RADIATIVE PROCESSES in ASTROPHYSICS


5.1—Compute the integral in equation (5.64) and prove the relation.

5.2—In this exercise we will try to obtain a more accurate expression for the bremsstrahlung emissivity. Consider a Cartesian coordinate system (with unit vectors along the axes $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) with the electron moving along the $z$-axis with velocity $\mathbf{u} = u(0, 0, 1)$. Assume the observer is in the direction $\mathbf{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, i.e. the angle between $\mathbf{k}$ and $\mathbf{u}$ is $\theta$. The heavy ion is at position $\mathbf{b} = b(1, 0, 0)$, where $b$ is the impact parameter. We assume that deviation of the electron trajectory from the straight line are small.

1) Compute the Coulomb force $\mathbf{F}$ (which is a vector) acting on the electron as a function of time $t$ (with $t = 0$ corresponding to the electron passing the ion at the closest distance). This force has two components along $x$- and $z$-axes. Compute corresponding acceleration $\ddot{\mathbf{r}}$ of the electron.

2) Compute the electric field at the position of the observer at distance $r$ from the charges as a function of time

$$\mathbf{E}(t) = \frac{1}{c^2 r} \left[ (\ddot{d} \times \mathbf{k}) \times \mathbf{k} \right]$$

where $\ddot{d} = e\ddot{r}$. Show that the result can be represented as a sum of two terms

$$\mathbf{E}(t) = \mathbf{E}_1(t) + \mathbf{E}_2(t),$$

with

$$\mathbf{E}_1(t) = \frac{Ze^3}{m_e c^2 r \sqrt{b^2 + (ut)^2}} \left( \sin \theta \cos \phi \mathbf{k} - \mathbf{e}_x \right),$$

$$\mathbf{E}_2(t) = \frac{Ze^3}{m_e c^2 r \sqrt{b^2 + (ut)^2}} \left( \sin \theta \cos \phi \mathbf{e}_x - \cos \theta \sin \phi \mathbf{e}_y + \sin \theta \mathbf{e}_z \right).$$

3) Compute the Fourier transform of both terms:

$$\hat{\mathbf{E}}_{1,2}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \mathbf{E}_{1,2}(t) dt.$$  

Hint:

$$\int_0^{\infty} \cos ax \frac{dx}{\sqrt{b^2 + x^2}} = \frac{a}{b} K_1(ab), \quad \int_0^{\infty} \frac{x \sin ax}{\sqrt{b^2 + x^2}} \frac{dx}{\sqrt{b^2 + x^2}} = aK_0(ab),$$

where $K_n(x)$ are the modified Bessel functions

$$K_n(x) = \int_0^{\infty} e^{-x} \cosh(nt) \, dt.$$  

4) Compute the square of the Fourier transform:

$$|\hat{\mathbf{E}}(\omega)|^2 = |\hat{\mathbf{E}}_1(\omega) + \hat{\mathbf{E}}_2(\omega)|^2$$
to obtain

$$|\vec{E}(\omega)|^2 = \frac{Z^2e^6}{m_e^2c^4r^2} \frac{1}{\pi^2} (\frac{\omega b}{u}) \left[ (1 - \sin^2 \theta \cos^2 \phi)K_1^2(\omega b/u) + \sin^2 \theta K_0^2(\omega b/u) \right]. \quad (9)$$

Make sure that the cross-term \( \propto K_0K_1 \) disappears.

5) Integrate the previous expression over the surface of the sphere of radius \( r \) (with \( \Omega \) being the solid angle),

$$\frac{dW}{d\omega} = P_\omega = c \int |\vec{E}(\omega)|^2 r^2 d\Omega, \quad (10)$$
to get the the total energy (per units frequency) radiated in all directions passing through that sphere

$$P_\omega = \frac{8}{3\pi} \frac{Z^2e^6}{m_e^2c^3} \frac{1}{(bu)^2} \left( \frac{\omega b}{u} \right)^2 \left[ K_1^2(\omega b/u) + K_0^2(\omega b/u) \right]. \quad (11)$$

6) Compare this expression to that derived in class (equation 6.10). Remember that \( P_\nu = 2\pi P_\omega \).

Using the asymptotic expansions of \( K_n(x) \) for small and large \( x \):

$$K_0(x) \sim K_1(x) \sim \sqrt{\frac{\pi}{2x}} \exp(-x), \quad x \gg 1, \quad (12)$$

$$K_0(x) \sim -\ln x, \quad K_1(x) \sim 1/x, \quad x \ll 1, \quad (13)$$

compute the limiting expression for \( P_\omega \) for \( \omega \ll u/b \) and \( \omega \gg u/b \). How much wrong we were in our derivation of the power in class?

5.3–Suppose X-rays are received from a source of known distance \( L \) with a flux \( F \) (erg s\(^{-1}\) cm\(^{-2}\)). The X-ray spectrum has the form as sketched in the figure below.

![X-ray spectrum graph](image)

It is proposed that these X-rays are due to bremsstrahlung from an optically thin, hot plasma cloud, which is in hydrostatic equilibrium around a central mass \( M \). This means that the pressure must balance the gravity, or \( 3kT \sim GmM/R \), where \( m \) is the typical mass of the gas particles. Assume that the cloud thickness \( \Delta R \) is roughly its radius, \( \Delta R \sim R \). Find \( R \) and the density of the cloud \( \rho \) in terms of the known observations and the conjectured mass \( M \).

a. If \( F = 10 \text{ erg s}^{-1} \text{ cm}^{-2} \), \( L = 10 \text{ kpc} \), what are the constraints on \( M \) such that the source would
indeed be effectively optically thin (for self-consistency)?
b. Does electron scattering play any role?

5.4—An ultrarelativistic, $\gamma \gg 1$, electron emits synchrotron radiation. Solve Eq. (7.12) and show that its energy decreases with time according to

$$\gamma = \gamma_0 (1 + A \gamma_0 t)^{-1}, \quad A = \frac{2e^4 B_\perp^2}{3m^3 c^5}. $$

Here $\gamma_0$ is the initial value of $\gamma$, $B_\perp = B \sin \alpha$, and $\alpha$ is the pitch angle. Show that the time for the electron to lose half its energy is

$$t_{1/2} = (A \gamma_0)^{-1} = \frac{5.1 \times 10^8}{\gamma_0 B_\perp^2}. $$

How does one reconcile the decrease of $\gamma$ here with the result of constant $\gamma$ implied by Eq. (7.1)?

5.5—Prove that the charged particle trajectory in the homogeneous magnetic field is a helical curve given by Eq. (7.6).

5.6—The Radio Lobes of Cygnus A. Read the section Application to Radio Galaxies in Shu, p. 179-181. The observational data consist of (i) a radio map at 6 cm of Cyg A (Fig.18.5 in Shu), and (ii) the radio spectrum of the lobes of Cyg A (shown below).

Hubble’s law is given by $v = cz = H_0 d$ km/s, where $v$ is the expansion velocity, $H_0 = 50$ km/s/Mpc, $d$ is the distance, and $z = \lambda_{\text{obs}} / \lambda_{\text{lab}} - 1$ is the redshift. The redshift was determined to be $z = 0.0566$ from the optical spectrum.

a) Determine the distance to Cyg A.

b) Make an estimate of the size of the radio lobes. Assume spherical lobes. Calculate the volume.

c) Assume that the number density of electrons has the energy distribution $N(\gamma) = n_0 \gamma^{-p}$ cm$^{-3}$. Calculate an expression for the volume emissivity, $j_\nu$, from the $N(\gamma)$ electrons, under the assumption that one single electron radiates $4\pi j_\nu = \langle P_{\text{em}} \rangle \delta(\nu - \gamma^2 \nu_L)$ erg/s/Hz. Determine $p$, $\nu_{\text{min}}$, and $\nu_{\text{max}}$ using the observed radio spectrum.

d) Calculate $n_0$, $\gamma_{\text{min}}$, $B$ using the following three relations:

1) Assume that the radio lobes contains the minimum possible energy, i.e. assume equipartition. Use Eq. (18.14). Note the sign error in Eq. (18.13).

2) Relate $\gamma_{\text{min}}$ to $\nu_{\text{min}}$ and $B$.

3) Determine using $j_\nu$ and the volume $V$, an expression for the total monochromatic radio luminosity, $L_\nu$ (see Eq. 18.12), as a function of $n_0$, $B$, and $\nu_{\text{min}}$ (or $\gamma_{\text{min}}$).

e) Calculate the total energy in the radio lobes.

f) The center of the galaxy (i.e. the central black hole in Cyg A) radiates $\sim 10^{44}$ erg/s from radio to gamma ray wavelengths. Assume that the black hole feeds the same power into the radio lobes through the jets. How long time has it taken to fill the lobes with the energy that we
deduce to be contained in magnetic fields and electrons? Assume that the lobes have not lost much energy through radiation.

g) Calculate the cooling time (i.e. the radiative lifetime) for electrons at $\gamma_{\text{min}}$? At $\gamma_{\text{max}}$? Compare with the results from f). Discuss the implications.

![Radio spectrum of Cygnus A radio galaxy. $1 \text{ Jy} = 10^{-23} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$.](image_url)