BESSEL POTENTIAL SPACES WITH VARIABLE EXPONENT

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Abstract. We show that a variable exponent Bessel potential space coincides with the variable exponent Sobolev space if the Hardy-Littlewood maximal operator is bounded on the underlying variable exponent Lebesgue space. Moreover, we study the Hölder type quasi-continuity of Bessel potentials of the first order.

1. Introduction

The (classical) Bessel potential space $\mathcal{L}^\alpha,p(\mathbb{R}^n)$, $1 < p < \infty$, consists of all functions $u$, $u = g_\alpha * f$, where $f \in L^p(\mathbb{R}^n)$. Here $g_\alpha$ is the Bessel kernel of the order $\alpha \geq 0$. It is well know that when $\alpha$ is a natural number the space $\mathcal{L}^\alpha,p(\mathbb{R}^n)$ (with the norm of $u$ defined as $\|f\|_{L^p(\mathbb{R}^n)}$) coincides with the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ and the corresponding norms are equivalent. The aim of this paper is to study this question in variable exponent case, that is, when the exponent $p$ is a measurable function $p: \mathbb{R}^n \to [p_*, p^*]$, $1 < p_* < p^* < \infty$.

If the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined (see the next section), the variable exponent Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $u \in L^{p(\cdot)}(\mathbb{R}^n)$ whose distributional derivatives up to the order $k$ belong to $L^{p(\cdot)}(\mathbb{R}^n)$. These spaces have attracted steadily increasing interest over the past five years. The research was motivated by the differential equations with non-standard growth and coercivity conditions arising from modeling certain fluids called electrorheological (cf. [21]).

We define the variable exponent Bessel potential space $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ as in the classical situation. Assuming that the Hardy-Littlewood maximal operator $M$ is bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ we show that the variable exponent Bessel potential space $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)$ and the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$, $k \in \mathbb{N}$, coincide and their norms are equivalent.

As an application we study the Hölder type quasi-continuity of Bessel potentials of the first order. More precisely, we show that each function $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ coincides pointwise outside a small set (measured by the Bessel capacity) with a Hölder continuous function $w \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and the norm of the difference $u - w$ in $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ is small.

2. Variable exponent spaces

Let $G$ be a measurable subset of $\mathbb{R}^n$ (with respect to $n$-dimensional Lebesgue measure), by $|G|$ we mean its $n$-volume and $\chi_G$ will represent the characteristic function of $G$. For $r \in (0, \infty)$ and $x \in \mathbb{R}^n$ let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ of radius $r$ and center $x$.

By the symbol $\mathcal{P}(\mathbb{R}^n)$ we denote the family of all measurable functions $p(\cdot): \mathbb{R}^n \to [1, \infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ put

$$p_* := \text{ess inf}_{x \in \mathbb{R}^n} p(x), \quad p^* := \text{ess sup}_{x \in \mathbb{R}^n} p(x).$$

Furthermore, we introduce a class $\mathcal{B}(\mathbb{R}^n)$ by

$$\mathcal{B}(\mathbb{R}^n) := \{p \in \mathcal{P}(\mathbb{R}^n); 1 < p_* \leq p^* < \infty\}.$$
Let \( p(\cdot) \in B(\mathbb{R}^n) \). Consider the functional
\[
\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx
\]
on all measurable functions \( f \) on \( \mathbb{R}^n \). The **Lebesgue space with variable exponent** \( L^{p(\cdot)}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that, for some \( \lambda > 0 \),
\[
\varrho_{p(\cdot)}(f/\lambda) < \infty,
\]
equipped with the norm
\[
\|f\|_{p(\cdot)} = \inf \{ \lambda > 0; \varrho_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]
Recall that (cf. \([14, (2.9)](2.10)\))
\[
\varrho_{p(\cdot)}(f/\|f\|_{p(\cdot)}) = 1 \quad \text{for every} \quad f \quad \text{with} \quad 0 < \|f\|_{p(\cdot)} < \infty, \tag{2.1}
\]
\[
\text{if} \quad \|f\|_{p(\cdot)} \leq 1 \quad \text{then} \quad \varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}, \tag{2.2}
\]
\[
\varrho_{p(\cdot)}(f) \leq 1 \quad \text{if and only if} \quad \|f\|_{\varrho(\cdot)} \leq 1, \tag{2.3}
\]
\[
\varrho_{p(\cdot)}(f) \to 0 \quad \text{if and only if} \quad \|f\|_{p(\cdot)} \to 0. \tag{2.4}
\]
The **Hardy-Littlewood maximal operator** \( M \) is defined on locally integrable functions \( f \) on \( \mathbb{R}^n \) by the formula
\[
Mf(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy. \tag{2.6}
\]

**Definition 2.1.** By \( \mathcal{M}(\mathbb{R}^n) \) denote the class of all functions \( p \in B(\mathbb{R}^n) \) for which the operator \( M \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \), that is,
\[
\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)} \tag{2.7}
\]
equipped with a positive constant \( C \) independent of \( f \).

**Remark 2.2.** For example, \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \) if the following two conditions are satisfied:
\[
|p(x) - p(y)| \leq \frac{c}{-\log(|x - y|)}, \quad |x - y| \leq 1/2, \tag{2.8}
\]
\[
|p(x) - p(y)| \leq \frac{c}{\log(e + |x|)}, \quad |y| > |x|. \tag{2.9}
\]
For more details see \([2, 3, 5, 15, 17, 18]\) where various sufficient conditions for \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \) can be found.

Let \( p(\cdot) \in B(\mathbb{R}^n) \) and \( k \in \mathbb{N} \). We define the **Sobolev space with variable exponent** \( W^{k,p(\cdot)}(\mathbb{R}^n) \) by
\[
W^{k,p(\cdot)}(\mathbb{R}^n) := \{ u; \; D^\beta u \in L^{p(\cdot)}(\mathbb{R}^n) \quad \text{if} \quad |\beta| \leq k \},
\]
equipped with the norm
\[
\|u\|_{W^{k,p(\cdot)}} = \sum_{|\beta| \leq k} \|D^\beta u\|_{p(\cdot)},
\]
where \( \beta \in \mathbb{N}_0^n \) is a multi-index, \( |\beta| = \beta_1 + \cdots + \beta_n \) and \( D^\beta = \frac{\partial^{|eta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \).

The **Bessel kernel** \( g_\alpha \) of order \( \alpha, \alpha > 0 \), is defined by
\[
g_\alpha(x) = \frac{\pi^{n/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-s - (\pi^2|x|^2)/s} s^{(\alpha - n)/2} \, ds, \quad x \in \mathbb{R}^n. \tag{2.10}
\]

Let \( p(\cdot) \in B(\mathbb{R}^n) \) and \( \alpha \geq 0 \). The **Bessel potential space with variable exponent** \( L^{\alpha,p(\cdot)}(\mathbb{R}^n) \) is, for \( \alpha > 0 \), defined by
\[
L^{\alpha,p(\cdot)}(\mathbb{R}^n) := \{ u = g_\alpha * f; \; f \in L^{p(\cdot)}(\mathbb{R}^n) \},
\]
and is equipped with the norm
\[
\|u\|_{\alpha,p(\cdot)} := \|f\|_{p(\cdot)}. \tag{2.11}
\]
If \( \alpha = 0 \) we put \( g_0 * f := f \) and \( L^{0,p(\cdot)}(\mathbb{R}^n) := L^{p(\cdot)}(\mathbb{R}^n) \) (normed by \( (2.1) \)).
We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leq cB$ (or $cA \geq B$) for some positive constant $c$ independent of appropriate quantities involved in the expressions $A$ and $B$, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

3. RELATIONSHIP BETWEEN SOBOLEV AND BESSEL POTENTIAL SPACES

The main result of this section is the following theorem.

**Theorem 3.1.** Let $p \in \mathcal{M}(\mathbb{R}^n)$ and let $k \in \mathbb{N}$. Then

$$L^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$$

and the corresponding norms are equivalent.

Before we prove the main theorem we shall need some auxiliary results. First we introduce some notation.

If $f$ belongs to the Schwartz class $\mathcal{S}$, the Fourier transform of $f$ is the function $\mathcal{F}f$ or $\hat{f}$ defined by

$$\mathcal{F}f(x) = \hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i x y} dy.$$

Let us summarize the basic properties of the Bessel kernel $g_{\alpha}$, $\alpha > 0$:

$$g_{\alpha} \text{ is nonnegative, radially decreasing and } \int_{\mathbb{R}^n} g_{\alpha}(y) dy = 1,$$

$$\hat{g}_{\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^n,$$

$$g_{\alpha} * g_{\beta} = g_{\alpha+\beta}, \quad \alpha, \beta > 0.$$

Let $\delta_0$ denotes the Dirac delta measure at zero. For $\alpha > 0$ we define the measure $\mu_{\alpha}$ on measurable sets $E \subset \mathbb{R}^n$ by

$$\mu_{\alpha}(E) = \delta_0(E) + \sum_{k=1}^{\infty} b(\alpha, k) \int_E g_{2k}(y) dy,$$

where $b(\alpha, k) = (-1)^k (\alpha/2)^k = (-1)^k \prod_{j=0}^{k-1} ((\alpha/2) - j), k = 1, 2, \ldots$. Since

$$\sum_{k=1}^{\infty} |b(\alpha, k)| < \infty,$$

the measure $\mu_{\alpha}$ is a finite signed Borel measure on $\mathbb{R}^n$. For $\alpha = 0$ we set $\mu_0 = \delta_0$. This construction uses the Taylor expansion of the function $t \mapsto (1 - t)^{\alpha/2}$, $\alpha > 0$, $t \in (0, 1]$, to give

$$\frac{|x|^\alpha}{(1 + |x|^2)^{\alpha/2}} = \left(1 - \frac{1}{1 + |x|^2}\right)^{\alpha/2} = 1 + \sum_{k=1}^{\infty} b(\alpha, k) (1 + |x|^2)^{-2k/\alpha}, \quad x \in \mathbb{R}^n,$$

which implies that (for $\alpha > 0$)

$$\mu_{\alpha}(x) = \frac{|x|^\alpha}{(1 + |x|^2)^{\alpha/2}}.$$

Obviously, (3.5) holds for $\alpha = 0$, too. (For more details see [19, p. 32] and [23, p. 134].)

We define the Riesz transform $\mathcal{R}_j f$, $j = 1, \ldots, n$, of a function $f \in \mathcal{S}$ by the formula

$$\mathcal{R}_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \lim_{\varepsilon \to 0_+} \int_{|y| > \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy.$$

Recall that (cf. [23]),

$$\mathcal{F}(\mathcal{R}_j f)(x) = \frac{-i \varepsilon y_j}{|x|} \hat{f}(x).$$

Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ be a multi-index. Then the multi-Riesz transform $\mathcal{R}_\beta$ is defined as

$$\mathcal{R}_\beta f = \mathcal{R}_{\beta_1} \circ \cdots \circ \mathcal{R}_{\beta_n} f.
Let \( f \in \mathcal{S} \). Then it is easy to verify (cf. [19]) that
\[
\mathcal{F}(\mathcal{R}_\beta f)(x) = \left( \frac{-ix_1}{|x|} \right)^{\beta_1} \cdots \left( \frac{-ix_n}{|x|} \right)^{\beta_n} \hat{f}(x),
\]
(3.7)
\[
\mathcal{F}(\mathcal{R}_\beta(D^\beta f))(x) = \left( \frac{-2\pi i x_1}{|x|} \right)^{\beta_1} \cdots \left( \frac{-2\pi i x_n}{|x|} \right)^{\beta_n} \hat{f}(x),
\]
(3.8)
\[
\mathcal{F}(D^\beta f)(x) = (-2\pi i)^{|\beta|} x^\beta \hat{f}(x)
\]
(3.9) \((x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}).\)

**Lemma 3.2.** Let \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \). Then there exists a positive constant \( C \) such that, for any \( \alpha \geq 0 \) and \( f \in L^{p(\cdot)}(\mathbb{R}^n), \)
\[
\|g_\alpha \ast f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.
\]
(3.10)

**Proof.** Using (3.1) and putting \( \varepsilon = 1 \) and \( g_\alpha = \varphi = \psi \) in Theorem 2 (a) on page 62 of [23] we obtain a point-wise estimate
\[
(g_\alpha \ast f)(x) \leq Mf(x), \quad x \in \mathbb{R}^n \quad (\alpha \geq 0).
\]
Hence, by (2.7) the inequality (3.10) follows. \( \square \)

**Lemma 3.3.** Let \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n), \alpha \geq 0 \) and \( \beta \in \mathbb{N}_0^n \). Then there exists a positive constant \( C \) such that, for any \( f \in L^{p(\cdot)}(\mathbb{R}^n), \)
\[
\|g_\alpha \ast f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)},
\]
(3.11)
\[
\|\mathcal{R}_\beta f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.
\]
(3.12)

**Proof.** It is easy to calculate
\[
(\mu_\alpha \ast f)(x) = f(x) + \sum_{k=1}^\infty b(\alpha, k) (g_{2k} \ast f)(x).
\]
Then, by (3.10) and (3.4),
\[
\|\mu_\alpha \ast f\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} + \sum_{k=1}^\infty |b(\alpha, k)| \|g_{2k} \ast f\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \left( 1 + C \sum_{k=1}^\infty |b(\alpha, k)| \right) \lesssim \|f\|_{p(\cdot)}
\]
which proves (3.11).

To prove (3.12) we use the results of L. Diening and M. Růžička [8, Prop. 4.3] and L. Diening [6, Thm. 8.14] that under the assumption \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \) there exists a positive constant \( c \) such that, for any \( f \in L^{p(\cdot)}(\mathbb{R}^n), \)
\[
\|\mathcal{R}_j f\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}, \quad j = 1, \ldots, n.
\]
Applying (3.6) and iterating this inequality we obtain (3.12) with \( C = c|\beta| \). \( \square \)

**Lemma 3.4.** Let \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \). Then
(i) \( C_0^\infty(\mathbb{R}^n) \) is dense in \( W^{k,p(\cdot)}(\mathbb{R}^n), k \in \mathbb{N} \);
(ii) the Schwartz class \( \mathcal{S} \) is dense in \( L^{a,p(\cdot)}(\mathbb{R}^n), \alpha \geq 0. \)

**Proof.** The density in (i) follows from the assumption \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \) by [7, Cor. 2.5].

Let us prove (ii). If \( \alpha = 0 \), the result follows from density of \( C^\infty_0(\mathbb{R}^n) \) in \( L^{p(\cdot)}(\mathbb{R}^n) \) (cf. [14, Thm. 2.11]). Let \( \alpha > 0 \) and \( u \in L^{a,p(\cdot)}(\mathbb{R}^n) \). Then there is a function \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) such that \( u = g_\alpha \ast f \). By density of \( C^\infty_0(\mathbb{R}^n) \) in \( L^{p(\cdot)}(\mathbb{R}^n) \) we can find a sequence \( (f_j)_{j=1}^\infty \subset C^\infty_0(\mathbb{R}^n) \subset \mathcal{S} \) converging to \( f \) in \( L^{p(\cdot)}(\mathbb{R}^n) \). Since the mapping \( f \mapsto g_\alpha \ast f \) maps \( \mathcal{S} \) onto \( \mathcal{S} \) (cf. [23]), the functions \( u_j := g_\alpha \ast f_j, j \in \mathbb{N}, \) belong to \( \mathcal{S} \). Moreover,
\[
\|u - u_j\|_{a,p(\cdot)} = \|f - f_j\|_{p(\cdot)} \to 0 \quad \text{as} \quad j \to \infty
\]
and the assertion follows. \( \square \)
Lemma 3.5. Let \( f \in \mathcal{S} \) and \( k \in \mathbb{N} \). Then

\[
\hat{f} = g_k \ast \sum_{m=0}^{k} \binom{k}{m} g_{k-m} \ast \mu_m \ast (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \mathcal{R}_\beta(D^\beta f),
\]

where \( \binom{m}{\beta} = \frac{m!}{\beta_1! \beta_2! \cdots \beta_n!} \).

Proof. (Cf. [19, Lemma 5.15]) Let \( f \in \mathcal{S} \). Using the Binomial Theorem we have

\[
1 = \sum_{m=0}^{k} \binom{k}{m} \sum_{|\beta|=m} \binom{m}{\beta} x_1^{2\beta_1} \cdots x_n^{2\beta_n}
\]

\[
= \frac{1}{(1+|x|^2)^{k/2}} \sum_{m=0}^{k} \binom{k}{m} \frac{1}{(1+|x|^2)^{2\beta_1}} (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} (-\frac{2\pi x_1^2}{|x|^2})^{\beta_1} \cdots (-\frac{2\pi x_n^2}{|x|^2})^{\beta_n}.
\]

Consequently, by (3.8), (3.5) and (3.2), we obtain

\[
\hat{f}(x) = \hat{g}_k(x) \sum_{m=0}^{k} \binom{k}{m} \hat{g}_{k-m}(x) \hat{\mu}_m(x) (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \mathcal{F}(\mathcal{R}_\beta(D^\beta f))(x).
\]

The result then follows by applying the inverse Fourier transform. \( \square \)

Lemma 3.6. Let \( f \in \mathcal{S} \), \( k \in \mathbb{N} \) and \( \beta \in \mathbb{N}_0^n \), \( |\beta| \leq k \). Then

\[
D^\beta (g_k \ast f) = (2\pi)^{|\beta|} g_{k-|\beta|} \ast \mu_{|\beta|} \ast \mathcal{R}_\beta f.
\]

Proof. (Cf. [19, Lemma 5.17]) Let \( f \in \mathcal{S} \). By (3.9), (3.2) and (3.5),

\[
\mathcal{F}(D^\beta (g_k \ast f))(x) = (-2\pi i)^{|\beta|} x^\beta \hat{g}_k(x) \hat{f}(x)
\]

\[
= (2\pi)^{|\beta|} \left( \frac{1}{(1+|x|^2)^{k-|\beta|}} \right) (-\frac{ix_1}{|x|})^{\beta_1} \cdots (-\frac{ix_n}{|x|})^{\beta_n} \hat{f}(x)
\]

The result now follows by applying the inverse Fourier transform. \( \square \)

Proof of Theorem 3.1. (i) Let \( f \in L^{k,p}(\mathbb{R}^n) \). In view of Lemma 3.4 we can assume that \( f \in \mathcal{S} \). Then there is a function \( h \in \mathcal{S} \) such that \( f = g_a \ast h \). By (2.11), Lemma 3.5 and Lemma 3.2,

\[
\| f \|_{W^{k,p}(\mathbb{R}^n)} = \sum_{|\beta| \leq k} \| D^\beta f \|_{p(\cdot)} = \sum_{|\beta| \leq k} \| D^\beta (g_k \ast h) \|_{p(\cdot)}
\]

\[
= \sum_{|\beta| \leq k} \| (2\pi)^{|\beta|} g_{k-|\beta|} \ast \mu_{|\beta|} \ast \mathcal{R}_\beta h \|_{p(\cdot)} \leq c \| h \|_{p(\cdot)} = c \| f \|_{k,p(\cdot)},
\]

where \( c > 0 \) is a suitable constant independent of \( f \).

(ii) We prove the reverse inequality. Let \( f \in W^{k,p}(\mathbb{R}^n) \). Again, by Lemma 3.4, we can assume that \( f \in \mathcal{S} \). Then, by Lemma 3.5 and Lemma 3.2,

\[
\| f \|_{k,p(\cdot)} = \left\| \sum_{m=0}^{k} g_{k-m} \ast \mu_m \ast (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \mathcal{R}_\beta(D^\beta f) \right\|_{p(\cdot)}
\]

\[
\leq c \sum_{|\beta| \leq k} \| D^\beta f \|_{p(\cdot)} = c \| f \|_{W^{k,p(\cdot)}}
\]

with a suitable constant \( c > 0 \) independent of \( f \). \( \square \)
4. Capacity

Let \( E \subset \mathbb{R}^n \), \( \alpha > 0 \) and \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). Define a capacity in \( \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n) \) by

\[
\text{Cap}_{\alpha,p(\cdot)}(E) = \inf g_{p(\cdot)}(f),
\]

where the infimum is taken over all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) with \( g_{\alpha \cdot f} \geq 1 \) on \( E \). Since \( g_{\alpha} \) is non-negative (cf. (3.1)) we can assume that \( f \geq 0 \).

**Lemma 4.1.** Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). The capacity \( \text{Cap}_{\alpha,p(\cdot)} \) is an outer measure. That is,

(i) \( \text{Cap}_{\alpha,p(\cdot)}(\emptyset) = 0 \);

(ii) if \( E_1 \subset E_2 \), then \( \text{Cap}_{\alpha,p(\cdot)}(E_1) \leq \text{Cap}_{\alpha,p(\cdot)}(E_2) \);

(iii) if \( E_i \subset \mathbb{R}^n, i = 1, 2, \ldots \), then

\[
\text{Cap}_{\alpha,p(\cdot)} \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \text{Cap}_{\alpha,p(\cdot)}(E_i).
\]

**Proof.** The property (i) immediately follows on putting \( f \equiv 0 \). The property (ii) follows from the fact that every test function of \( E_2 \) is also a test function of \( E_1 \).

Next we prove (iii), following [13]. We may assume that \( \sum_{i=1}^{\infty} \text{Cap}_{\alpha,p(\cdot)}(E_i) < \infty \). Let \( \varepsilon > 0 \) be fixed. For every \( i \in \mathbb{N} \) choose \( f_i \in L^{p(\cdot)}(\mathbb{R}^n) \) such that \( g_{\alpha \cdot f_i} \geq 1 \) on \( E_i \) and

\[
\int_{\mathbb{R}^n} |f_i(x)|^{p(x)} \, dx \leq \text{Cap}_{\alpha,p(\cdot)}(E_i) + \frac{\varepsilon}{2}.
\]

Put \( f := \sup_{i} f_i \) and \( E := \bigcup_{i=1}^{\infty} E_i \). If \( x \in E \) then \( x \in E_i \) for some \( i \in \mathbb{N} \) and \( (g_{\alpha \cdot f})(x) \geq (g_{\alpha \cdot f_i})(x) \geq 1 \). Thus, \( f \) is a test function for \( E \). Set \( h_k = \max_{1 \leq i \leq k} f_i \) and define \( X_i = \{ x \in \mathbb{R}^n; h_k(x) = f_i(x) \} \). Consequently, \( \mathbb{R}^n = \bigcup_{i=1}^{k} X_i \) and

\[
\int_{\mathbb{R}^n} |h_k(x)|^{p(x)} \, dx \leq \sum_{i=1}^{k} \int_{X_i} |f_i(x)|^{p(x)} \, dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} |f_i(x)|^{p(x)} \, dx \leq \sum_{i=1}^{\infty} \text{Cap}_{\alpha,p(\cdot)}(E_i) + \varepsilon.
\]

Since \( h_k \uparrow f \), by the Monotone Convergence Theorem we have

\[
\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx \leq \sum_{i=1}^{\infty} \text{Cap}_{\alpha,p(\cdot)}(E_i) + \varepsilon.
\]

Letting \( \varepsilon \to 0 \) we obtain the assertion. \( \square \)

The ordinary Sobolev capacity is defined by

\[
C_{p(\cdot)}(E) = \inf g_{W^{1,p(\cdot)}}(u)
\]

where

\[
g_{W^{1,p(\cdot)}}(u) := \int_{\mathbb{R}^n} (|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}) \, dx
\]

and the infimum is taken over all \( u \in W^{1,p(\cdot)}(\mathbb{R}^n) \) for which there is an open set \( G \supset E \) such that \( u \geq 1 \) a. e. on \( G \).

It is possible to show that if \( p \in \mathcal{B}(\mathbb{R}^n) \) then \( C_{p(\cdot)} \) is an outer measure and an Choquet capacity [12, Corollaries 3.3 and 3.4]. Relationship between the capacities \( \text{Cap}_{1,p(\cdot)} \) and \( C_{p(\cdot)} \) is formulated in the next lemma.

**Lemma 4.2.** Assume that \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \) and \( E \subset \mathbb{R}^n \). Then

\[
\text{Cap}_{1,p(\cdot)}(E) \leq c \max\{C_{p(\cdot)}(E)^{\frac{1}{p^*}}, C_{p(\cdot)}(E)^{\frac{1}{p}}\}
\]

and

\[
C_{p(\cdot)}(E) \leq C \max\{\text{Cap}_{p(\cdot)}(E)^{\frac{1}{p^*}}, \text{Cap}_{p(\cdot)}(E)^{\frac{1}{p}}\}.
\]

Here \( c \) and \( C \) are positive constants independent of \( E \).
Proof. Let \( u \in W^{1,p(\cdot)}(\mathbb{R}^n) \), \( u \geq 1 \) on an open neighborhood of \( E \), be a test function for \( C_{p(\cdot)}(E) \). By Theorem 3.1 there exists \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) so that \( u = g_1 \ast f \), and

\[
\|u\|_{W^{1,p(\cdot)}} \approx \|f\|_{p(\cdot)}.
\]

Obviously, \( f \) is a test function for \( Cap_{p(\cdot)}(E) \) and

\[
Cap_{p(\cdot)}(E) \leq \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx.
\]

By (2.2) it is easy to see that for a function \( g \in L^{p(\cdot)}(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} |g(x)|^{p(x)} \, dx \leq \max \left\{ \|g\|_{p_1(\cdot)}^{p^*}, \|g\|_{p_2(\cdot)}^{p^*} \right\}
\]

and

\[
\|g\|_{p(\cdot)} \leq \max \left\{ \left( \int_{\mathbb{R}^n} |g(x)|^{p(x)} \, dx \right)^{1/p^*}, \left( \int_{\mathbb{R}^n} |g(x)|^{p(x)} \, dx \right)^{1/p_1(\cdot)} \right\}.
\]

Applying (4.2), (4.3), (4.1) and (4.4), we arrive at

\[
Cap_{p(\cdot)}(E) \leq \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx \leq \max \left\{ \|f\|_{p_1(\cdot)}^{p^*}, \|f\|_{p_2(\cdot)}^{p^*} \right\}
\]

\[
\leq \max \{ \|u\|_{W^{1,p_1(\cdot)}}^{p^*}, \|u\|_{W^{1,p_2(\cdot)}}^{p^*} \} \leq \max \{ \varrho_{W^{1,p_1(\cdot)}}(u)^{p^*}, \varrho_{W^{1,p_2(\cdot)}}(u)^{p^*} \},
\]

and the first inequality follows.

Let \( \varepsilon > 0 \). Take \( f \geq 0, f \in L^{p(\cdot)}(\mathbb{R}^n) \), such that \( g_1 \ast f \geq 1 \) on \( E \) and

\[
\varrho_{p(\cdot)}(f) \leq Cap_{1,p(\cdot)}(E) + \varepsilon.
\]

Since \( f \geq 0 \), the function \( g_1 \ast f \) is lower semi-continuous and so, the set \( E_{\varepsilon} = \{ x \in \mathbb{R}^n; \frac{g_1 \ast f}{\varepsilon} > 1 \} \) is open and contains \( E \). Thus,

\[
C_{p(\cdot)}(E) \leq \varrho_{W^{1,p_1(\cdot)}}(\frac{g_1 \ast f}{\varepsilon}) \leq (1 - \varepsilon)^{-p^*} \varrho_{W^{1,p_1(\cdot)}}(g_1 \ast f).
\]

Letting \( \varepsilon \to 0_+ \), we obtain

\[
C_{p(\cdot)}(E) \leq \varrho_{W^{1,p(\cdot)}}(g_1 \ast f).
\]

Now, by Theorem 3.1, we have

\[
C_{p(\cdot)}(E) \leq \varrho_{W^{1,p_1(\cdot)}}(g_1 \ast f) \leq \max \{ \|g_1 \ast f\|_{p_1(\cdot)}^{p^*}, \|g_1 \ast f\|_{p_2(\cdot)}^{p^*} \}
\]

\[
\leq \max \{ \|g_1 \ast f\|_{p_1(\cdot)}^{p^*}, \|g_1 \ast f\|_{p_2(\cdot)}^{p^*} \} \leq \max \{ \varrho_{p(\cdot)}(f)^{p^*}, \varrho_{p(\cdot)}(f)^{p^*} \}
\]

which completes the proof.

\[ \square \]

5. Hölder type quasi-continuity

In this section we study point-wise behavior of functions from \( L^{1,p(\cdot)}(\mathbb{R}^n) \). First we investigate a quasi-continuity of such functions.

**Proposition 5.1.** Let \( p(\cdot) \in \mathcal{M}(\mathbb{R}^n) \). Every \( u \in L^{1,p(\cdot)}(\mathbb{R}^n) \) is quasi-continuous. That is, for every \( \varepsilon > 0 \), there exists a set \( F \subset \mathbb{R}^n \), \( Cap_{1,p(\cdot)}(F) \leq \varepsilon \), so that \( u \) restricted to \( \mathbb{R}^n \setminus F \) is continuous.

**Proof.** Let \( u = g_1 \ast f \in L^{1,p(\cdot)}(\mathbb{R}^n) \). Then, by Lemma 3.4, there is a sequence \( u_i = g_1 \ast f_i \in S \cap L^{1,p(\cdot)}(\mathbb{R}^n) \) converging to \( u \) in \( L^{1,p(\cdot)}(\mathbb{R}^n) \). We may assume without loss of generality, by considering a subsequence if necessary, that

\[
\|u_i - u_{i+1}\|_{1,p(\cdot)} \leq 4^{-i}, \quad i = 1, 2, \ldots
\]

Put

\[
E_i = \{ x \in \mathbb{R}^n; |u_i(x) - u_{i+1}(x)| > 2^{-i} \}, \quad i = 1, 2, \ldots, \quad \text{and} \quad F_j = \bigcup_{i=j}^{\infty} E_i.
\]
Continuity of the functions \( u_i \) implies that the sets \( E_i, F_j, i, j = 1, 2, \ldots \), are open. By Theorem 2.2 of [12], the functions \( |u_i - u_{i+1}| \) belong to \( W^{1,p_i}(\mathbb{R}^n) \) if \( u_i \in W^{1,p_i}(\mathbb{R}^n), i = 1, 2, \ldots \), and so, using Theorem 3.1, we have that \( 2^i|u_i - u_{i+1}| \in L^{p_i}(\mathbb{R}^n) \). Hence, for every \( i = 1, 2, \ldots \), there is a function \( h_i \in L^{p_i}(\mathbb{R}^n) \) such that \( 2^i|u_i - u_{i+1}| = g_i \cdot h_i \). Using (5.1), (2.3) and definition of the norm (2.11), we obtain

\[
\text{Cap}_{1,p_i}(E_i) \leq \int_{\mathbb{R}^n} h_i(x)^{p_i} \, dx \leq \|h_i\|_{p_i} = 2^i\|u_i - u_{i+1}\|_{1,p_i} \leq 2^{-i}, \quad i = 1, 2, \ldots .
\]

Given \( \varepsilon > 0 \), choose \( j \in \mathbb{N} \) so that \( 2^{1-j} < \varepsilon \). Now, Lemma 4.1 (iii) implies that

\[
\text{Cap}_{1,p_i}(F_j) \leq \sum_{i=j}^{\infty} \text{Cap}_{1,p_i}(E_i) \leq \sum_{i=j}^{\infty} 2^{-i} \leq 2^{1-j} < \varepsilon .
\]

Moreover, for every \( x \in \mathbb{R}^n \setminus F_j \) and every \( k, l \in \mathbb{N}, k > l \geq j \),

\[
|u(x) - u_k(x)| \leq \sum_{i=l}^{k-1} |u_i(x) - u_{i+1}(x)| \leq \sum_{i=l}^{k-1} 2^{-i} < 2^{1-l}.
\]

Hence, the sequence \( \{u_k\}_i \) is uniformly convergent in \( \mathbb{R}^n \setminus F_j \) which implies that the function \( v := \lim_{i \to \infty} u_i \) restricted to \( \mathbb{R}^n \setminus F_j \) is continuous. Put

\[
G := \{x \in \mathbb{R}^n; |u(x) - v(x)| > 0\}, \quad F := F_j \cup G .
\]

Obviously, the function \( u \) restricted to \( \mathbb{R}^n \setminus F \) is continuous, and so, it remains to prove that

\[
\text{Cap}_{1,p_i}(G \setminus F_j) = 0 .
\]

Given \( k \in \mathbb{N} \), by the uniform convergence of \( \{u_i\}_i \) to \( v \) on \( \mathbb{R}^n \setminus F_j \) we find \( i_0 \in \mathbb{N} \) such that, for any \( i \geq i_0 \),

\[
\{x \in \mathbb{R}^n \setminus F_j; |u(x) - v(x)| > 2/k\} \subseteq \{x \in \mathbb{R}^n \setminus F_j; |u(x) - u_i(x)| > 1/k\} .
\]

Moreover, for any \( i \geq i_0 \),

\[
\{x \in \mathbb{R}^n \setminus F_j; |u(x) - u_i(x)| > 1/k\} \subseteq \{x \in \mathbb{R}^n \setminus F_j; g_1 \cdot |f - f_i|(x) > 1\} .
\]

Consequently, using (ii) of Lemma 4.1 and the definition of capacity \( \text{Cap}_{1,p_i} \), we obtain, for any \( i \geq i_0 \),

\[
\text{Cap}_{1,p_i}(\{x \in \mathbb{R}^n \setminus F_j; |u(x) - v(x)| > 2/k\}) \leq \text{Cap}_{1,p_i}(\{x \in \mathbb{R}^n \setminus F_j; g_1 \cdot |f - f_i|(x) > 1\}) \leq \varrho_{p_i}(k|f - f_i|).
\]

Letting \( i \to \infty \), we obtain, in view of the fact that \( ||f - f_i||_{p_i} = ||u - u_i||_{1,p_i} = 0 \) and (2.5), that

\[
\text{Cap}_{1,p_i}(\{x \in \mathbb{R}^n \setminus F_j; |u(x) - v(x)| > 2/k\}) = 0 .
\]

Since \( k \in \mathbb{N} \) was arbitrary, the assertion (5.2) now follows by (iii) of Lemma 4.1.

Our second aim is to generalize the result of J. Malý [16] on Hölder type quasi-continuity. The idea of proof of Malý can be applied in spaces \( L^{1,p}((\mathbb{R}^n), too).\)

**Definition 5.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Say that \( u : \Omega \to \mathbb{R} \) is an \( \alpha \)-Hölder-continuous function on \( \Omega \) if

\[
\sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty .
\]

First, let us formulate the result.

**Theorem 5.3.** Let \( p \in \mathcal{M}(\mathbb{R}^n) \) and let \( p^* \leq n \). Suppose that

\[
0 < \alpha < \beta \frac{p_s(p_s - \varepsilon_0)}{p^*(p^* - \varepsilon_0)},
\]

where \( \varepsilon_0 \) is small enough.
where $0 < \varepsilon_0 < p_* - 1$ and $0 < \beta < \varepsilon_0/p_*$. If $u \in L^{1,p_c}({\mathbb R}^n)$ then, for any $\varepsilon > 0$, there exists an $\alpha$-Hölder continuous function $w$ on $\mathbb{R}^n$ such that
\[
\|u - w\|_{1,p_c} \leq \varepsilon, \quad \text{Cap}_{p_c - \varepsilon_0} \left( \{ x \in \mathbb{R}^n ; w(x) \neq u(x) \} \right) \leq \varepsilon.
\]

Before proving Theorem 5.3 we need some preliminary results. The following embedding theorem can be found in [24, Section 2.7.1].

**Lemma 5.4.** Let $r > n$ and let $u \in L^{1,r}({\mathbb R}^n)$. Then there is a positive constant $c = c(n,r)$ such that
\[
(5.4) \quad |u(x) - u(y)| \leq c \|u\|_{1,r} |x - y|^{1 - n/r} \quad \text{and} \quad |u(x)| \leq c \|u\|_{1,r}
\]
for all $x, y \in \mathbb{R}^n$.

**Lemma 5.5.** Assume that $p(\cdot) \in M(\mathbb{R}^n)$, $p^* \leq n$, $0 < \varepsilon_0 < p_* - 1$ and $0 < \beta < \varepsilon_0/p_*$. Let $u \in L^{1,p(\cdot)}({\mathbb R}^n)$ with $\|u\|_{1,p(\cdot)} \leq 1$ and $u = g_1 * f$. Then there exist bounded functions $u_R \in L^{1,p(\cdot)}({\mathbb R}^n)$, $0 < R < 1$, $u_R = g_1 * f_R$, so that
\[
(5.5) \quad \lim_{R \to 0^+} \|u - u_R\|_{1,p(\cdot)} = 0,
\]
and
\[
(5.6) \quad \|u_R(x) - u_R(y)\| \leq cR^{p_0 - \varepsilon_0}/p^*,
\]
and
\[
(5.7) \quad |u_R(x) - u_R(y)| \leq C|x - y|^\beta \text{ if } |x - y| \leq R.
\]

**Proof.** Given $u = g_1 * f \in L^{1,p(\cdot)}({\mathbb R}^n)$, where $f \in L^{p(\cdot)}({\mathbb R}^n)$, $\|u\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} \leq 1$. Let $R \in (0,1)$ and $\lambda_R = R^{-\beta(p_* - \varepsilon_0)/\varepsilon_0}$. Put
\[
M_R = \{ x \in \mathbb{R}^n ; |f(x)| \geq \lambda_R \}.
\]

Clearly, (2.3) yields
\[
(5.8) \quad |M_R| \leq \|f\|_{\lambda_R p_*} \leq \lambda_R^{-p_*}.
\]
We set
\[
f_R(x) := \begin{cases} 
0 & \text{if } x \in M_R \\
f(x) & \text{if } x \in \mathbb{R}^n \setminus M_R
\end{cases}
\]
and $u_R := g_1 * f_R$.

Obviously, the functions $u_R$ are bounded and belong to $L^{1,p(\cdot)}({\mathbb R}^n)$. Since $|M_R| \to 0$ as $R \to 0^+$ (cf. (5.8)), we have by (2.5)
\[
\|u - u_R\|_{1,p(\cdot)} = \|f - f_R\|_{p(\cdot)} = \|f\chi_{M_R}\|_{p(\cdot)} \to 0 \quad \text{as } R \to 0^+.
\]
By the Hölder inequality (cf. [14, Thm. 2.1]), (2.4) and $0 < \varepsilon_0 < p_* - 1$ we arrive at
\[
\|u_R(x) - u_R(y)\| \leq cR^{p_0 - \varepsilon_0}/p^*.
\]

Hence and from (5.8) we obtain (5.6).

It remains to prove (5.7). Put
\[
r = \frac{n\varepsilon_0 - \beta p_* (p_* - \varepsilon_0)}{\varepsilon_0 - \beta p_*}.
\]

Since $0 < \beta < \varepsilon_0/p_*$ and $p_* < p^* \leq n$ we find that $r > n$ and $r - p(x) \geq 0$, $x \in \mathbb{R}^n$. As the functions $|f_R|$ are bounded by $\lambda_R$ and $\|f_R\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 1$, it implies
\[
\|f_R\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 1,
\]
that is, $u_R \in L^{1,r}(\mathbb{R}^n)$ and $\|u_R\|_{1,r} \leq \lambda_R^{r - p_*}/r$ for all $R \in (0,1)$. Consequently, using (5.4), we obtain,
\[
|u_R(x) - u_R(y)| \leq c|x - y|^1 \|u_R\|_{1,r} \leq c|x - y|^1 \lambda_R^{r - p_*}/r
\]
\[
\leq c|x - y|^1 \lambda_R^{r - p_*} R^{-\beta(p_* - \varepsilon_0)(r - p_*)/(r\varepsilon_0)} \leq c|x - y|^{1 - n/r - \beta(p_* - \varepsilon_0)(r - p_*)/(r\varepsilon_0)}
\]
\[
eq c|x - y|^\beta.
\]
and (5.7) is verified. □

Similarly as in the classical case, the space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is also closed under truncation.

**Lemma 5.6.** Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and let $\tau \geq 0$. Then the function $v := \max \{-\tau, \min\{\tau, u\}\}$ belongs to $W^{1,p(\cdot)}(\mathbb{R}^n)$ and satisfies $\|u - v\|_{W^{1,p(\cdot)}} \leq \|u\|_{W^{1,p(\cdot)}}$.

**Proof.** The assertions immediately follow from the fact that the space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is a lattice (see [12, Thm. 2.2]). □

**Proof of Theorem 5.3.** Fix $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$. We may assume that $\|u\|_{1,p(\cdot)} \leq 1$.

Let $u_R = g_1 * f_R$ have the same meaning as in Lemma 5.5 and let $\alpha$ be a number from (5.3).

Denote $u_j := u_{R_j}$ and $f_j := f_{R_j}$, where the sequence $\{R_j\} \subset (0, 1]$ is chosen so that

\[(5.9) \quad R_0 = 1, \quad R_{j+1}^\alpha \leq \frac{1}{2} R_j^\alpha, \quad j = 0, 1, \ldots,
\]

and (cf. (5.5))

\[(5.10) \quad \sum_{j=1}^{\infty} \|u_{j+1} - u_j\|_{1,p(\cdot)} < \infty.
\]

For $j \in \mathbb{N}$ put

\[E_j = \{x \in \mathbb{R}^n; |u_{j+1}(x) - u_j(x)| > R_j^\alpha\}.
\]

Then, using the inequality

\[g_1 * (R_j^\alpha |f_{j+1} - f_j|)(x) \geq R_j^\alpha |g_1 * (f_{j+1} - f_j)(x)| = R_j^\alpha |u_{j+1}(x) - u_j(x)| > 1, \quad x \in E_j,
\]

and (5.6), we get

\[Cap_{1,p(\cdot) - \varepsilon_0}(E_j) \leq \varepsilon_{p(\cdot) - \varepsilon_0}(R_j^\alpha |f_{j+1} - f_j|, f_j) \leq R_j^{-\alpha(p^*-\varepsilon_0)} \varepsilon_{p(\cdot) - \varepsilon_0}(|f - f_j| + |f_{j+1} - f|) \leq 2^{p^*-1} 2^c R_j^{-\alpha(p^*-\varepsilon_0)} R_j^{\alpha p^*(p^*-\varepsilon_0)/p^*} = C_R^\alpha R_j^{\alpha p^*(p^*-\varepsilon_0)/(p^*-\alpha(p^*-\varepsilon_0))}.
\]

Since the exponent on $R_j$ is positive, we can find by (5.9) a number $j_0 \in \mathbb{N}$ such that

\[(5.11) \quad \sum_{j=j_0}^{\infty} Cap_{1,p(\cdot) - \varepsilon_0}(E_j) < \varepsilon
\]

and (cf. (5.5) and (5.10))

\[(5.12) \quad \|u - u_{j_0}\|_{1,p(\cdot)} + \sum_{j=j_0}^{\infty} \|u_{j+1} - u_j\|_{1,p(\cdot)} \leq \varepsilon.
\]

For $j \in \mathbb{N}$ define the functions

\[v_j = \max \{-R_j^\alpha, \min\{R_j^\alpha, u_{j+1} - u_j\}\}.
\]

The functions $v_j$ belong (by Lemma 5.6 and Theorem 3.1) to $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and it is easy to see that

\[(5.13) \quad E_j = \{x \in \mathbb{R}^n; v_j(x) \neq u_{j+1}(x) - u_j(x)\}, \quad j \in \mathbb{N}.
\]

Set

\[w := u_{j_0} + \sum_{j=j_0}^{\infty} v_j \quad \text{and} \quad E := \bigcup_{j=j_0}^{\infty} E_j.
\]

Then, by (5.13), $w = u$ outside $E$, that is,

\[E \supset \{x \in \mathbb{R}^n; u(x) \neq v(x)\}.
\]

Moreover, by (5.11) and Lemma 4.1 (iii),

\[Cap_{1,p(\cdot) - \varepsilon_0}(E) \leq \varepsilon.
\]

Concerning the $\alpha$-Hölder continuity of $w$, observe that it is enough to estimate $|w(x) - w(y)|$ for $|x - y| \leq R_{j_0}$ as the function $w$ is bounded. Choose $x, y \in \mathbb{R}^n$ such that $0 < |x - y| \leq R_{j_0}$ and find $k \in \mathbb{N}$, $k \geq j_0$, so that

\[(5.14) \quad R_{k+1} < |x - y| \leq R_k.
\]
By (5.7),
\[ |w_{j_{0}}(x) - u_{j_{0}}(y)| \leq C|x - y|^\beta. \]
If \( j_{0} \leq j \leq k \) we deduce from (5.7) that
\[ |v_j(x) - v_j(y)| \leq C|x - y|^\beta. \]
If \( j > k \) then, by (5.9) and (5.14),
\[ |v_j(x) - v_j(y)| \leq |v_j(x)| + |v_j(y)| \leq 2R_j^\alpha \leq 2^{k-j+2}R_{k+1}^\alpha \leq 2^{k-j+2}|x - y|^\alpha. \]
Using (5.9), we obtain
\[ k \leq \frac{\alpha}{\log 2} \log \frac{1}{R_k} \leq \frac{\alpha}{(\beta - \alpha) \log 2} R_k^{\alpha - \beta} \leq \frac{\alpha}{(\beta - \alpha) \log 2} |x - y|^{\alpha - \beta}. \]
Consequently,
\[ |w(x) - w(y)| \leq |u_{j_{0}}(x) - u_{j_{0}}(y)| + \sum_{j = j_{0}}^{k} |v_j(x) - v_j(y)| + \sum_{j = k + 1}^{\infty} |v_j(x) - v_j(y)| \leq C(k + 1)|x - y|^\beta + \sum_{j = k + 1}^{\infty} 2^{k-j+2}|x - y|^\alpha \lesssim |x - y|^\alpha \left( 1 + \sum_{j = k + 1}^{\infty} 2^{k-j+2} \right) \lesssim |x - y|^\alpha. \]
Finally, by (5.12),
\[ \|u - w\|_{1;p(\cdot)} \leq \|u - u_{j_{0}}\|_{1;p(\cdot)} + \sum_{j = j_{0}}^{\infty} \|v_j\|_{1;p(\cdot)} \leq \|u - u_{j_{0}}\|_{1;p(\cdot)} + \sum_{j = j_{0}}^{\infty} \|u_{j+1} - u_j\|_{1;p(\cdot)} \leq \epsilon \]
and the assertions are verified. \( \square \)

**Remark 5.7.** The assumption \( p^* \leq n \) in Theorem 5.3 is quite natural. If \( p_\ast > n \) then every Sobolev class has a continuous representative and the classical Morey’s inequality [10, Theorem 3, on p. 143] together with [14, Theorem 2.8] implies that
\[ |u(y) - u(z)| \leq Cr^{1 - \frac{n}{p_\ast}} \left( \int_{B(x,r)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \leq C(1 + |B(x,r)|)r^{1 - \frac{n}{p_\ast}} \|\nabla u\|_{p(\cdot)} \]
for every \( u \in W^{1,p(\cdot)}(\mathbb{R}^n) \), \( r > 0 \) and \( y, z \in B(x,r) \). Thus every Sobolev class has a locally \((1 - \frac{n}{p_\ast})\)-Hölder continuous representative which, by [11, Theorem 4.7], is
\[ v(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \, dy. \]
The last claim follows by an observation that if two continuous functions coincide almost everywhere then they actually coincide everywhere.

Related results concerning Sobolev spaces \( W^{1,p(\cdot)}(\Omega) \) on an open bounded subset \( \Omega \) of \( \mathbb{R}^n \) with \( p(x) > n \) for all \( x \in \Omega \) are derived in [9].

**Remark 5.8.** One of the referees pointed out to us a recent preprint [1] where the spaces of Bessel potentials over the spaces \( L^{p(\cdot)}(\mathbb{R}^n) \) were investigated. Among other results the authors independently proved Theorem 3.1 and showed that \( C^\infty_0(\mathbb{R}^n) \) is dense in \( L^{p(\cdot)}(\mathbb{R}^n) \) if \( \alpha \geq 0 \), \( p(\cdot) \) satisfies conditions (2.8), (2.9) and \( 1 < p_\ast \leq p^* < \frac{n}{\alpha} \) (cf. Lemma 3.4 (iii)).

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References