Chapter 3

Radiation fields

3.1 Definitions of electric and magnetic fields

The electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be defined through their effect on a charge $q$ at location $\vec{r}$ moving with velocity $\vec{v}$. This is the operational definition (by doing experiments). Consider behaviour of a charged particle of charge $q$:

$$\vec{F} = q\vec{E}.$$  \hspace{1cm} (3.1)

If acceleration is parallel to the velocity then the force is parallel the velocity. Let us define the electric field as force per unit charge:

If, on the other hand, the acceleration is perpendicular to the velocity, then the force is also perpendicular to the velocity.
We define then the magnetic field as follows:

\[ \vec{F} = q \frac{\vec{v}}{c} \times \vec{B} \]  

(3.2)

The total force is called Lorentz force:

\[ \vec{F} = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \]  

(3.3)

### 3.2 Work performed by the field

The work performed moving one particle per unit time is:

\[ \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} = q \left( \vec{E} \cdot \vec{v} + \frac{\vec{v}}{c} \cdot \vec{v} \times \vec{B} \right) = q \vec{E} \cdot \vec{v}, \]  

(3.4)

where \( d\vec{r} \) is the displacement and \( \vec{F} \cdot d\vec{r} \) is the work.

Let us write Newton’s law for a non-relativistic particle:

\[ \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} m \vec{v}. \]  

(3.5)

The work per unit time is then

\[ q\vec{v} \cdot \vec{E} = \vec{F} \cdot \vec{v} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{d}{dt} \frac{1}{2} mv^2 = \frac{d}{dt} \epsilon_{\text{mech}}. \]  

(3.6)

Thus the work per unit time done by the field on the particle is equal to the change of the kinetic (mechanic) energy per unit time.

Now let us consider a collection of particles. We can define the charge density as:

\[ \rho_e(\vec{r}, t) = \sum_i q_i \delta(\vec{r} - \vec{r}_i(t)), \]  

(3.7)

and similarly the current density:

\[ \vec{j}_e(\vec{r}, t) = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t)). \]  

(3.8)
3.3 Maxwell’s equations in differential form

Notice that
\[ \int \delta(\vec{r} - \vec{r}_i(t)) \, d^3r = 1. \]  \hspace{1cm} (3.9)

The proper total charge is given by the integral over the volume:
\[ q = \int \rho \, d^3r = \sum_i \int q_i \delta(\vec{r} - \vec{r}_i(t)) \, d^3r, \]  \hspace{1cm} (3.10)

and similarly the proper total current is
\[ q\vec{v} = \int \vec{j} \, d^3r = \sum_i \int q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t)) \, d^3r. \]  \hspace{1cm} (3.11)

The work per unit time done by the fields at position \( \vec{r} \) per unit volume is then:
\[ \sum_i \delta(\vec{r} - \vec{r}_i(t)) \, q_i \vec{v}_i \cdot \vec{E} = \vec{j}_e \cdot \vec{E}. \]  \hspace{1cm} (3.12)

### 3.3 Maxwell’s equations in differential form

**Coulomb’s law**
\[ \nabla \cdot \vec{D} = 4\pi \rho. \]  \hspace{1cm} (3.13)

**Guilbert’s "law".** No magnetic charges (= monopoles).
\[ \nabla \cdot \vec{B} = 0. \]  \hspace{1cm} (3.14)

**Faraday’s law of induction**
\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}. \]  \hspace{1cm} (3.15)
Maxwell’s generalization of Ampere’s law:

\[ \nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}, \]  

(3.16)

where \( \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \) is called the displacement current.

The fields are related as

\[ \vec{B} = \mu \vec{H}, \quad \vec{D} = \epsilon \vec{E}, \]  

(3.17)

where \( \epsilon \) and \( \mu \) are the dielectric constant and magnetic permeability of the medium. In vacuum \( \epsilon = \mu = 1 \).

Notes

1) Note the invariance for \( \vec{E} \rightarrow \vec{B} \) and \( \vec{B} \rightarrow -\vec{E} \) if \( \rho = 0, \vec{j} = 0 \) and \( \epsilon = \mu = 1 \).

2) if \( \nabla \cdot \vec{A} \) and \( \nabla \times \vec{A} \) are known, then \( \vec{A} \) is uniquely specified (to arbitrary constant). Helmholtz theorem (ch 1.15 in Arfken 2nd ed.)

Thus if \( \rho, \vec{j} \) (the sources of the field) are known, then \( \vec{E}, \vec{B} \) uniquely determined through Maxwell’s equations.

Definitions:

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \]

\[ \text{div} \ A = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \]

\[ \text{curl} \ A = \nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right). \]
3.4 Conservation law for electric charge (Benjamin Franklin)

The conservation law for electric charge follows directly from Maxwell’s equations. Take $\nabla \cdot$ on Ampere’s law:

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \left( \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \vec{D}) \right)$$

$$= 0$$

$$4\pi \rho$$

$$\Rightarrow \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{3.19}$$

Let us prove the conservation law in a simple way. Let us consider a current (flow) of charges through a volume of size $dx \, dy \, dz$, with the flow in the $x$ direction only.

$$\int_{x+dx}^{x} j(x+dx) \, dz - \int_{x}^{x+dx} j(x) \, dz = -\frac{\partial \rho}{\partial t} \, dx \, dy \, dz. \tag{3.20}$$

Thus we get

$$\frac{\partial j}{\partial x} \, dx$$

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0.$$  

3.5 Energy density, flux of electromagnetic field, energy conservation

Consider the work done per unit volume and unit time. Using Ampere’s law we get

$$\vec{j} = \frac{1}{4\pi} \left[ c(\nabla \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} \right] \tag{3.21}$$
and
\[
\vec{j} \cdot \vec{E} = \frac{1}{4\pi} \left[ c\vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right].
\] (3.22)

The first term in the brackets (see problem set 2):
\[
\vec{E} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}).
\] (3.23)

Substitute \( \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \) from Faraday’s law:
\[
\vec{j} \cdot \vec{E} = \frac{1}{4\pi} \left[ -\frac{1}{2\mu} \frac{\partial \vec{B}^2}{\partial t} - \frac{\varepsilon}{2} \frac{\partial \vec{E}^2}{\partial t} - c \nabla \cdot (\vec{E} \times \vec{H}) \right].
\] (3.24)

We obtain thus Poynting’s theorem (conservation of energy)
\[
\vec{j} \cdot \vec{E} + \frac{1}{8\pi} \frac{\partial}{\partial t} \left( \varepsilon \vec{E}^2 + \frac{1}{\mu} \vec{B}^2 \right) = -\nabla \cdot \left( \frac{c}{4\pi} \vec{E} \times \vec{H} \right).
\] (3.25)

We can define the energy density of the electromagnetic field as
\[
U_{\text{field}} = \frac{1}{8\pi} \left( \varepsilon \vec{E}^2 + \frac{\vec{B}^2}{\mu} \right).
\] (3.26)

The Poynting flux
\[
\vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{H})
\] (3.27)
determines the energy flux of the electromagnetic field.

Equation (3.25) allows a simple interpretation: the change of mechanical energy and field energy is equal to the minus of the divergence of flux.

We can integrate over the volume
\[
\int_V dV \frac{\partial}{\partial t} \left( \varepsilon_{\text{volume}} \vec{E}^2 + \frac{1}{\mu_{\text{volume}}} \vec{B}^2 \right) = -\int_V dV \nabla \cdot \vec{S}.
\] (3.28)

Using Gauss’s theorem one gets
\[
\frac{d}{dt} \left( \varepsilon_{\text{volume}} + \varepsilon_{\text{field}} \right) = -\int_{\Sigma} d\vec{A} \cdot \vec{S}.
\] (3.29)

Thus the change of the total energy in the volume is equal to the inward (sign \(-\)) energy flux through surface.
If surface area $\Sigma \rightarrow \infty$ then the electrostatic and magnetostatic fields (check your old textbooks) depend on distance as $E \propto 1/r^2$ and $H \propto 1/r^2$. Therefore, the Poynting flux

$$\vec{S} \propto \vec{E} \times \vec{H} \propto \frac{1}{r^3}. \quad (3.30)$$

The total energy escaping to infinity is then

$$\int \vec{S} \cdot d\vec{A} \propto \frac{1}{r^4} r^2 \rightarrow 0. \quad (3.31)$$

We shall later find that the time-dependent fields depend on distance as $\propto \frac{1}{r}$, therefore the energy escaping to infinity

$$\int \vec{S} \cdot d\vec{A} \propto \frac{1}{r^2} r^2 \rightarrow \text{finite.} \quad (3.32)$$

This way the radiation escapes to infinity.

### 3.6 Maxwell’s equations in vacuum

Vacuum mean that there is no charges, no currents. $\epsilon = \mu = 1$.

$$\rho = 0, \quad \vec{j} = 0.$$ 

There exists a trivial solution: $\vec{E} = \text{const}$ and $\vec{B} = \text{const}$. Why would non-trivial solutions exist?

**Faraday’s induction law**

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}. $$
Thus time varying $\vec{B}$-field gives rise to $\vec{E}$-field that in turn gives rise to $\vec{B}$-field through Ampere’s law

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}. \quad (3.32)$$

Thus continuously varying $\vec{E}$ and $\vec{B}$ fields can generate each other for ever even though $\rho = 0$, $\vec{j} = 0$, i.e. no sources for the fields. It was hard to accept in 1860s. Waves in vacuum required introduction of aether. It is now less surprising if the radiation is considered as particles (photons).

### 3.6.1 Wave equation in vacuum

Looking at non-trivial solutions, waves carrying energy & momentum. Take curl of Faraday’s law:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c} \frac{\partial}{\partial t} \nabla \times \vec{B} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (3.33)$$

$$= \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

$= 0$

(see exercise 2.3). Thus we get the homogeneous wave equations:

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = 0, \quad (3.34)$$

and by symmetry

$$\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = 0. \quad (3.35)$$

Here $\nabla^2 \equiv \Delta$ is Laplacian operator.

### 3.6.2 Solution of the wave equation

The general solution of the wave equations

$$\vec{E} = \vec{a}_1 E_0 e^{i(k \cdot \vec{r} - \omega t)}, \quad \vec{B} = \vec{a}_2 B_0 e^{i(k \cdot \vec{r} - \omega t)}, \quad (3.36)$$

where $\vec{a}_1$ and $\vec{a}_2$ are the unit vectors, $E_0$ and $B_0$ are complex constants, $k = k \hat{n}$ and $\omega$ are the wave-vector and frequency. Such a solution represent waves traveling in
the $\vec{n}$ direction. The most general solution of Maxwell equations can then be constructed by superposition of wave of various frequencies and traveling in different directions.

Substituting (3.36) into Maxwell’s equation, we get:

\[
\begin{align*}
    ik \cdot \vec{a}_1 E_0 &= 0, \\
    ik \cdot \vec{a}_2 B_0 &= 0, \\
    ik \times \vec{a}_1 E_0 &= \frac{i \omega}{c} \vec{a}_2 B_0, \\
    ik \times \vec{a}_2 B_0 &= -\frac{i \omega}{c} \vec{a}_1 E_0.
\end{align*}
\] (3.37)

The top two equations tell us that $\vec{a}_1$ and $\vec{a}_2$ are both perpendicular to $\vec{k}$. With that knowledge, the bottom two equations tell us that $\vec{a}_1$ and $\vec{a}_2$ are perpendicular to each other. Thus $\vec{a}_1$, $\vec{a}_2$ and $\vec{k}$ for the right-hand triad of mutually perpendicular vectors.

We thus can get the relation between $E_0$ and $B_0$:

\[
E_0 = \frac{\omega}{kc} B_0, \quad B_0 = \frac{\omega}{kc} E_0,
\] (3.38)

so that

\[
\omega^2 = c^2 k^2
\] (3.39)

and

\[
E_0 = B_0.
\] (3.40)

Taking $k$ and $\omega$ positive we get

\[
\omega = ck.
\] (3.41)

The phase velocity of the waves is

\[
v_{\text{ph}} = \omega / k = c
\] (3.42)

and the group velocity is also $c$:

\[
v_{\text{gr}} = \frac{\partial \omega}{\partial k} = c.
\] (3.43)

We can now compute the energy flux of these waves. Since $\vec{E}$ and $\vec{B}$ vary sinusoidally with time, the Poynting vector fluctuate. We can take the time average, which is normally measured. It is shown in problem set 2 that for two quantities

\[
A(t) = A e^{i \omega t}, \quad B(t) = B e^{i \omega t},
\] (3.44)
the time average of their product of their real parts is

\[
\langle \text{Re}A(t) \text{Re}B(t) \rangle = \frac{1}{2} \text{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2} \text{Re}(\mathcal{A}^*\mathcal{B}), 
\]

where * denotes complex conjugate. Thus we get

\[
\langle S \rangle = \frac{c}{8\pi} \text{Re}(E_0 B_0^*) = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2, 
\]

where we used \( E_0 = B_0 \).

## 3.7 Radiation spectrum

From the time variation of the electrical field (and, analogously, the magnetic field) follows the spectrum of the radiation. The spectrum is the amount of energy per unit area per unit time per unit frequency interval, and is most easily derived through a Fourier transformation. Let us consider a pulse of radiation that passes by an observer. For a pulse of radiation, \( E(t) \to 0 \) and \( B(t) \to 0 \) as \( t \to \pm\infty \). We only consider the \( E \)-field along one axis, \( E(t) = \hat{a} \cdot \vec{E}(t) \). The Fourier transform and its inverse are now defined as

\[
\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt, \quad (3.47)
\]

\[
E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega) e^{-i\omega t} d\omega. \quad (3.48)
\]

The quantity \( \hat{E}(\omega) \) contains the full frequency information of \( E(t) \). The amount of energy \( dW \) passing through a surface element \( dA \) per time \( dt \) is given by the Poynting vector \( \vec{S} \):

\[
\frac{dW}{dA \, dt} = |\vec{S}(t)| = \frac{c}{4\pi} E^2(t). \quad (3.49)
\]

The total energy per unit area is

\[
\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt. \quad (3.50)
\]
3.7. RADIATION SPECTRUM

From Parseval’s theorem \(^1\) we get

\[
\frac{dW}{dA} = c \int_0^\infty |\hat{E}(\omega)|^2 d\omega, \quad (3.51)
\]

because \(|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2\). Thus we can define the energy per unit area per unit frequency (for the entire pulse):

\[
\frac{dW}{dA \, d\omega} = c |\hat{E}(\omega)|^2. \quad (3.52)
\]

If pulses repeat on the time-scale \(T\), we can introduce power per unit time:

\[
\frac{dW}{dA \, dt \, d\omega} = \frac{c}{T} |\hat{E}(\omega)|^2. \quad (3.53)
\]

In this expression we need to measure the emission pulse over lengths of time \(T\) that is sufficiently long to sample all relevant frequencies \(\omega\) (\(\omega T > 1\)), but short compared to the duration of the whole signal (the properties of \(E(t)\) remain approximately constant, i.e. process is stationary). The fact that the time variation of the electrical field and its spectrum are related through a Fourier transform makes it very convenient to derive a spectral shape from the characteristics of \(E(t)\).

As an example, let us consider a pulse described by an exponential:

\[
E(t) = e^{-t/\tau}, \quad t > 0, \quad (3.54)
\]

where \(\tau\) is the decay constant. Compute the Fourier transform

\[
\hat{E}(\omega) = \frac{1}{2\pi} \int_0^\infty e^{-t/\tau} e^{i\omega t} dt = \frac{1}{2\pi} \frac{1}{\frac{1}{\tau} - i\omega}, \quad (3.55)
\]

and therefore the spectrum is

\[
|\hat{E}(\omega)|^2 = \frac{1}{4\pi^2} \frac{1}{\frac{1}{\tau} - i\omega} \frac{1}{\frac{1}{\tau} + i\omega} = \frac{1}{4\pi^2} \left( \frac{1}{\frac{1}{\tau} + \omega^2} \right) = \begin{cases} \text{const} = \frac{\tau^2}{4\pi}, & \omega \ll 1/\tau, \\ \propto \omega^{-2}, & \omega \gg 1/\tau. \end{cases} \quad (3.56)
\]

Other examples are shown in Fig. 3.1.

\(^1\)The Parseval’s theorem for Fourier pairs is stated as:

\[
\int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega.
\]
Figure 3.1: EM pulses (left) and associated radiation spectra (right) for three pulse shapes. Top: a pulse of duration $T$ has a spectrum stretching over a bandwidth of $\sim 1/T$. Middle: A periodic signal with frequency $\omega_0$ for a total duration of time $T$ will have a spectrum of width $\sim 1/T$ centered on a frequency $\omega_0$. Bottom: A similar periodic signal with a decay time of $T$ (damped oscillator) will produce a spectrum of bandwidth $\sim 1/T$ centered on a frequency $\omega_0$, but without the higher and lower frequency ‘wiggles’ found in the previous example.
3.8. STOKES PARAMETERS

Any (single-frequency) $\vec{E}$ wave can be decomposed into two orthogonal waves with amplitudes $\vec{E}_1$ and $\vec{E}_2$ and the same frequency (but arbitrary phase difference). The resulting composite $\vec{E}$ traces out an ellipse.

3.8 Stokes parameters

Consider a plane wave in the $z$-direction. The Fourier decomposition:

$$\vec{E}_+(z-ct) = \int_{-\infty}^{\infty} \vec{e}_+(k) e^{i k (z - ct)} dk.$$ \hspace{1cm} (3.57)

Here $k$ - wavenumber; $\omega \equiv kc$ - angular frequency; $\omega = 2\pi \nu$, $\nu$ is frequency.

So far we only considered oscillation in one plane (linearly polarized). Most general wave is superposition of oscillations in 2 perpendicular planes. It is convenient to consider $\vec{e}_+$ complex. $\vec{e}_+$ should be transverse to $\hat{z}$:

$$\vec{e}_+(k) = \hat{x} E_x(k) e^{i \phi_x(k)} + \hat{y} E_y(k) e^{i \phi_y(k)},$$ \hspace{1cm} (3.58)

where $E_{x,y}$ are real amplitudes; $\phi_{x,y}$ are phases. ($e^{ix} = \cos x + i \sin x$).

The Fourier components are the real parts of $\vec{e}_+(k) e^{i k (z - ct)}$:

$$\vec{E}_k = \hat{x} E_x(k) \cos[k(z - ct) + \phi_x(k)] + \hat{y} E_y(k) \cos[k(z - ct) + \phi_y(k)].$$ \hspace{1cm} (3.59)

Thus an arbitrary polarized monochromatic wave (i.e. a given $k$) is described by 4 real parameters (instead of just one intensity $I_\nu$). It is inconvenient to use
CHAPTER 3. RADIATION FIELDS

\[ \mathbf{E}(t) = E_x(t) \mathbf{e}_x + E_y(t) \mathbf{e}_y = E_0 \cos(\omega t - \phi_x) \mathbf{e}_x + E_0 \sin(\omega t - \phi_y) \mathbf{e}_y. \]

(3.60)

(The four parameters are \( E_x, E_y, \phi_x, \phi_y \))

Here real amplitudes

\[ E_x(t) = E_0 \cos(\omega t \cos \phi_x + \sin \omega t \sin \phi_x), \]
\[ E_y(t) = E_0 \cos(\omega t \cos \phi_y + \sin \omega t \sin \phi_y). \]

(3.61)

The \( \mathbf{E}_k \) vector traces an ellipse: elliptically polarized wave. The principal axis of the ellipse has a tilt \( \chi \) (polarization angle) with respect to the \( x \)-axis.

In the new coordinate system \( x', y' \) which is rotated by angle \( \chi \) relative to the old \( x, y \) system, the ellipse equation is given by the following relations:

\[ E_1(t) = E_0 \cos \beta \cos \omega t, \quad E_2(t) = E_0 \sin \beta \sin \omega t, \]

where we introduced the ellipticity parameter \( \beta \). Rotation is counter clockwise for \( 0 < \beta < \pi/2 \) (left handed polarization). \( \beta = \pm \pi/4 \) means circular polarization, while for \( \beta = 0 \) or \( \beta = \pi/2 \) the polarization is linear.

Since the coordinates in two systems are related as

\[ E_x = E_1 \cos \chi - E_2 \sin \chi, \quad E_y = E_1 \sin \chi + E_2 \cos \chi, \]

Figure 3.3: Geometry for elliptically polarized wave propagating in \( z \)-direction.

\( \mathcal{E}, \mathcal{E}_x, \mathcal{E}_y, \phi_x, \phi_y \) since they have different units. Stokes (1852) found a convenient set of 4 parameters describing polarized light.

Let us consider fixed position, say, \( z = 0 \):

\[ \mathbf{E}_k(t) = \hat{x}E_x(t) + \hat{y}E_y(t) = \hat{x}E_x \cos(\omega t - \phi_x) + \hat{y}E_y \cos(\omega t - \phi_y). \]
we get

\[
E_x(t) = E_0(\cos \beta \cos \chi \cos \omega t - \sin \beta \sin \chi \sin \omega t),
\]

\[
E_y(t) = E_0(\cos \beta \sin \chi \cos \omega t + \sin \beta \cos \chi \sin \omega t).
\]

Identifying coefficients in front of \( \cos \omega t \) and \( \sin \omega t \) in equations (3.61) and (3.62), we get:

\[
E_x \cos \phi_x = E_0 \cos \beta \cos \chi,
\]

\[
E_x \sin \phi_x = -E_0 \sin \beta \sin \chi,
\]

\[
E_y \cos \phi_y = E_0 \cos \beta \sin \chi,
\]

\[
E_y \sin \phi_y = E_0 \sin \beta \cos \chi,
\]

where we have three new parameters \( E_0, \beta \) and \( \chi \) describing completely 100% polarized monochromatic wave instead of the previous four: \( E_x, E_y, \phi_x, \phi_y \). Among the phases, it is only the difference \( \phi_y - \phi_x \) that matters. Stokes (1852) defined 4 practical quantities to characterise a wave (Stokes parameters):

\[
I = E_x^2 + E_y^2 = E_0^2,
\]

\[
Q = E_x^2 - E_y^2 = E_0^2 \cos 2\beta \cos 2\chi,
\]

\[
U = 2E_xE_y \cos(\phi_y - \phi_x) = E_0^2 \cos 2\beta \sin 2\chi,
\]

\[
V = 2E_xE_y \sin(\phi_y - \phi_x) = E_0^2 \sin 2\beta.
\]

Here we again have 4 parameters, but they are not independent since for a completely polarized wave:

\[
I^2 = Q^2 + U^2 + V^2.
\]

Sometimes alternatively one uses 3 parameters:

\[
E_0 = \sqrt{I},
\]

\[
\sin 2\beta = V/I,
\]

\[
\tan 2\chi = U/Q,
\]

where \( \beta \) is the ellipticity parameter and \( \chi \) the polarization angle.

One of the possible ways of presenting polarization is on the Poincare sphere (see Fig. 3.8).

Light is normally not monochromatic and not 100% polarized. Different part of the object have different polarizations and different phases. Therefore, in reality
CHAPTER 3. RADIATION FIELDS

Figure 3.4: Representation of Stokes parameters using the Poincare sphere.

$I^2 \geq Q^2 + U^2 + V^2$ and four parameters are needed to characterize the polarization. An important property of Stokes parameters that they are additive for a superposition of independent waves (i.e. those that do not have permanent phase relations).

\[ I = \sum I^{(k)}, \quad Q = \sum Q^{(k)}, \quad U = \sum U^{(k)}, \quad V = \sum V^{(k)}. \]  
\[ (3.66) \]

Therefore any Stokes vector can be represented as a sum of one unpolarized (first term) and one completely polarized (second term) parts:

\[
\begin{pmatrix}
  I \\
  Q \\
  U \\
  V 
\end{pmatrix} = \begin{pmatrix}
  I - \sqrt{Q^2 + U^2 + V^2} \\
  0 \\
  0 \\
  0
\end{pmatrix} + \begin{pmatrix}
  \sqrt{Q^2 + U^2 + V^2} \\
  Q \\
  U \\
  V
\end{pmatrix}.
\]
\[ (3.67) \]

Polarization degree is then defined as

\[ \Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \]
\[ (3.68) \]

One can also define linear $\Pi_{\text{lin}} = \sqrt{Q^2 + U^2}/I$ and circular polarizations $\Pi_{\text{cir}} = V/I$. 