Let X_1, ..., X_n be a random sample from a p-variate distribution and assume that the multivariate observations are generated by the location-scatter model.

The full-rank p × p matrix Λ is called the mixing matrix, and parameter μ is the location center. We wish to test the hypotheses

H_0: μ = 0 vs. H_1: μ ≠ 0

Naturally, the test statistic one should use depends on the assumptions on the distribution of the standard variable Z_i

These are possible assumptions for Z_i:

- Normal Model
  - X_i ~ N(μ, Λ)
- Elliptic Model
  - X_i is spherical distributed around the origin with Med(X_i) = Λ^{-1}μ
- Symmetric IC Model
  - X_i is symmetric around the origin (Z_i ~ Z_i)

For most of the different models, the literature offers a wide variety of possible tests. To name only a few tests which are all asymptotically valid (after some possible adjustments):

- Hotelling’s T^2, optimal in the normal model (requires however always finite 2nd moments).
- Signed-rank test of Hettmansperger et al. based on Oja signs and ranks, affine invariant.
- One sample signed-rank tests in the symmetric IC model

The test we are proposing has the following idea:

1. Recover the underlying components:
   - Easy when Λ is known, Z_i(Λ) = Λ^{-1}X_i. But if unknown any estimate ̂Λ that is p×p consistent and not affected under individual sign changes of observations can be used instead.

2. Compute for each component the marginal signs and ranks:
   - These will be denoted by S_i(Λ) = (S_{1i}(Λ), ..., S_{pi}(Λ)) and R_i(Λ) = (R_{1i}(Λ), ..., R_{pi}(Λ)), where R_{pi}(Λ) denotes the marginal rank of |Z_{pi}| among |Z_{pi}|, ..., |Z_{pi}|
   - Choose an appropriate score function for each component:
     - The score vector is defined as a vector of (2 + 1) rescaled score functions (β > 0), K(u) = (K_1(u), ..., K_p(u))
     - Combine the marginal scores to form a test statistic:

The proposed test statistic is given by:

Q_{A}(Λ) = (T_{A}(Λ))^T K^T Λ^T T_{A}(Λ),

where

- T_{A}(Λ) = n^{-1/2} \sum_{i=1}^{n} T_{i}(Λ) = n^{-1/2} \sum_{i=1}^{n} S_i(Λ) \otimes K(u)
- (K(u) = diag(K_1(u), ..., K_p(u)) is under Λ, \tilde{M}(Z_i(Λ)) = \tilde{M}(Z_i(Λ)) among Λ, K(u) = \tilde{M}(Z_i(Λ))
- under H_0 Q_{A}(Λ) is asymptotically chi-square distributed with p degrees of freedom.

If an estimate ̂Λ is used, the statistics will be denoted correspondingly as Q_{A}(Λ)

Important particular cases are:

- Sign test:
  - Q_{A} = T_{A}^T ̂T_{A} / n^{-1} \sum_{i=1}^{n} S_i(Λ) \otimes K(u)
- Wilcoxon type test:
  - Q_{A} = T_{A}^T ̂T_{A} / n^{-1} \sum_{i=1}^{n} S_i(Λ) \otimes K(u)
- Van der Waerden type test:
  - Q_{A} = T_{A}^T ̂T_{A} / n^{-1} \sum_{i=1}^{n} S_i(Λ) \otimes K(u)

where \Phi^{-1}(u) = φ^{-1}(u(1 + 1/2))

The test has the following properties:

- It is affine invariant, given Λ is affine equivariant.
- In the case of H_0: μ = μ_0 against H_1(μ) = μ_0 + n^{-1/2}ε, the efficiencies of our tests compared to Hotelling’s T^2 are weighted univariate efficiencies of the univariate scores compared to Student’s t-test, where the weights depend on the shift μ through the “standardized” shift 1/√n - 1/√p.

Some selected univariate efficiencies of signed-rank test compared to Student’s t-test:

<table>
<thead>
<tr>
<th>μ</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptotic</td>
<td>1</td>
<td>0.888</td>
<td>0.883</td>
<td>0.881</td>
<td>0.880</td>
</tr>
<tr>
<td>Efficiency</td>
<td>1</td>
<td>0.886</td>
<td>0.882</td>
<td>0.880</td>
<td>0.879</td>
</tr>
</tbody>
</table>

The non-parametric signed-rank test we suggest here is affine invariant, robust and optimal in the symmetric IC model and does not depend on the choice of an estimate for the mixing matrix Λ. Furthermore, this test is also still asymptotically valid in the case since given such a rotation the p components still converge in distribution to Z because (possibly redefined) uncorrelated Gaussian variables with a common scale are independent. The 3rd Figure on the left shows simulation results when setting in the previous setting all components to N(0,1).