Chapter 5

Relativistic covariance and kinematics

5.1 Lorentz transformations

The special theory of relativity is based on two postulates:

1. The laws of nature are the same in two reference frames in uniform relative motion with no rotation.

2. The speed of light is constant \( c \) in all such frames.

Consider two coordinate systems \( K \) and \( K' \) that moves with relative velocity \( V \) along the \( x \)-axis. From the postulates one can show that the coordinates in the two systems are related through Lorentz transformations (LT):

\[
\begin{align*}
x' &= \gamma (x - \beta ct), \\
y' &= y, \\
z' &= z, \\
ct' &= \gamma (ct - \beta x),
\end{align*}
\]

(5.1)

where \( \gamma = \frac{1}{\sqrt{1-\beta^2}} \), \( \beta = V/c \). This also can be written using the Lorentz transformation tensor

\[
\Lambda = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(5.2)
as

\[
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix} = \Lambda \begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix},
\]

(5.3)

The inverse transform can be obtained changing \(\beta \rightarrow -\beta\):

\[
\begin{align*}
x &= \gamma(x' + \beta ct'), \\
y &= y', \\
z &= z', \\
ct &= \gamma(ct' + \beta x').
\end{align*}
\]

(5.4)

The Lorentz transformation with arbitrary velocity \(\vec{\beta}\) of a 4-vector \(\mathbf{a} = \{a_0, \vec{a}\}\) is given by:

\[
\begin{align*}
a'_0 &= \gamma(a_0 - \vec{\beta} \cdot \vec{a}), \\
\vec{a}' &= \vec{a} - a_0 \gamma \vec{\beta} + (\gamma - 1) \vec{\beta} (\vec{\beta} \cdot \vec{a}) / \beta^2.
\end{align*}
\]

(5.5)

### 5.1.1 Proper time

Some quantities are Lorentz invariants, i.e. they have the same value in all Lorentz-frames. Proper time, \(d\tau\), between events with time- and spatial distances \(dt, dx, dy, dz\), is defined as

\[
c^2 d\tau^2 \equiv c^2 dt^2 - (dx^2 + dy^2 + dz^2).
\]

(5.6)

One can prove that \(d\tau\) is the Lorentz invariant by using the Lorentz transformation, \(d\tau = d\tau'\). Prove this at home!

Proper time \(\tau\) is the time shown by the clocks that observers carry along, i.e. \(\tau\) is the time in the rest frame of the observer, where \(dx = dy = dz = 0\). In this system we have \(d\tau = dt\).

### 5.1.2 Lorentz-Fitzgerald length contraction

The \(K'\)-observer carries a rod of length \(L' = x_2' - x_1'\). In \(K\)-system the rod’s length is measured by determining the coordinates of the ends of the rod at the same time \(t\). Therefore

\[
L' = x_2' - x_1' = \gamma(x_2 - Vt - x_1 + Vt) = \gamma(x_2 - x_1) = \gamma L,
\]

(5.7)
5.1. LORENTZ TRANSFORMATIONS

where we used LT. Here \( L \) is the length in \( K \)-system. Thus,

\[
L = (1 - \beta^2)^{1/2} L' = L'/\gamma. \tag{5.8}
\]

The rod is shorter for the \( K \)-observer.

If the \( K \)-observer carries the rod, then the \( K' \)-observer finds it to be shorter. The effect is symmetric. The reason for differences is that the measurements are not simultaneous between the two frames.

5.1.3 Time dilation

Consider a clock at rest in \( K' \) (a comoving clock, \( x' = 0 \)) that measures a time interval \( T' = t'_2 - t'_1 \). In the \( K \)-system (using clocks in the \( K \)-system), one measures

\[
T = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma T'. \tag{5.9}
\]

Here we used LT with \( x' = 0 \). For \( K \), the clock in \( K' \) seems to be slower. On the contrary, for \( K' \) the clocks in \( K \) seem slower.

5.1.4 Velocity transformation

LT can be written in differential form:

\[
\begin{align*}
\frac{dx}{dt} &= \gamma(dx' + \beta c dt'), \\
\frac{dy}{dt} &= dy', \\
\frac{dz}{dt} &= dz', \\
\frac{cdt}{dt} &= \gamma(cdt' + \beta dx').
\end{align*} \tag{5.10}
\]

We get then for the velocities

\[
\begin{align*}
u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + \beta c dt')}{\gamma(dt' + \beta dx'/c)} = \frac{u'_x + V}{1 + \beta u'_x/c}, \\
u_y &= \frac{dy}{dt} = \frac{u'_y}{\gamma(1 + \beta u'_x/c)}, \\
u_z &= \frac{dz}{dt} = \frac{u'_z}{\gamma(1 + \beta u'_x/c)}. \tag{5.11}
\end{align*}
\]
or rewriting this for velocities parallel and perpendicular to $V$:

$$u_\parallel = \frac{u'_\parallel + V}{1 + \beta u'_\parallel / c} \tag{5.12}$$

$$u_\perp = \frac{u'_\perp}{\gamma(1 + \beta u'_\parallel / c)} \tag{5.13}$$

### 5.1.5 Transformation of velocity directions = aberration

The angle $\theta$ the velocity makes to some direction can be defined via the ratio of the projections of the velocity to the direction and perpendicular to it:

$$\tan \theta = \frac{u_\perp}{u_\parallel} = \frac{u'_\perp}{\gamma(u'_\parallel + V)} = \frac{u' \sin \theta'}{\gamma(u' \cos \theta' + V)}. \tag{5.14}$$

Note that the azimuth angle does not change.

To determine aberration of light, we need to substitute $u' = c$:

$$\tan \theta = \frac{\sin \theta'}{\gamma(\cos \theta' + \beta)}. \tag{5.15}$$

At home: use $\cos^2 \theta = \frac{1}{1 + \tan^2 \theta}$ to show that

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}, \tag{5.16}$$

$$\sin \theta = \frac{\sin \theta'}{\gamma(1 + \beta \cos \theta')}. \tag{5.17}$$

Example: How does $\theta' = \pi/2$ transforms?

$$\theta' = \pi/2 \Rightarrow \tan \theta = 1/(\gamma \beta) \Rightarrow \cos \theta = \beta \Rightarrow \sin \theta = 1/\gamma. \tag{5.18}$$

If $\gamma \gg 1 \Rightarrow \theta = 1/\gamma$.

Isotropic emission in $K'$ becomes "beamed emission" in $K$-frame, where the particle moves with Lorentz factor $\gamma$.

### 5.1.6 Doppler effect

There are three different intervals to keep track of:

1) the time interval $\Delta t'$ in the moving particle frame $K'$ (e.g. related to the frequency of emitted radiation $\Delta \nu' = 2\pi/\omega'$);
2) the time interval $\Delta t$ in the observer’s system $K$: $\Delta t = \gamma \Delta t'$ (time dilation);
3) the time interval $\Delta t_{\text{Arrival}}$, during which a pulse is received by the observer.

A particle of velocity $V$ emits photons at 1 and 2 with time interval $\Delta t$, towards the observer. When the 2nd photon is emitted, the 1st one has travelled a distance $c \Delta t$. From the figure we get

$$c \Delta t_A = c \Delta t - V \Delta t \cos \theta \Rightarrow \Delta t_A = \Delta t(1 - \beta \cos \theta). \quad (5.19)$$

If $\Delta t_A$ is the time interval for receiving one wavelength, the observed frequency becomes:

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{2\pi}{\Delta t(1 - \beta \cos \theta)} = \frac{2\pi}{\gamma \Delta t'(1 - \beta \cos \theta)} = \frac{\omega'}{\gamma(1 - \beta \cos \theta)}. \quad (5.20)$$

**Relativistic Doppler formula:**

$$\omega' = \omega \gamma (1 - \beta \cos \theta), \quad \omega = \omega' \gamma (1 + \beta \cos \theta'). \quad (5.21)$$

Show that $\gamma (1 - \beta \cos \theta) = \frac{1}{\gamma(1 + \beta \cos \theta')}$ using the $\cos \theta \cdot \cos \theta'$ formula.

For small $\theta$ and large $\gamma$ (i.e. $\beta = \sqrt{1 - 1/\gamma^2} \sim 1 - 1/2\gamma^2$ ),

$$\omega = \frac{\omega'}{\gamma[1 - (1 - 1/2\gamma^2)(1 - \theta'^2/2)]} = \frac{\omega' 2\gamma}{1 + \theta'^2 \gamma^2}. \quad (5.22)$$
5.1.7 4-vectors

Examples:

- position \((ct, \vec{x})\)
- 4-velocity \(\gamma(c, \vec{u})\)
- 4-momentum for a photon \(k = \frac{\hbar \omega}{c} (1, \vec{n})\)
  where \(\vec{n}\) is the unit vector in the direction of the photon propagation
- 4-current density \((\rho c, \vec{j})\)
- 4-potential \((\phi, \vec{A})\)
- 4-momentum for particle \(p = (E/c, \vec{p})\)

All Lorentz transformed in the same way as the "position" vector, e.g. for the 0th component:

\[
ct' = \gamma (ct - \beta x),
\]

(5.23)

particle energy

\[
E'/c = \gamma (E/c - \beta p_x),
\]

(5.24)

photon energy

\[
\frac{\hbar \omega'}{c} = \frac{\hbar \omega}{c} (1 - \beta n_x) = \gamma \frac{\hbar \omega}{c} (1 - \beta \cos \theta) \Rightarrow
\]

(5.25)

\[
\omega' = \omega \gamma (1 - \beta \cos \theta), \text{ i.e. we obtained the relativistic Doppler effect formula directly from the LT.}
\]

5.1.8 Lorentz invariants

Scalar products of 4-vectors are Lorentz invariants. For example:

a)

\[
(ct, \vec{x}) \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = -c^2 t^2 + x^2 = \text{const.}
\]

(5.26)

Minus sign appears by rule in space-time metric. One can instead use standard definition of the scalar product, but introduce imaginary \(i\) infront of the 0th element of the 4-vector, i.e.

b)

\[
(ict, \vec{x}) \begin{pmatrix} ict \\ \vec{x} \end{pmatrix} = (ict)^2 + x^2 = -c^2 t^2 + x^2.
\]

(5.27)

\[
(E/c, \vec{p}) \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = p^2 - E^2/c^2 = \text{const.}
\]

(5.28)

What constant? Consider its value in the rest frame where \(\vec{p} = 0\), then \(0 - \frac{(mc)^2}{c^2} = -(mc)^2\). This means that \(E^2 - (pc)^2 = (mc^2)^2\). If one introduces energy \(E = \gamma mc^2\) and momentum \(\vec{p} = \gamma \vec{u} \vec{n}\), one gets \(\gamma^2 - \beta^2 \gamma^2 = 1\), i.e. \(\gamma = 1/\sqrt{1 - \beta^2}\).
5.1. LORENTZ TRANSFORMATIONS

The dependence between $E$ and $pc$ (hyperbola $E^2 = (pc)^2 + (mc^2)^2$) is the same as we had when discussing proper time. Here the constant is the rest mass energy $mc^2$.

5.1.9 Electro-magnetic field transformation

The electric or magnetic fields cannot be represented as 4-vectors. Instead one can introduce the electro-magnetic field tensor:

$$
F = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_x & -B_y \\
-E_y & -B_x & 0 & B_z \\
-E_z & B_y & -B_z & 0
\end{pmatrix}.
$$

(5.29)

The LT to the system $K'$ moving with velocity $V = c\beta$ along the x-axis can be written using the LT tensor as

$$F' = \Lambda^T F \Lambda.$$

(5.30)

This can be rewritten in the form

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel}, \quad \vec{B}_{\parallel}' = \vec{B}_{\parallel},$$

(5.31)

$$\vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \beta \times \vec{B}), \quad \vec{B}_{\perp}' = \gamma(\vec{B}_{\perp} - \beta \times \vec{E}).$$

(5.32)

The immediate consequence is that the concept of pure electric or magnetic field is not Lorentz invariant. If in one frame the field is purely electric ($\vec{B} = 0$), in some other frame it will be, in general, a mixed electric and magnetic field. Thus the general term electro-magnetic field.

Note that $\vec{B}^2 - \vec{E}^2$ and $\vec{E} \cdot \vec{B}$ are Lorentz invariants.
5.2 Radiation from relativistic charges

Multipole expansion is an expansion in $u/c$, which cannot be used for $u \approx c$. For $u/c \leq 1$, start from the exact potentials for one particle, the Lienard-Wiechert potentials:

$$\begin{pmatrix} \phi(\vec{x}, t) \\ \vec{A}(\vec{x}, t) \end{pmatrix} = \begin{bmatrix} \frac{q}{R - \vec{R} \cdot \vec{u}/c} \\ \frac{1}{u/c} \end{bmatrix} t_{\text{ret}},$$

(5.33)

where $t_{\text{ret}} = t - R(t_{\text{ret}})/c$, $\vec{R}(t_{\text{ret}}) = \vec{x} - \vec{r}(t_{\text{ret}})$, and $\vec{r}(t_{\text{ret}})$ is the position of particle at $t_{\text{ret}}$.

All is valid in both near and wave zone. Just to remind the notations:
- $\vec{k} = \vec{R}/R$ - direction to the observer;
- $\vec{\beta} = \vec{u}/c$ velocity;
- $\dot{\vec{\beta}} = \dot{\vec{u}}/c$ acceleration;
- $R$ distance to the observer;
- $\gamma = 1/(1 - \beta^2)^{1/2}$ Lorentz factor.

Lienard-Wiechert Potentials

1) if velocity $\vec{u}/c = 0$ (or $\vec{u}/c \ll 1$), then $\phi(\vec{x}, t) = q/R$, \hspace{1em} $\vec{A}(\vec{x}, t) \approx 0$. 
2) if velocity $\vec{u} \neq 0$ and a reasonable fraction of $c$, then $R - \vec{R} \cdot \vec{u}/c = R(1 - \beta \cos \theta)$ causes "beaming" in the $\vec{u}$ direction. Here $\vec{A}$ is parallel to $\vec{u}$ everywhere and is largest in forward direction due to Doppler factor.

$\vec{E}$ and $\vec{B}$ fields

As usual, $\vec{E}$ and $\vec{B}$ are obtained through

$$\vec{B} = \nabla \times \vec{A},$$

$$\vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}.$$ \hspace{1cm} (5.34)

In home exercise 3.2 we have shown that

$$\vec{E}(\vec{x}, t) = \frac{q}{(R - \vec{R} \cdot \vec{\beta})^3} \left\{ (1 - \beta^2)(\vec{R} - R\vec{\beta}) + \frac{\vec{R}}{c} \times [(\vec{R} - R\vec{\beta}) \times \vec{\beta}] \right\}_{\text{tot}},$$

$$\vec{B}(\vec{x}, t) = \frac{\vec{R}}{R} \times \vec{E}. \hspace{1cm} (5.35)$$

The first term in curved brackets goes as $\propto R/R^3 \propto R^{-2}$ which is as in Coulomb field. The second term $\propto R^2/R^3 \propto R^{-1}$ makes transverse, radiation field. $\vec{B} \perp \vec{E}$ and $\vec{B} \perp \vec{k}$ in both near and wave zone.

Expression $(R - \vec{R} \cdot \vec{\beta}) = R(1 - \vec{k} \cdot \vec{\beta}) = R(1 - \beta \cos \theta)$ contains the Doppler factor $\kappa = 1 - \beta \cos \theta$. It appears in $\vec{E}$ and $\vec{B}$ partly due to

$$\frac{\partial t}{\partial t_{\text{ret}}} = 1 - \beta \cos \theta,$$ \hspace{1cm} (5.36)

which is identical to

$$\frac{\Delta t_A}{\Delta t} = 1 - \beta \cos \theta$$ \hspace{1cm} (5.37)

discussed in the section 5.1.6 with different notations $\Delta t_A \leftrightarrow \Delta t$ and $\Delta t \leftrightarrow \Delta t_{\text{ret}}$. 

\hspace{1cm}
5.2.1 Electromagnetic field from charge with constant velocity

1) If velocity $\vec{\beta} = 0$, then one recovers the usual static Coulomb field

\[ \vec{E} = \frac{q}{R^3} \vec{R} = \frac{q}{R^2} \hat{k}, \quad \vec{B} = \hat{k} \times \vec{E} = 0, \quad \text{Coulomb field.} \quad (5.38) \]

Uniform motion, $\dot{\vec{\beta}} = 0 \Rightarrow$ only the Coulomb term left.

\[ \vec{E}(\vec{x}, t) = q \left[ \frac{(1 - \beta^2)(R - R\vec{\beta})}{(R - R \cdot \vec{\beta})^3} \right]_{\text{ret}}. \quad (5.39) \]

One can ask a question where does $\vec{E}$-vector points?

The vector $\vec{R}_t \equiv \vec{R}(t_{\text{ret}}) - (t - t_{\text{ret}})\vec{u} = \vec{R}(t_{\text{ret}}) - \frac{R(t_{\text{ret}})}{c} \vec{u}$ points towards the observer from the present position of the charge. Thus, $\vec{E} \propto (R - R\vec{\beta}) = \vec{R}_t$ points away from the charge's present position, although the field is caused by what the charge did at time $t_{\text{ret}}$!

The denominator (i.e. Doppler factor) can be written in terms of present angle $\theta_t$ and distance $R_t$:

\[ R - R \cdot \vec{\beta} = R_t(1 - \beta^2 \sin^2 \theta_t)^{1/2}. \quad (5.40) \]
5.2. **RADIATION FROM RELATIVISTIC CHARGES**

Proof: Note that the length AD is $R\beta \cos \theta$. Then DC is $R - R\beta \cos \theta$, which is just the factor we want to express in terms $\theta_t$ and $R_t$. Now by Pythagoras' theorem for triangle BCD we get

$$ (R - \vec{R} \cdot \vec{\beta})^2 = R_{t}^2 - R_{t}^2 \sin^2 \phi. \quad (5.41) $$

Considering triangle ABC, and using the fact that the sine for an angle divided by the length of the opposite side is a constant, gives $\sin \phi / R\beta = \sin \theta_t / R$, or

$$ \sin \phi = \beta \sin \theta_t. \quad (5.42) $$

Substituting $\sin \phi$ taking a square root of both sides one gets the needed equality. QED.

Now we have

$$ \vec{E}(\vec{x}, t) = \frac{q}{\gamma^2 R_t^3(1 - \beta^2 \sin^2 \theta_t)^{3/2}} \vec{R}_t, \quad (5.43) $$

$$ \vec{B}(\vec{x}, t) = \frac{\vec{R}}{R} \times \vec{E} = (\vec{\beta} + \frac{\vec{R}}{R}) \times \vec{E} = \vec{\beta} \times \vec{E}. \quad (5.44) $$

a) If $\vec{\beta} = 0$,

$$ \vec{E} = q \frac{\vec{R}_t}{R_t^3}, \quad \vec{B} = 0, \quad \text{i.e. Coulomb field.} \quad (5.45) $$

b) If $\vec{\beta} \neq 0$, $\vec{R} - R\vec{\beta}$ gives rise to a beaming effect.

c) Sudden deceleration (Bremsstrahlung): Consider a charge with a constant velocity that rapidly stops during time $dt$ at time $t = 0$. This gives rise to a spherical transverse pulse that propagates outwards with speed of light.
The thickness of the transverse layer $\text{cd}t$. Number of flux lines through a ring is constant. The area is $2\pi R\text{d}R$, the thickness $\text{d}R = \text{cd}t$. Field strength = (number of flux lines)/area = constant/ $R$, i.e. radiation! Average static field strength = (number of flux lines)/area = const/$4\pi R^2$.

d) $\vec{E}$ and $\vec{B}$ fields from relativistic particle in uniform motion;

$$\vec{E} = \frac{q(1 - \beta^2)\vec{R}_t}{R_t^3(1 - \beta^2 \sin^2 \theta_t)^{3/2}}, \quad \vec{B} = \vec{\beta} \times \vec{E}. \quad (5.46)$$

When $\beta \to 1$, we have $|\vec{B}| \approx |\vec{E}|$. Furthermore, $\vec{B} \perp \vec{E}$ always. This is similar to radiation! Consider a field at a point located at distance $b$ from the track of the charge. The charge passes the origin at $t = 0$. 
5.2. RADIATION FROM RELATIVISTIC CHARGES

Here we define \( R_t = b/\sin \theta_t = b/\cos \phi \). The field can be decomposed into a \( \parallel \) and a \( \perp \) fields:

\[
\begin{align*}
E_\perp &= \frac{q(1 - \beta^2)}{b^2(1 - \beta^2 \sin^2 \theta_t)^{3/2}} = \frac{q(1 - \beta^2) \cos^3 \phi}{b^2(1 - \beta^2 \cos^2 \phi)^{3/2}}, \quad (5.47) \\
E_\parallel &= \frac{q(1 - \beta^2) \sin^2 \theta_t \cos \theta_t}{b^2(1 - \beta^2 \sin^2 \theta_t)^{3/2}} = \frac{q(1 - \beta^2) \cos \phi \sin \phi}{b^2(1 - \beta^2 \cos^2 \phi)^{3/2}}. \quad (5.48)
\end{align*}
\]

For \( \gamma \gg 1 \), the denominator becomes \( 1 - \beta^2 \cos^2 \phi = 1 - (1 - 1/\gamma^2)(1 - \sin^2 \phi) \approx 1 + \sin^2 \phi = \frac{1 + \gamma^2 \sin^2 \phi}{\gamma^2} \). The denominator is small for \( \gamma^2 \sin^2 \phi \ll 1 \), i.e. \( \sin \phi \ll 1/\gamma \ll 1 \), i.e. \( \phi \ll 1/\gamma \) or \( \frac{u}{b} \ll 1/\gamma \), i.e. for times \( t \ll \frac{b}{\gamma u} \).

The observer sees a pulse \( \vec{E}(t) \):

\[
E_\perp \text{ has same sign and has maximum at } \phi = 0: \quad E_\perp = \frac{q(1 - \beta^2)}{b^2(1 - \beta^2)^{3/2}} = \frac{q \gamma}{b^2} \approx \gamma \times \text{static field.} \quad (5.49)
\]

\( E_\parallel \) changes sign and has smaller amplitude \( \sim q/b^2 \approx \text{static field:} \)

\[
E_\parallel \approx \frac{q}{b^2} \frac{\gamma^3 \sin \phi}{\gamma^2(1 + \gamma^2 \sin^2 \phi)^{3/2}}, \quad (5.50)
\]

and maximum \( \approx \frac{q}{b^2} \) occurs at \( \phi \approx \frac{1}{\gamma} \). The field lines are thus concentrated within an angle \( 1/\gamma \) relative to the transverse direction. The observer thus sees a transverse field with \( |E| \approx |B| \) and \( \vec{E} \perp \vec{B} \).

This Coulomb field can be Fourier-decomposed and be considered as a field consisting of virtual photons. (This is used in semi-classical calculations, e.g. Jackson ch 15.4, Weizsäcker-Williams method). When charge is accelerated, one can consider the emitted photons to be virtual photons that have been shaken off.
5.2.2 Electromagnetic field from accelerated charge in the wave zone, i.e. radiation field

In the wave zone $\vec{R} \approx \vec{k}x$ and $R = x$, where $x$ is mean distance to charge. Acceleration field (radiation field) is given by

$$
\vec{E}(\vec{x}, t) \approx \frac{q}{(R - \vec{R} \cdot \vec{\beta})^3} \left\{ \frac{\vec{R}}{c} \times [(\vec{R} - R\vec{\beta}) \times \vec{\beta}] \right\}_{\text{ret}} \\
\vec{B}(\vec{x}, t) = \frac{\dot{\vec{R}}}{R} \times \vec{E} = \vec{k} \times \vec{E},
$$

(5.51)

where $\kappa = 1 - \vec{k} \cdot \vec{\beta}$. We see that $\vec{E} \propto \vec{k} \times [..] \Rightarrow \vec{E} \perp \vec{k}$. Also $\vec{B} \perp \vec{k}$ and $|\vec{E}| = |\vec{B}| \propto \frac{1}{x}$, i.e. radiation.

Let us consider a few special cases.

1) Non-relativistic motion $\beta \ll 1$, $\kappa \approx 1$:

$$
\vec{B} = \vec{k} \times \vec{E} = \frac{q}{c^2 x} [\vec{k} \times (\vec{k} \times \vec{u})] = \frac{q}{c^2 x} [\vec{k} \times (\vec{k} \cdot \vec{u}) - \vec{u}(\vec{k} \cdot \vec{k})] \\
= - \frac{q}{c^2 x} (\vec{u} \times \vec{k}) = \frac{\vec{d} \times \vec{k}}{c^2 x},
$$

(5.52)

i.e. the classical Larmour formula for "accelerating" displacement. Radiation pattern $\frac{dP}{d\Omega} = \frac{c}{4\pi} (Bx)^2 = \frac{\vec{d}^2}{4\pi} \sin^2 \theta$.

2) Relativistic motion $\beta \to 1$, $\gamma \gg 1$:

Note two things.

a) The Doppler factor $\kappa = 1 - \beta \cos \theta$ can be very small when $\beta \sim 1$ and for certain angles $\theta \sim 0$. Then $1/\kappa$ is very large. For $\gamma \gg 1$, $\beta = 1 - \frac{1}{2\gamma^2}$ and

$$
\kappa = 1 - \beta \cos \theta \approx 1 - (1 - \frac{1}{2\gamma^2})(1 - \frac{\theta^2}{2}) \approx \frac{1 + \gamma^2 \theta^2}{2\gamma^2}
$$

(5.53)

and

$$
\frac{1}{\kappa} = \frac{2\gamma^2}{1 + \gamma^2 \theta^2}.
$$

(5.54)
This is large when $\theta \ll 1/\gamma \ll 1$. Due to the $1/\kappa^3$ factor the radiation field is beamed, i.e. concentrated towards the $\theta = 0$ direction.

b) For observers with $\vec{k}$, such that $(\vec{k} - \vec{\beta}) \parallel \dot{\vec{u}}$, $\vec{E} = 0$, i.e. no radiation. Note that $(\vec{k} - \vec{\beta}) \parallel (\vec{R} - R\vec{\beta}) = \vec{R}_r$, thus $\vec{E} = 0$ if $\dot{\vec{u}}(t_{ret})$ parallel to the direction from the observer to the location where the charge would have been at time $t$ if it had been in uniform motion.

3) The general case:

Angular distribution of radiated power. As before, the radiated power passing on area $x^2 d\Omega$ in direction $\vec{k}$, becomes

$$dP = \frac{c}{4\pi} |\vec{E} \times \vec{B}| x^2 d\Omega = \frac{cB^2}{4\pi} (x^2 d\Omega),$$

(5.55)
i.e. the received power per unit solid angle becomes

$$\frac{dP}{d\Omega_{\text{received}}} = \frac{c(xE)^2}{4\pi} = \frac{q^2}{4\pi c^3} \left\{ \frac{\vec{k} \times [(\vec{k} - \vec{\beta}) \times \dot{\vec{u}}]}{\kappa^3} \right\}^2.$$  

(5.56)

Denote the expression in $\{}$ as $|\vec{g}|$. Then

$$\vec{g} = \frac{1}{\kappa^3} [(\vec{k} \cdot \dot{\vec{u}})(\vec{k} - \vec{\beta}) - \vec{k} \cdot (\vec{k} - \vec{\beta}) \dot{\vec{u}}].$$

(5.57)

Then

$$g^2 = \frac{1}{\kappa^6} [(\vec{k} \cdot \dot{\vec{u}})^2 (\vec{k} - \vec{\beta})^2 + \kappa^2 |\dot{\vec{u}}|^2 - 2\kappa (\vec{k} \cdot \dot{\vec{u}}) (\vec{k} - \vec{\beta}) \cdot \dot{\vec{u}}]$$

$$= \frac{1}{\kappa^6} [(\vec{k} \cdot \dot{\vec{u}})^2 (1 + \beta^2 - 2\vec{k} \cdot \vec{\beta}) + \kappa^2 |\dot{\vec{u}}|^2 - 2\kappa [(\vec{k} \cdot \dot{\vec{u}})^2 - (\vec{k} \cdot \dot{\vec{u}})(\vec{\beta} \cdot \dot{\vec{u}})]$$

(5.58)

$$= \frac{1}{\kappa^4} |\dot{\vec{u}}|^2 + \frac{2}{\kappa^5} (\vec{k} \cdot \dot{\vec{u}})(\vec{\beta} \cdot \dot{\vec{u}}) - \frac{1}{\kappa^6} (\vec{k} \cdot \dot{\vec{u}})^2 (1 - \beta^2),$$

where we used $\kappa = 1 - \vec{k} \cdot \vec{\beta}$.

Define a coordinate system: $\vec{u} = (0, 0, u), \vec{\beta} = |\dot{\vec{u}}|(\sin i, 0, \cos i), \vec{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$
Then $\mathbf{k} \cdot \dot{\mathbf{u}} = |\dot{\mathbf{u}}| (\sin \theta \cos \phi \sin i + \cos \theta \cos i)$, and $\dot{\mathbf{u}} \cdot \dot{\beta} = |\dot{\mathbf{u}}| \beta \cos i$.

Special case $\dot{\mathbf{u}} \parallel \dot{\mathbf{u}}$, i.e. $i = 0$, then $\mathbf{k} \cdot \dot{\mathbf{u}} = |\dot{\mathbf{u}}| \cos \theta$ and $\dot{\mathbf{u}} \cdot \dot{\beta} = |\dot{\mathbf{u}}| \beta$. Then (show this!)

$$g^2 = |\dot{\mathbf{u}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^6}. \quad (5.59)$$

Then the received power is

$$\frac{dP}{d\Omega_{\text{received}}} = \frac{q^2}{4\pi c^3} g^2 \approx \frac{16q^2 |\dot{\mathbf{u}}|^2}{\pi c^3} \gamma^{10} \frac{\gamma^2 \theta^2}{(1 + \gamma^2 \theta^2)^6}. \quad (5.60)$$

Radiation pattern (angular distribution of $dP/d\Omega$). If $\beta \sim 1$, $\gamma \gg 1$, and $\mathbf{d} \parallel \dot{\mathbf{u}}$, the torus becomes very elongated. Maximum at $\theta \sim 1/\gamma$, no radiation at $\theta = 0$. It is simply the non-relativistic torus (applicable in the instantaneous rest frame of the charge) that has been Lorentz transformed (see Rybicki & Lightman pp. 140-143 for details). The relativistic result can be obtained by Lorentz transforming $dP/d\Omega$, $\sin \theta$, and $|\dot{\mathbf{u}}|$.
5.2. RADIATION FROM RELATIVISTIC CHARGES

Special case \( \hat{u} \bot \hat{u}, \) i.e. \( i = \pi / 2, \) then \( \vec{k} \cdot \hat{u} = |\hat{u}| \sin \theta \cos \phi \) and \( \hat{u} \cdot \vec{\beta} = 0. \) Then

\[
\frac{g^2}{|\hat{u}|^2} = \frac{1}{\kappa^4} - \frac{(1 - \beta^2)}{\kappa^6} \sin^2 \theta \cos^2 \phi.
\] (5.61)

As previously discussed, the radiation observer’s time interval, \( dt \) (denoted \( \Delta t_A \) previous lecture), is not equal to the particle observer’s time interval \( dt_{\text{ret}} \) (denoted \( \Delta t \) before). We have \( \frac{dt}{dt_{\text{ret}}} = 1 - \beta \cos \theta = \kappa. \)

The emitted power per unit solid angle

\[
\frac{dP}{d\Omega_{\text{emitted}}} = \frac{dW}{dt_{\text{ret}} d\Omega} = \left( \frac{dt}{dt_{\text{ret}}} \right) \frac{dW}{d\Omega} = \left( \frac{dt}{dt_{\text{ret}}} \right) \frac{dP}{d\Omega_{\text{received}}} = \kappa \frac{dP}{d\Omega_{\text{received}}},
\] (5.62)

i.e. emitted power is not equal to the received power as the same amount of energy \( dW \) is emitted and received during different time intervals.

**Radiated power**

Integrate \( dP/d\Omega \) over \( d\Omega. \) One must choose if it the received or emitted power that is of interest. To compute local energy losses in the gas, requires a knowledge of the emitted power.

\[
\frac{dP}{d\Omega_{\text{emitted}}} = \kappa \frac{dP}{d\Omega_{\text{received}}} = \frac{q^2 \kappa g^2}{4\pi c^3},
\] (5.63)

\[
P_{\text{emitted}} = \frac{q^2}{4\pi c^3} \int \kappa g^2 d\Omega = \frac{2e^2}{3c^3} \gamma^6 [||\hat{u}||^2 - (\hat{u} \times \vec{\beta})^2] = \frac{2e^2}{3c^3} \gamma^6 ||\hat{u}||^2 (1 - \beta^2 \sin^2 i).\] (5.64)

**Parallel acceleration** \( a_\parallel \equiv \frac{\hat{u}_\parallel}{i}, i = 0, \)

\[
P_{\text{emitted,}} = \frac{2e^2}{3c^3} \gamma^6 a_\parallel^2.
\] (5.65)
Perpendicular acceleration $a_\perp \equiv \hat{a}_\perp$, $i = \pi/2$,

$$P_{\text{emitted,}\perp} = \frac{2e^2}{3c^2} \gamma^6 (1 - \beta^2) a_\perp^2 = \frac{2e^2}{3c^2} \gamma^4 a_\perp^2. \tag{5.66}$$

In a general case, $\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$:

$$P_{\text{emitted}} = \frac{2e^2}{3c^2} \gamma^4 (a_\perp^2 + \gamma^2 a_\parallel^2). \tag{5.67}$$

This is relativistic Larmor formula. For a given acceleration, a relativistic particle radiates a factor $\gamma^4$ or $\gamma^6$ more than a non-relativistic.

### 5.2.3 Lorentz invariance of the radiated power $P$

The expression for $P_{\text{emitted}}$ can be derived very elegantly using Lorentz invariance of $P$. Lienard who derived $P_{\text{emitted}}$ in 1898 did not have access to special relativity.

$K'$ is the instantaneous rest frame. During a short moment electron is at rest in this system, and in a short time interval before and after the velocity of electron is non-relativistic in $K'$.

For a non-relativistic charge one can use Larmor formula (dipole radiation). Consider the energy $dW'$ that is radiated during $dt'$. Corresponding total momentum change is $dp' = 0$ due to the symmetry of the torus.

Now Lorentz transform to $K$ (energy transforms as time):

$$dW = \gamma(dW' + \beta cp') = \gamma dW', \tag{5.68}$$

since $dp' = 0$ and the time interval

$$dt = \gamma(dt' + \beta dx'/c) = \gamma dt', \tag{5.69}$$

since $dx' = 0$ in $K'$. The power

$$P = \frac{dW}{dt} = \frac{\gamma dW'}{\gamma dt'} = P', \tag{5.70}$$

i.e. total power is Lorentz invariant for processes with symmetry in the rest frame.

We have

$$P' = \frac{2q^2}{3c^2} |a'|^2 = \frac{2q^2}{3c^2} (a^2_\perp + a^2_\parallel) \tag{5.71}$$

in the instantaneous rest frame. In $K$ we have

$$P = \frac{2q^2}{3c^2} \gamma^4 (a^2_\perp + \gamma^2 a^2_\parallel). \tag{5.72}$$

Since $P = P'$, the acceleration must Lorentz transform as $a'_\perp = \gamma^2 a_\perp$ and $a'_\parallel = \gamma^3 a_\parallel$. 