ is the amount of energy passing through area element $dA$ in time $dt$ within solid angle $d\Omega$ and $\sigma \, ds$ is the probability of scattering with the volume. For isotropic scattering this energy is emitted in all direction $4\pi j_\nu \, dV \, dt$. Thus

$$j_\nu = \sigma_\nu \frac{1}{4\pi} \int I_\nu \, d\Omega = \sigma_\nu J_\nu,$$

$$S_\nu = \frac{j_\nu}{\sigma_\nu} = J_\nu,$$

and

$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu).$$

This is integro-differential equation.

- Mean free path (for scattering only).

The average distance a photon can travel without being scattered. Let $d\tau = \sigma \, ds$ is the optical depth for both processes. The probability that a photon propagates optical depth $\tau$ is $\exp(-\tau)$. The mean optical depth is then

$$\langle \tau \rangle = \int_0^\infty \tau \exp(-\tau) \, d\tau = 1.$$

The mean free path is then

$$l = \frac{\langle \tau \rangle}{\sigma} = \frac{1}{\sigma}.$$
• Random walk.
Scattering can be viewed as a random walk of photons within the medium.
Let the displacement of the photon at step \( i \) is \( \vec{r}_i \). The net displacement after \( N \) steps is
\[
\vec{R} = \vec{r}_1 + \vec{r}_2 + \ldots + \vec{r}_N.
\]
Let us find the mean square displacement:
\[
\vec{l}^2 = \langle \vec{R}^2 \rangle = \langle \vec{r}_1^2 \rangle + \langle \vec{r}_2^2 \rangle + \ldots + \langle \vec{r}_N^2 \rangle + 2\langle \vec{r}_1 \cdot \vec{r}_2 \rangle + 2\langle \vec{r}_1 \cdot \vec{r}_3 \rangle \ldots
\]
(2.60)
The terms involving scalar products give zero, because for isotropic scattering the average of cosine of the scattering angle is zero. Each term involving square of a displacement gives approximately the square of mean free path \( l^2 \) (to be more exact, the mean square of the free path, \( \langle r^2 \rangle = 2l^2 \)). Therefore
\[
l^2 \approx N l^2,
\]
(2.61)
\[
l_\ast \approx \sqrt{N} l.
\]
(2.62)

• Mean number of scatterings.
A photon injected in the medium of size \( R \) should on average travel that distance, in order to escape the medium, i.e. \( R \approx \sqrt{N} l \). Thus, the mean number of scatterings is
\[
N \approx (R/l)^2 = \tau^2, \quad \tau \gg 1,
\]
(2.63)
where \( \tau \) is the optical thickness of the medium. When \( \tau \) is small, photons mostly escape without interactions. The probability that they interact within the medium is \( 1 - \exp(-\tau) \approx \tau \ll 1 \), and
\[
N \approx \tau, \quad \tau \ll 1.
\]
(2.64)
A more general formula working for any \( \tau \) is then
\[
N \approx \tau + \tau^2, \quad \text{or} \quad N \approx \max[\tau, \tau^2].
\]
(2.65)

• Escape time.
This is time it takes for a photon to diffuse from the medium:
\[
t_{\text{esc}} = \frac{Nl}{c} = \frac{NR}{\tau c} = \begin{cases} \frac{R}{\tau c}, & \tau \gg 1, \\ \frac{R}{c}, & \tau \ll 1. \end{cases}
\]
(2.66)
2.6 Scattering and absorption

- Combined scattering and absorption (in a thermal medium)

\[ \frac{dI}{ds} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu) \] (2.67)

with

\[ S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \] (2.68)

- Mean free path (for absorption and scattering).
  
The average distance a photon can travel without being absorbed or scattered. The extinction coefficient \( \alpha_\nu + \sigma_\nu \) and the optical depth for both processes is \( d\tau_\nu = (\alpha_\nu + \sigma_\nu)ds \). The mean free path is then

\[ l_\nu = \frac{\langle \tau_\nu \rangle}{\alpha_\nu + \sigma_\nu} = \frac{1}{\alpha_\nu + \sigma_\nu}. \] (2.69)

- A chance that after the free path the photon will be absorbed is \( \epsilon_\nu = \alpha_\nu / (\alpha_\nu + \sigma_\nu) \); chance that it will be scattered \( = 1 - \epsilon_\nu = \sigma_\nu / (\alpha_\nu + \sigma_\nu) \). The quantity \( 1 - \epsilon_\nu \) is called the single-scattering albedo. Source function is then

\[ S_\nu = (1 - \epsilon_\nu)J_\nu + \epsilon_\nu B_\nu. \] (2.70)

- Thermalization length.
  
A photon is created by thermal emission of an atom. It scatters many times, but at some point it can get absorbed by some other atom. The total path between creation and absorption is called thermalization length. Because the probability of getting absorbed in each interaction act (i.e. in the end of each free path) is \( \epsilon \), a photon on average has \( N = 1/\epsilon \) scatterings before absorption. Thus we have

\[ l_2^2 = \frac{l^2}{\epsilon}, \quad l_s = \frac{l}{\sqrt{\epsilon}}, \] (2.71)

and

\[ l_s = \frac{1}{\sqrt{\alpha_\nu (\alpha_\nu + \sigma_\nu)}} \] (2.72)
2.7. RADIATIVE DIFFUSION

- Effective optical thickness of the medium.

\[ \tau_s = \sqrt{\tau_a (\tau_a + \tau_s)}, \quad (2.73) \]

where \( \tau_a = \alpha_v R \) and \( \tau_s = \sigma_v R \) are the optical thickness of the medium of size \( R \) for absorption and scattering separately. If \( \tau_s \gg 1 \), the medium is effectively optically thick. The radiation field is then close to thermalization with the matter and \( I_\nu \approx B_\nu, S_\nu \approx B_\nu \).

2.7 Radiative diffusion

Solving the equation of radiative transfer in near-homogenous media. We will use plane-parallel geometries, with \( \mu = \cos \theta \).

- Rosseland approximation.

Consider a star. Assume that the medium is near-homogeneous and that the opacity is large so that the intensity is close to the Planck function. We can rewrite the radiative transfer equation in the following form:

\[ \mu \frac{dI_\nu}{dz} = -(\alpha_v + \sigma_v)(I_\nu - S_\nu), \quad (2.74) \]

where \( z \) is the vertical coordinate and \( \mu = \cos \theta \), and \( \theta \) is the polar (zenith) angle. Rewrite RTE:

\[ I_\nu = S_\nu - \mu \frac{1}{\alpha_v + \sigma_v} \frac{dI_\nu}{dz}. \quad (2.75) \]

Inside the star, intensity is close to the Planck function and the source function too. Thus we can approximate in the rhs \( I = B \) and \( S = B \):

\[ I_\nu = B_\nu - \mu \frac{1}{\alpha_v + \sigma_v} \frac{dB_\nu}{dz}. \quad (2.76) \]

Find the flux:

\[ F_\nu(z) = \int I_\nu \cos \theta d\Omega = 2\pi \int_{-1}^{1} I_\nu(z, \mu) \mu d\mu = -\frac{2\pi}{\alpha_v + \sigma_v} \frac{dB_\nu}{dz} \int_{-1}^{1} \mu^2 d\mu \]
\[ = -\frac{4\pi}{3(\alpha_v + \sigma_v)} \frac{dB_\nu(T)}{dz} = -\frac{4\pi}{3(\alpha_v + \sigma_v)} \frac{dB_\nu(T)}{dT} \frac{dT}{dz}. \quad (2.77) \]
Then the total flux at height $z$ is

$$F(z) = \int_0^\infty F(\nu) d\nu = -\frac{4\pi}{3} \frac{dT}{dz} \int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1} \frac{dB_\nu(T)}{dT} d\nu.$$  \hspace{1cm} (2.78)

Define the Rosseland mean opacity as

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}.$$  \hspace{1cm} (2.79)

Since

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial B(T)}{\partial T} = \sigma S B T^3 / \pi$$  \hspace{1cm} (2.80)

we get

$$F(z) = -\frac{16\sigma S B T^3}{3\alpha_R} \frac{dT}{dz} = \frac{4\pi}{3} \frac{\partial B}{\partial \tau_R},$$  \hspace{1cm} (2.81)

where $d\tau_R = -\alpha_R dz$ This means that the energy flux deep inside a star, for example, only depends on the temperature gradient and a single, weighted mean of all opacities.

- Eddington approximation.
  
  Now assume that in a near-homogenous medium the intensity is almost isotropic, but no longer assume that total opacity is large. Expanding the intensity into first-order terms of $\mu$:

$$I_\nu(\tau, \mu) = a_\nu(\tau) + b_\nu(\tau)\mu.$$  \hspace{1cm} (2.82)

We can evaluate the three moments of the intensity (equivalent with the mean intensity, the flux, and the pressure) as (suppressing the $\nu$ subscripts for clarity)

$$J \equiv \frac{1}{2} \int_{-1}^1 I d\mu = a,$$  \hspace{1cm} (2.83)

$$H \equiv \frac{1}{2} \int_{-1}^1 I\mu d\mu = b/3,$$  \hspace{1cm} (2.84)

$$K \equiv \frac{1}{2} \int_{-1}^1 I\mu^2 d\mu = a/3.$$  \hspace{1cm} (2.85)
2.7. RADIATIVE DIFFUSION

The latter equation for \( K = J/3 \) is the Eddington approximation (equivalent with expansion to first order in \( \mu \)). The RTE:

\[
\mu \frac{dI}{d\tau} = I - S. \tag{2.86}
\]

where \( d\tau = -(\alpha + \sigma)dz \) (note minus sign). Integrating over \( \mu \) we get:

\[
\frac{dH}{d\tau} = J - S. \tag{2.87}
\]

Multiplying (2.86) by \( \mu \) before integrating, we get using Eddington approximation:

\[
\frac{dK}{d\tau} = H = \frac{1}{3} \frac{dJ}{d\tau}. \tag{2.88}
\]

The last two equations can be combined to give second-order equation for \( J \) (radiative diffusion equation) which we can hope to solve:

\[
\frac{1}{3} \frac{\partial^2 J}{\partial \tau^2} = \epsilon (J - B). \tag{2.89}
\]

where we used \( S = (1 - \epsilon)J + \epsilon B \) from equation (2.70). If we have the temperature structure of the medium, i.e. \( B(T) \), we can solve this equation for \( J \) taking proper boundary conditions. We thus get \( J \), then using (2.70) we obtain \( S_{\nu} \) and then \( I_{\nu} \) by applying the formal solution of the RTE.

- Introducing optical depth

\[
\tau_* = \sqrt{3} \epsilon \tau = \sqrt{3} \tau_a (\tau_a + \tau_s), \tag{2.90}
\]

we get a different form of the diffusion equation

\[
\frac{\partial^2 J}{\partial \tau_*^2} = J - B. \tag{2.91}
\]

- Two-stream approximation.

In the Eddington approximation, let’s approximate \( I_{\nu}(\mu, z) \) with \( I_{\nu} \) along two directions only, \( \mu = \pm 1/\sqrt{3} \).

\[
I^+ = I(\tau, \mu = 1/\sqrt{3}), \tag{2.92}
\]

\[
I^- = I(\tau, \mu = -1/\sqrt{3}). \tag{2.93}
\]
The expression for $J$, $H$, and $K$ now become

$$J = \frac{1}{2}(I^+ + I^-), \quad (2.94)$$

$$H = \frac{1}{2\sqrt{3}}(I^+ - I^-), \quad (2.95)$$

$$K = \frac{1}{6}(I^+ + I^-) = J/3, \quad (2.96)$$

Our choice of $\mu = \pm 1/\sqrt{3}$ is explained by the fact that the Eddington approximation still holds, $K = J/3$. In other words, if the intensity is near-isotropic, the intensity can be approximated by taking into account only the intensity along angles $\mu = \pm 1/\sqrt{3}$.

With some more algebra we solve equation (2.94), (2.95) using (2.88):

$$I^+ = J + \frac{1}{\sqrt{3}} \frac{\partial J}{\partial \tau}, \quad (2.97)$$

$$I^- = J - \frac{1}{\sqrt{3}} \frac{\partial J}{\partial \tau}. \quad (2.98)$$

This gives us our two boundary conditions for $J$ and $\partial J/\partial \tau$, if we know what $I^+$ and $I^-$ are at a given locations $\tau_1$ and $\tau_2$ in the source. For example, if no radiation is entering from outside we have:

$$I^-(\tau = \tau_1) = 0, \quad I^+(\tau = \tau_2) = 0. \quad (2.99)$$

The Eddington approximation is often used for stellar atmospheres. In that case, the inner boundary conditions (inside the star), can be written as

$$F = 4\pi H = \frac{4\pi \frac{\partial B(T)}{\partial \tau}}{3} = \frac{4\pi \frac{\partial J(\tau)}{\partial \tau}}{3} \quad (2.100)$$

at very large $\tau$ where the temperature structure is known.