Decomposition properties of John domains and uniform domains in real normed vector spaces

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1 Notations and preliminaries

2 References
Uniform domain and John domain

A domain $D$ in Banach spaces $E$ is called $c$-uniform if there exists a constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\alpha$ in $D$ satisfying

\[
\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c \, d_D(z) \quad \text{for all } z \in \alpha, \text{ and}
\]
\[
\ell(\alpha) \leq c \, |z_1 - z_2|.
\]

$D$ is called $c$-John domain if (1) in the definition of $c$-uniformity is satisfied, not necessarily (2).

Example

Let $D = \mathbb{B}(0, 1) \setminus (0, 1)$, then $D$ is a John domain but not a uniform domain.
**Remark**

We call a domain $D$ a *uniform domain* (resp. *John domain*) if it is a $c$-uniform (resp. $c$-John) domain for some positive constant $c$. A simply connected $c$-uniform domain (resp. $c$-John domain) $D \subset \mathbb{R}^2$ with at least two boundary points will be called a $c$-quasidisk (resp. $c$-John disk).
A homeomorphism $f$ is $K$-Qc if for any curve family $\Gamma$:

$$\frac{M(\Gamma)}{K} \leq M(f(\Gamma)) \leq K M(\Gamma),$$

where

$$M(\Gamma) = \inf_{\rho \in A(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm$$

and

$$A(\Gamma) = \{\rho \in Bor(\mathbb{R}^n) : \rho > 0 \text{ and } \int_{\gamma} \rho ds > 1 \text{ for all } \gamma \in \Gamma\}.$$
Quasiconformally decomposition in $\mathbb{R}^2$

A domain $D$ in $\mathbb{R}^2$ is said to be quasiconformally decomposable if there exists a constant $K$ with the following property: for each pair $x_1, x_2 \in D$, there exists a subdomain $D_0$ of $D$ such that $x_1, x_2 \in \overline{D_0}$ and $\partial D_0$ is a $K$-quasiconformal circle, i.e., the image of the unit circle under a $K$-quasiconformal mapping of $\overline{\mathbb{R}^2}$ onto itself, where $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$.

Theorem (Gehring and Osgood, 1979)

A domain $D \subset \mathbb{R}^2$ is a uniform domain if and only if it is quasiconformally decomposable.
**Uniform Domain Decomposition Property**

A domain $D \subset \mathbb{R}^n$ is said to have the *uniform domain decomposition property* if there exists a positive constant $c$ with the following property: For each pair of points $z_1, z_2$ in $D$, there exists a subdomain $D_0$ of $D$ such that $z_1, z_2 \in D_0$ and $D_0$ is a simply connected $c$-uniform domain.

**Theorem (G. Martin, 1985)**

Let $D$ be a uniform domain in $\mathbb{R}^n$. Then there is a constant $L$, depending only on the constant of uniformity for $D$, such that for each pair of points $x_1, x_2$ in $D$ there is an $L$-bi-Lipschitz embedding $f : \mathbb{B}^n(0, |x_1 - x_2|) \hookrightarrow D$ with $\{x_1, x_2\} \subset f(\mathbb{B}^n(0, |x_1 - x_2|))$. 
A domain in $\mathbb{R}^n$ is $b$-uniform if and only if it has the $c$-uniform domain decomposition property, where the constant $c$ depends on $b$ and $n$ and the constant $b$ depends on $c$. 
JOHN DOMAIN DECOMPOSITION PROPERTY

A domain $D \subset \mathbb{R}^n$ is said to have the *John domain decomposition property* if there exists a positive constant $c$ with the following property: for each pair $z_1, z_2 \in D$, there exists a subdomain $D_0$ of $D$ such that $z_1, z_2 \in \overline{D_0}$ and $D_0$ is a simply connected $c$-John domain.

THEOREM (M. HUANG, S. PONNUSAMY AND X. WANG, 2008)

A domain in $\mathbb{R}^n$ is a $b$-John domain if and only if it has the $c$-John domain decomposition property, where the constant $c$ depends on $b$ and $n$ and the constant $b$ depends on $c$. 
We have the generalization of above results in real normed vector spaces $E$.

**Theorem (M. Huang and X. Wang)**

A domain in $E$ is $b$-uniform if and only if it has the $c$-uniform domain decomposition property, where the constants $b$ and $c$ depend only on each other.

**Theorem (M. Huang and Y. Li)**

A domain in $E$ is a $b$-John domain if and only if it has the $c$-John domain decomposition property, where the constants $b$ and $c$ depend only on each other.


REFERENCES II


M. Huang and X. Wang, Decomposition properties of Uniform domains in real normed vector spaces, (submitted)

M. Huang and Y. Li, Decomposition properties of John domains in real normed vector spaces, (submitted)
Let $\psi : [0, \infty) \to [0, \infty]$ be a homeomorphism. A domain $D$ in $E$ is called QH $\psi$-uniform if

$$k_D(z_1, z_2) \leq \psi\left(\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\right)$$

for all $z_1, z_2 \in D$, where $d_D(z_1)$ denotes the distance from $z_1$ to the boundary $\partial D$ of $D$.

Are the results above true for $\psi$-uniform domain?
THANK YOU