An embedding into an Orlicz space for $L^1$-functions from irregular domains

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For the occasion of David Shoikhet’s 60th birthday

Abstract. We prove an embedding into an Orlicz space for $L^1$-functions defined in irregular bounded John domains of the Euclidean $n$-space. We show that the result is essentially sharp. We study Orlicz embeddings for $L^p$-functions, $1 < p < n$, too.

1. Introduction

We study the embeddings for $L^1$-functions defined in bounded irregular $\phi$-John domains of the Euclidean $n$-space. The space $L^1$ is the space of distributions with the first-order derivatives in the space $L^1$. Suppose that $\alpha \in [1, 1 + \frac{1}{n-1})$ and $\beta \geq 0$. Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be the function $\phi(0) = 0$ and

$$\phi(t) = \frac{t^\alpha}{\log(e + t^{-1})}. \quad (1.1)$$

A bounded domain $D$ in $\mathbb{R}^n, n \geq 2$, is a $\phi$-John domain if there exist a constant $c_J > 0$ and a point $x_0 \in D$ such that each point $x \in D$ can be joined to $x_0$ by a rectifiable curve $\gamma : [0, l] \rightarrow D$, parametrized by its arc length, such that $\gamma(0) = x$, $\gamma(l) = x_0$, $l \leq c_J$, and

$$\phi(t) \leq c_J \text{ dist} \left( \gamma(t), \partial D \right)$$

for all $t \in [0, l]$.

Examples of these domains are classical John domains [10, 2.1, p. 384], and hence Lipschitz domains and uniform domains, when $\phi(t) = t$. More generally, all $s$-John domains are $\phi$-John domains with $\phi(t) = t^s$, $s > 1$, [16, Definition, p. 85]. Examples of $\varphi$-John domains with

$$\varphi(t) = \frac{t}{\log(e + t^{-1})} \quad (1.2)$$

are domains with some cusps, [7, Example 6.5], and mushrooms-type domains, [7, Example 6.2].

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Embeddings into Orlicz spaces of exponential type for domains with a cone condition are well known \cite[Theorem 1, Theorem 2]{18}; we also refer to \cite{20}, \cite{14}, \cite{13}.

A. Cianchi has proved sharp results for Orlicz-Sobolev spaces defined in domains satisfying relative isoperimetric inequalities in \cite[Theorem 2 and Example 1]{3}. His work covers Orlicz spaces of exponential type and more. Especially, classical John domains satisfy the so called Trudinger’s inequality, \cite[Example 1]{3}. Results with weights for $s$-John domains can be found in \cite[Theorem 2]{19}.

There is an embedding $W^{1,p}(D) \hookrightarrow L^{np/(n-p)}(D)$ for a classical John domain $D$ with $1 \leq p < n$, \cite[Lemma 3 and Lemma 4, (6)]{1} and \cite[Theorem]{15}. An embedding into Orlicz spaces of logarithmic type has been proved for $L^1_p$-functions, $1 < p < n$, defined in $\varphi$-John domain with condition (1.2), \cite[Theorem 1.1]{7}. The method there is partly a modification of L. I. Hedberg’s optimization argument with the Riesz potential, \cite[Lemma, (3)]{8}. The use of the fact that the Hardy-Littlewood maximal operator is bounded from $L^p \rightarrow L^p$, when $1 < p < n$, is critical in the proof of \cite[Theorem 1.1]{7}. Hence, the $L^1_p$-functions need to be considered in a different way to \cite{7}.

We prove an Orlicz embedding for $L^1_p$-functions defined in irregular $\varphi$-John domains. Our main result is the following.

**Theorem 1.1.** Suppose that $\alpha \in [1, 1 + 1/n]$ and $\beta \geq 0$. Let $\Phi_1 : [0, \infty) \rightarrow [0, \infty)$ be the function

$$\Phi_1(t) = \left( \frac{t}{\log^{\beta(n-1)}(m+t)} \right)^{\frac{n}{\alpha(n-1)}}$$

with $m \geq e$. If $D$ is a $\varphi$-John domain in $\mathbb{R}^n$ defined in (1.1), then there exists a constant $C$ depending on $\alpha$, $\beta$, $n$, and $D$ only such that the inequality

$$\|u - u_D\|_{L^{1}(D)} \leq C\|\nabla u\|_{L^{1}(D)}$$

holds for all $u \in L^1(D)$.

We give the proof for Theorem 1.1 in Section 5 and we show that the result is essentially sharp in Section 6. We give auxiliary results in Section 2 and Section 3. Further, we study Orlicz embeddings for $L^1_p$-functions in Section 4 also when $1 < p < n$.

**2. Preliminaries**

An open ball with a center $x$ and radius $r > 0$ is written as $B(x, r)$. The corresponding closed ball is denoted by $\overline{B}(x, r)$. Given any set $A$ in $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$, the distance between $x$ and the boundary $\partial A$ is written as $\text{dist}(x, \partial A)$, and $\text{diam}(A)$ stands for the diameter of $A$. The characteristic function of $A$ is denoted by $\chi_A$. When $A$ in $\mathbb{R}^n$ is a Lebesgue measurable set with positive $n$-Lebesgue measure $|A|$ we write the integral average of an integrable function $u$ in $A$ as

$$u_A = \int_A u(x) \, dx = |A|^{-1} \int_A u(x) \, dx.$$

We let $C(\ast, \cdots, \ast)$ denote a constant which depends on the quantities appearing in the parentheses only. In calculations from line to line we usually write $C$ for constants when it is not important to specify the dependence on the quantities.
appearing in the calculations. From line to line C might stand for a different constant.

Throughout this paper D is a bounded domain in \( \mathbb{R}^n, n \geq 2 \). The space \( L^1_p(D) \), \( 1 \leq p < \infty \), is the space of distributions with the first-order derivatives on D in the space \( L^p(D) \). The space \( L^1_p(D) \cap C^\infty(D) \) is dense in \( L^1_p(D) \), [11, 1.1.5, Theorem 1].

Let \( 1 \leq p < n \). Let \( \Phi_p : [0, \infty) \to [0, \infty) \) be the function

\[
\Phi_p(t) = \left( \frac{t}{\log^{\beta(n-1)}(m+t)} \right)^{\frac{\beta(n-1)}{\beta(m-1)}},
\]

where \( m \), depending on \( \alpha, \beta, n \), and \( p \), is chosen so that \( \Phi_p \) is a convex function. We study the Orlicz space \( L^{\Phi_p}(D) \) which means all measurable functions \( u \) in D such that

\[
\int_D \Phi_p(|\lambda u(x)|) \, dx < \infty
\]

with some fixed \( \lambda > 0 \). Since the function (2.1) satisfies the \( \Delta_2 \)-condition with some fixed constant \( C \), that is, \( \Phi_p(2t) \leq C\Phi_p(t) \), for all \( t \in [0, \infty) \), this space is equivalent to a space of all measurable functions \( u \) with

\[
\int_D \Phi_p(|u(x)|) \, dx < \infty.
\]

The space \( L^{\Phi_p}(D) \) equipped with the Luxemburg norm

\[
\|u\|_{L^{\Phi_p}(D)} = \inf \left\{ \lambda > 0 : \int_D \Phi_p\left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]

is a Banach space.

The point \( x_0 \) in the definition of the \( \varphi \)-John domain given in the beginning of Section 1 is called a John center of \( D \) and the constant \( c_J \) is a John constant of \( D \). If a domain is a \( \varphi \)-John domain with a John center \( x_0 \), then it is a \( \varphi \)-John domain with any other \( x \in D \), but with a different John constant. The function \( \varphi : [0, \infty) \to [0, \infty) \) in (1.1) is continuous and strictly increasing. There exists a constant \( C > 0 \) such that

\[
\varphi(t) \leq Ct \quad \text{and} \quad \varphi(t) \leq C\varphi(t/2)
\]

for all \( t \in [0, c_J] \); here \( c_J \) is the John constant.

In order to prove that an Orlicz embedding holds in the \( \varphi \)-John domains with the function (1.1) we need several auxiliary results.

We start with the following lemma which is a variation of [7, Lemma 3.5]. The idea is from [6, Theorem 9.3] where the classical John domains are considered. Originally, this type of the chaining argument idea with cubes seems to go back to [2, Lemma 2.1].

**Lemma 2.1.** Let \( D \) in \( \mathbb{R}^n, n \geq 2 \), be a \( \varphi \)-John domain with a constant \( c_J \) and let \( x_0 \in D \) be its John center. For every \( x \in D \setminus B(x_0, \text{dist}(x_0, \partial D)) \) there exists a sequence of balls \( B(x_i, r_i) \) with \( B(x_i, 2r_i) \subset D, i = 0, 1, \ldots, \) so that the following conditions hold for some constants \( C = C(c_J, \alpha, \beta), M = M(n), \) and \( N = N(n) \):

1. \( x_0 \) is the center of \( B\left(x_0, \frac{1}{2}\text{dist}(x_0, \partial D)\right) = B_0; \)
2. \( \varphi(\text{dist}(x, B_i)) \leq Cr_i \), and \( r_i \to 0 \) as \( i \to \infty; \)
3. no point of the domain \( D \) belongs to more than \( N \) balls \( B(x_i, r_i); \) and
Let $D$ in $\mathbb{R}^n$, $n \geq 2$, be a $\varphi$-John domain with a constant $c_J$ and $x_0 \in D$ being its John center. Then, there exists a constant $C = C(n, c_J)$ such that for all $u \in L^1_1(D)$ the inequality

$$
|u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}| \leq C \int_D \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy
$$

holds for almost every $x \in D$.

The proof of Theorem 2.2 is similar to the proof of the case $\alpha = \beta = 1$ in (1.1) which has been proved in [7, Theorem 3.6].

3. The modified Riesz potential and pointwise estimates

In order to obtain estimates for the integral of the integral representation in Theorem 2.2 we use the Hardy-Littlewood maximal operator. The classical centered Hardy-Littlewood maximal operator is written as

$$
Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| dy
$$

for locally integrable functions $f$ in $\mathbb{R}^n$ that is, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, [17, Section 1].

Lemma 3.1. Suppose that $\delta > 0$. Let $0 < \alpha < 1 + 1/(n-1)$ and $\beta \geq 0$ and $\varphi$ be given in (1.1). If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then the inequality

$$
\int_{B(x,\delta)} \frac{|f(y)|}{\varphi(|x-y|)^{n-1}} dy \leq C\delta^{n+1-n\alpha} \log^{\beta(n-1)}(e + 1/\delta) Mf(x)
$$

holds with a constant $C = C(\alpha, \beta, n)$ for every $x \in \mathbb{R}^n$.

Proof. Let $x \in \mathbb{R}^n$ be fixed and let $\delta$ be given. Let us divide the ball $B(x, \delta)$ into annuli and estimate

$$
\int_{B(x,\delta)} \frac{|f(y)|}{\varphi(|x-y|)^{n-1}} dy \leq \sum_{k=1}^{\infty} \varphi(\delta 2^{-k}) \int_{\{z:2^{-k}\delta \leq |x-z| < 2^{-k+1}\delta\}} |f(y)| dy
$$

$$
\leq C(n) \sum_{k=1}^{\infty} \varphi(\delta 2^{-k}) (2^{-k}\delta)^n \int_{\{z:|x-z| < 2^{-k+1}\delta\}} |f(y)| dy
$$

$$
\leq C(n) Mf(x) \sum_{k=1}^{\infty} \frac{(\delta 2^{-k})^{\alpha(1-n)}}{\log^{\beta(1-n)}(e + \delta^{-1}2^k)} (2^{-k}\delta)^n
$$

$$
\leq C(n)\delta^{n+1-n\alpha} Mf(x) \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e + \delta^{-1}2^k)}{2^{k(\alpha(1-n)+n)}}.
$$

We analyze the last sum. If $\delta \geq 1$, then

$$
\sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e + \delta^{-1}2^k)}{2^{k(\alpha(1-n)+n)}} \leq \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e + 2^k)}{2^{k(\alpha(1-n)+n)}} = C < \infty
$$
whenever $\alpha < 1 + 1/(n - 1)$. If $0 < \delta < 1$ and $\beta(n - 1) \leq 1$, then
$$
\sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e + \delta^{-1}2^k)}{2^{k(\alpha(1-n)+n)}} \leq \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e\delta^{-1} + \delta^{-1}2^k)}{2^{k(\alpha(1-n)+n)}}
$$
$$
= \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(e + 2^k)}{2^{k(\alpha(1-n)+n)}} \leq 2^{\beta(n-1)} \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(1/\delta)}{2^{k(\alpha(1-n)+n)}}
$$
$$
\leq 2^{\beta(n-1)} \sum_{k=1}^{\infty} \frac{\log^{\beta(n-1)}(1/\delta)}{2^{k(\alpha(1-n)+n)}},
$$
whenever $\alpha < 1 + 1/(n - 1)$ and $\beta \in [0, 1/(n - 1)]$. If $\beta(n - 1) > 1$, the estimates follow in the similar way. The claim follows from the above inequalities. 

The following lemma is from [7, Lemma 4.2]. Since there was a small mistake we prove a correct versio here.

**Lemma 3.2.** Suppose that $1 \leq p < n$ and $\delta > 0$. Let $\varphi$ be an increasing function on $(0, \infty)$. If $\|f\|_{L^p(\mathbb{R}^n)} \leq 1$, then there is a constant $C = C(n, p)$ such that for every $x \in \mathbb{R}^n$ the inequality
$$
\int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{|f(y)|}{\varphi(|x - y|)^{n-1}} dy \leq C\varphi(\delta)^{1-n} \delta^n \uparrow
$$
holds.

**Proof.** The case $p = 1$ is straightforward. Let $\delta > 0$ be given. If $\|f\|_{L^1(\mathbb{R}^n)} \leq 1$, then an easy calculation gives
$$
\int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{|f(y)|}{\varphi(|x - y|)^{n-1}} dy \leq \varphi(|x - y|)^{1-n} \int_{\mathbb{R}^n \setminus B(x, \delta)} |f(y)| dy \leq \varphi(\delta)^{1-n}.
$$

Assume that $1 < p < n$. Let us write $p' = p/(p - 1)$. Let $x \in \mathbb{R}^n$ be fixed and let $\delta$ be given. By Hölder’s inequality
$$
\int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{|f(y)|}{\varphi(|x - y|)^{n-1}} dy \leq \|f\|_{L^p(\mathbb{R}^n)} \|\varphi^{1-n} \chi_{\mathbb{R}^n \setminus B(x, \delta)} \|_{L^{p'}(\mathbb{R}^n)}
$$
$$
\leq \left\|\chi_{\mathbb{R}^n \setminus B(x, \delta)} \varphi(|x - \cdot|)^{-n} \right\|_{L^{(n-1)/n}(\mathbb{R}^n)}.
$$

We obtain for every $y \in \mathbb{R}^n \setminus B(x, \delta)$
$$
\varphi(|x - y|)^{-n} \leq C(n) |B(y, \varphi(|x - y|))|^{-1}
$$
$$
= C(n) \int_{B(y, 2|x - y|)} |B(y, 2|x - y|)|^{-1} |B(y, 2|x - y|)|^{-1} dz.
$$

Since $\alpha \geq 1$ we have
$$
\frac{|B(y, 2t)|}{|B(y, \varphi(t))|} = \left( \frac{2t \log^{\beta}(e + 1/t)}{t^\alpha} \right)^n \leq 2^n \left( \frac{\delta}{\varphi(\delta)} \right)^n \leq 2^n \left( \frac{\delta}{\varphi(\delta)} \right)^n.
for every $t \geq \delta$. We continue our estimate and obtain
\[\varphi(|x - y|)^{-n} \leq C(n) \left(\frac{\delta}{\varphi(\delta)}\right)^n M\left(\chi_B(x, \delta)|B(x, \delta)|^{-1}\right)(y)\]
for every $y \in \mathbb{R}^n \setminus B(x, \delta)$.

Let us recall that $1 < p < n$. Since $1 < \frac{n-1}{n}p' < \infty$ and the Hardy-Littlewood maximal operator is bounded in the corresponding Lebesgue space, [17, Section 1, Theorem 1(c)], we obtain
\[\|\chi_{\mathbb{R}^n \setminus B(x, \delta)}\varphi(|x - \cdot|)^{-n}\|_{L^{(n-1)/n}(\mathbb{R}^n)} \leq C(n) \left(\frac{\delta}{\varphi(\delta)}\right)^n \|M\left(\chi_B(x, \delta)|B(x, \delta)|^{-1}\right)\|_{L^{(n-1)/n}(\mathbb{R}^n)}\]
\[\leq C(n, p) \left(\frac{\delta}{\varphi(\delta)}\right)^n \|\chi_B(x, \delta)|B(x, \delta)|^{-1}\|_{L^{(n-1)/n}(\mathbb{R}^n)}\]
\[\leq C(n, p) \varphi(\delta)^{1-n} \|\chi_B(x, \delta)\|_{L^{(n-1)/n}(\mathbb{R}^n)}\]
\[\leq C(n, p) \varphi(\delta)^{1-n} \delta^n.\]

This yields the claim together with the first estimate.

By combining Lemmata 3.1 and 3.2 we obtain a pointwise estimate which is crucial for the proof of the main theorem, Theorem 1.1.

**Theorem 3.3.** Let $1 \leq p < n$. Let $\Phi_p: [0, \infty) \to [0, \infty)$ and $\varphi: [0, \infty) \to [0, \infty)$ be the functions
\[\Phi_p : t \mapsto \left(\frac{t}{\log^\beta(n+1)(m+t)}\right)^{\frac{1}{np(n-1)+n-p}}\]
and
\[\varphi : 0 \mapsto 0 \quad \text{and} \quad t \mapsto \frac{t^\alpha}{\log^\beta(e+t^{-1})} \quad \text{for} \quad t > 0\]
with $\alpha \in [1, \frac{1}{n-1}]$, $\beta \geq 0$, and $m \geq e$. Let $D$ in $\mathbb{R}^n$ be an open set. If $\|f\|_{L^p(\mathbb{R}^n)} \leq 1$, then there exists a constant $C = C(\alpha, \beta, n, p)$ such that the inequality
\[\Phi_p \left(\int_D \varphi(|x - y|)^{-n} dy\right) \leq C(Mf(x))^p\]
holds for every $x \in \mathbb{R}^n$.

**Proof.** Let us write
\[\Psi(z) = \frac{z}{\log^\beta(n+1)(m+z)}, \quad z \in [0, \infty).\]
Note that $\Psi$ is increasing since $\Phi_p$ is convex and $\Phi_p(0) = 0$. We have to show that there exists a constant $C$ such that the inequality
\[\Psi \left(\int_D \varphi(|x - y|)^{-n} dy\right) \leq C(Mf(x))^{\frac{np(n-1)+n-p}{n}}\]
holds. We may assume that $Mf(x) > 0$ for every $x$, since otherwise $f(x) = 0$ for almost every $x \in D$. 

\[\square\]
By choosing $\delta = (Mf(x))^{-\frac{1}{\beta}}$ we estimate

$$
\int_D \frac{|f(y)|}{\varphi(|x-y|)^{n-1}} \, dy \leq C\delta^{n+(1-n)p} \log^{\beta(n-1)}(e + (Mf(x))^\frac{1}{\beta}) + C\varphi((Mf(x))^{-\frac{1}{\beta}}) \log^{\beta(n-1)}(e + (Mf(x))^\frac{1}{\beta}) + C\varphi((Mf(x))^{-\frac{1}{\beta}}) \log^{\beta(n-1)}(e + (Mf(x))^\frac{1}{\beta}) \leq C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right) \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right) = e + (Mf(x))^\frac{1}{\beta}.
$$

By the definition of the function $\Psi$ we have

$$
\Psi \left( \int_D \frac{|f(y)|}{\varphi(|x-y|)^{n-1}} \, dy \right) \leq \Psi \left( C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \log^{\beta(n-1)}(t) \right) = C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right) \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right) \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right) \log^{\beta(n-1)} \left( m + C(Mf(x))^{\frac{ap(n-1)+n-p}{n}} \right).
$$

We estimate the last term and use l’Hôpital’s rule to obtain that the function

$$
z \mapsto \frac{\log(e + z^\frac{1}{\beta})}{\log \left( m + C z^{\frac{ap(n-1)+n-p}{n}} \right)}$$

is bounded on $[0, \infty)$. Hence, the previous inequalities imply that

$$
\Psi \left( \int_D \frac{|f(y)|}{\varphi(|x-y|)^{n-1}} \, dy \right) \leq C(Mf(x))^{\frac{ap(n-1)+n-p}{n}}.
$$
If diam(D) = ∞, the claim follows. If diam(D) < ∞, we have to consider the case (Mf(x))\(^{-\frac{1}{\alpha}}\) ≥ diam(D) and we choose δ = 2 diam(D). By Lemma 3.1 we obtain

\[
\Psi \left( \int_D \frac{|f(y)|}{\varphi((x-y)^n)^{n-1}} dy \right) \leq \int_{B(x,\delta)} \frac{|f(y)|}{\varphi((x-y)^n)^{n-1}} dy \\
\leq C\delta^{(1-n)\alpha \log\beta(n-1)(e+1/\delta)}Mf(x) \\
\leq C(Mf(x))^{1-\frac{\alpha p\gamma^p(n-1)}{n}} \leq C(Mf(x))^{\frac{\alpha p(n-1)+n-pn}{n}}.
\]

\[\square\]

4. The case \(1 < p < n\)

Theorem 2.2 and Theorem 3.3 imply the result in the case \(1 < p < n\) which coincides with [7, Theorem 1.1] when \(\alpha = \beta = 1\). We state the result for \(\alpha \geq 1\) and \(\beta \geq 0\).

**Theorem 4.1.** Suppose that \(1 < p < n\), \(\alpha \in [1, 1 + \frac{1}{n-1}]\), and \(\beta \geq 0\). Let \(\Phi_p : [0, \infty) \to [0, \infty)\) be the function

\[
\Phi_p(t) = \left( \frac{t}{\log\beta(n-1)(m+t)} \right)^{\frac{np}{\alpha p(n-1)+n-pn}}
\]

with \(m \geq c\). If \(D\) is a \(\varphi\)-John domain in \(\mathbb{R}^n\) defined in (1.1), then there exists a constant \(C\) depending on \(\alpha\), \(\beta\), \(n\), and \(D\) only such that the inequality

\[
\|u - u_D\|_{L^p(D)} \leq C\|\nabla u\|_{L^p(D)}
\]

holds for all \(u \in L^1_{1+p}(D)\).

Before the proof we give two remarks.

**Remark 4.2.** Theorem 4.1 coincides with [7, Theorem 1.1] in the case \(\alpha = \beta = 1\). It has been shown in [7, Theorem 6.3] that the result in this case is essentially sharp.

**Remark 4.3.** Theorem 4.1 coincides with the classical case when \(\alpha = 1\) and \(\beta = 0\). But, Theorem 4.1 with \(1 < \alpha\) is not sharp in the case \(\beta = 0\). Namely, the exponent in the Orlicz function \(\Phi_p\) should be \(\frac{np}{\alpha(n-1)-p+1}\), and not \(\frac{np}{\alpha(n-1)+n-pn}\), [5, p. 437], [9, Theorem 2.3]. We suggest a conjecture that \(\frac{np}{\alpha(n-1)-p+1}\) is the right exponent also in the case \(\beta > 0\).

**Proof.** Let us assume that \(\|\nabla u\|_{L^p(D)} \leq 1\). Then, by Theorem 2.2 and Theorem 3.3 the inequality

\[
\Phi_p \left( \|u(x) - u_B(x_0,\text{dist}(x_0,\partial D))\| \right) \leq \Phi_p \left( \int_D \frac{|\nabla u(y)|}{\varphi(|x-y|^n)} dy \right) \leq C(M|\nabla u|(x))^p
\]

holds for every \(x \in D\). We recall that \(M : L^p \to L^p\) is a bounded linear operator whenever \(p > 1\). Thus, integrating over \(D\) gives

\[
\int_D \Phi_p \left( \|u(x) - u_B(x_0,\text{dist}(x_0,\partial D))\| \right) dx \leq C \int_D (M|\nabla u|(x))^p dx \\
\leq C \int_D |\nabla u(x)|^p dx \leq C.
\]
Hence, we have
\[ \|u(x) - u_B(x_0, \text{dist}(x_0, \partial D))\|_{L^{p^*}D} \leq C \]
for every \( u \in L^{1,p}(D) \) with \( \|\nabla u\|_{L^p(D)} \leq 1 \). By using this inequality for \( u/\|\nabla u\|_{L^p(D)} \) we obtain
\[ \|u(x) - u_B(x_0, \text{dist}(x_0, \partial D))\|_{L^{p^*}D} \leq C\|\nabla u\|_{L^p(D)}. \]
The claim follows by the Hölder inequality and the Minkowski inequality.

5. The proof of the main theorem

Proof. Let us consider functions \( u \) such that \( \|u\|_{L^1(D)} \leq 1 \). Let \( B \) be the center ball in Theorem 2.2. It is enough to show that there is a constant \( c < \infty \) such that the inequality
\[ \int_D \Phi_1(|u - u_B|) \, dx < c \]
holds for every \( \|\nabla u\|_{L^1(D)} \leq 1 \). Inequality (5.1) implies that
\[ \|u - u_B\|_{L^{p^*}(D)} < C \]
with some constant \( C \) whenever \( \|\nabla u\|_{L^1(D)} \leq 1 \). By using inequality (5.2) to the function \( u/\|\nabla u\|_{L^1(D)} \) we obtain
\[ \|u - u_B\|_{L^{p^*}(D)} \leq C\|\nabla u\|_{L^1(D)}. \]
By the triangle inequality
\[ \|u - u_D\|_{L^{p^*}(D)} \leq \|u - u_B\|_{L^{p^*}(D)} + \|u_B - u_D\|_{L^{p^*}(D)}. \]
The latter term in (5.4) can be estimated:
\[ \|u_B - u_D\|_{L^{p^*}(D)} = |u_B - u_D| \|1\|_{L^{p^*}(D)} \leq |D|^{-1}\|1\|_{L^{p^*}(D)} \|u - u_B\|_{L^1(D)} \leq C|D|^{-1}\|1\|_{L^{p^*}(D)} \|u - u_B\|_{L^{p^*}(D)} \]
with some constant \( C \). Estimates (5.4), (5.5), and (5.3) yield the claim (1.3). Now we prove inequality (5.1). First we estimate
\[ \int_D \Phi_1(|u - u_B|) \, dx \leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : 2^j < |u(x) - u_B| \leq 2^{j+1}\}} \Phi_1(2^{j+1}) \, dx. \]
Let us define
\[ v_j(x) = \max\left\{ 0, \min\left\{ |u(x) - u_B| - 2^j, 2^j \right\} \right\} \]
for all \( x \in D \). If \( x \in \{x \in D : 2^j < |u(x) - u_B| \leq 2^{j+1}\} \), then \( v_{j-1}(x) \geq 2^{j-1} \). We obtain
\[ \int_D \Phi_1(|u - u_B|) \, dx \leq \sum_{j \in \mathbb{Z}^+} \int_{\{x \in D : v_j(x) \geq 2^j\}} \Phi_1(2^{j+2}) \, dx. \]
By the triangle inequality and Theorem 2.2 we have
\[ v_j(x) = |v_j(x) - (v_j)_B + (v_j)_B| \leq |v_j(x) - (v_j)_B| + |(v_j)_B| \leq C \int_D |\nabla v_j(y)| \varphi(|x - y|)^{n-1} \, dy + |(v_j)_B| \]
\[ \leq C \int_D \frac{|\nabla v_j(y)|}{\varphi(|x - y|)^{n-1}} \, dy + |(v_j)_B| \]
for almost every $x \in D$. By the Poincaré inequality in a ball, there exists a constant $C$ such that

$$|(v_j)_B| = (v_j)_B = \int_B v_j(x) \, dx \leq \int_B |u(x) - u_B| \, dx \leq C \int_B |\nabla u(x)| \, dx \leq |B|^{-1}.$$ 

Thus, $|(v_j)_B|$ is bounded by a constant depending on the distance between the John center and the boundary of $D$ only. Let us write

$$I(\nabla v_j)(x) = \int_D \frac{|\nabla v_j(y)|}{\varphi(|x - y|)^{n-1}} \, dy.$$ 

We continue to estimate the right side of inequality (5.6)

\begin{equation}
\int_D \Phi_1(|u - u_B|) \, dx \leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^j\}} \Phi_1(2^{j+2}) \, dx \\
\leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^{j-1}\}} \Phi_1(2^{j+2}) \, dx + \sum_{j = -\infty}^{j_0} \int_D \Phi_1(2^{j+2}) \, dx.
\end{equation}

(5.7)

Let us first estimate

\begin{equation}
\sum_{j = -\infty}^{j_0} \int_D \Phi_1(2^{j+2}) \, dx = |D| \sum_{j = -\infty}^{j_0} \Phi_1(2^{j+2}) \leq |D| \sum_{j = -\infty}^{j_0} 2^{n(j+2)} \leq C|D|.
\end{equation}

(5.8)

Then we estimate the sum

$$\sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^{j-1}\}} \Phi_1(2^{j+2}) \, dx.$$ 

Since $\|\nabla v_j\|_{L^1(D)} \leq \|\nabla u\|_{L^1(D)} \leq 1$, Theorem 3.3 implies that

$$\sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^{j-1}\}} \Phi_1(2^{j+2}) \, dx = \sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^{j-1}\}} \Phi_1(2^{j+2}) \, dx \\
\leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : CM(\nabla v_j)(x) \geq \Phi_1(2^{j-2})\}} \Phi_1(2^{j+2}) \, dx.$$ 

We choose for every $x \in \{x \in D : CM(\nabla v_j)(x) \geq \Phi_1(2^{j-2})\}$ a ball $B(x, r_x)$ such that

$$C \int_{B(x, r_x)} |\nabla v_j(y)| \, dy \geq \frac{1}{2} \Phi_1(2^{j-1}).$$ 

By the Besicovitch covering theorem (or the 5-covering theorem) we obtain a sub-covering $\{B_k\}_{k=1}^\infty$ so that we may estimate

$$\sum_{j \in \mathbb{Z}} \int_{\{x \in D : CI(\nabla v_j)(x) \geq 2^{j-1}\}} \Phi_1(2^{j+2}) \, dx \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^\infty \int_{B_k} \Phi_1(2^{j+2}) \, dx \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^\infty |B_k| \Phi_1(2^{j+2}) \\
\leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^\infty C|B_k| \frac{\Phi_1(2^{j+2})}{\Phi_1(2^{j-1})} \int_{B_k} |\nabla v_j(y)| \, dy \leq C \sum_{j \in \mathbb{Z}} \int_D |\nabla v_j(y)| \, dy \\
\leq C \int_D |\nabla u(y)| \, dy \leq C.$$
Estimates (5.7), (5.8), and the last estimate imply inequality (5.1) which was enough to prove. □

6. Example

We show that Theorem 1.1 is sharp by constructing a mushrooms-type domain. The construction shows also that the result in Theorem 4.1 is not sharp. Mushrooms-type domains can be found in [12] and [11]. We construct the mushrooms-type domain suitable to our purposes as in [7, Section 6].

Let \( Q_m, m = 1, 2, \ldots \), be a closed cube in \( \mathbb{R}^n \) with side length \( 2r_m \). Let \( P_m, m = 1, 2, \ldots \), be a closed rectangle in \( \mathbb{R}^n \) which has side length \( r_m \) for one side and \( 2\varphi(r_m) \) for the remaining \( n - 1 \) sides. Let \( Q_0 = [0, 1]^n \). We attach \( Q_m \) and \( P_m \), together creating 'mushrooms' which we then attach, as pairwise disjoint sets, to one of the sides of \( Q_0 \). We add the mushrooms to the side of \( Q_0 \) that lies in the hyperplane \( x_2 = 1/2 \). We refer to Figure 1 in the plane case. We wish to define a domain that is symmetric with respect to the hyperplane \( x_2 = 1/2 \). Thus, let \( Q^*_m \) and \( P^*_m \) be the images of the sets \( Q_m \) and \( P_m \), respectively, under the reflection across the hyperplane \( x_2 = 1/2 \). We define

\[
G = \text{int}\left( Q_0 \cup \bigcup_{m=1}^{\infty} (Q_m \cup P_m \cup Q^*_m \cup P^*_m) \right).
\]

We choose \( r_m = 2^{-2m}, m = 1, 2, \ldots \). We let \( \varphi \) be the function from (1.1). Then, the domain \( G \) is a \( \varphi \)-John domain. This can be seen as in [7, Lemma 6.2].

**Theorem 6.1.** Let \( G \) in \( \mathbb{R}^n, n \geq 2 \), be a mushrooms-type domain constructed as in (6.1). Let \( 1 \leq p < n \) and \( \xi \in (0, 1/p) \) be given. If

\[
\Psi_p : t \mapsto \left( \frac{t}{\log^{\varphi(n-1)(m+t)}} \right)^{\frac{n-1}{p}}
\]

with \( m \geq e \), then there does not exist a constant \( C \), independent of the function \( u \), so that the inequality

\[
\|u - u_G\|_{L^{\Psi_p}(G)} \leq C\|\nabla u\|_{L^p(G)}
\]

would be valid for every \( u \in L^1_p(G) \).

**Proof.** Let us define a sequence of piecewise linear continuous functions \( (u_k)_{k=1}^\infty \) by setting

\[
u_k(x) := \begin{cases} v_k & \text{in } Q_k, \\ -v_k & \text{in } Q^*_k, \\ 0 & \text{otherwise,} \end{cases}
\]

where \( v_k := \frac{r_k^{(p-1)/p}}{2n/p\varphi(r_k)^{(n-1)/p}} \).

Then the integral average of \( u_k \) over \( G \) is zero for each \( k \). By a straightforward calculation we obtain

\[
\|\nabla u_k\|_{L^p(G)} = \left( \int_{P_k \cup P^*_k} \left( \frac{v_k}{r_k^p} \right)^p \, dx \right)^{\frac{1}{p}} = 1
\]
for every \( k = 1, 2, \ldots \). If \( k \) is large enough, we estimate

\[
\int_G \left( \frac{|u_k(x)|}{\log^{2(n-1)} \left( m + |u_k(x)| \right)} \right)^{\frac{n}{n \beta (n-1) - p + 1}} \, dx \geq \int_{Q_k} \left( \frac{v_k}{\log^{2(n-1)} \left( m + v_k \right)} \right)^{\frac{n}{n \beta (n-1) - p + 1}} \, dx
\]

\[
> C \left( \frac{\log^{\beta(n-1)} \left( e + 1/r_k \right)}{\log^{\beta(n-1)} \left( m + v_k \right)} \right)^{\frac{n}{n \beta (n-1) - p + 1}} > C \left( \frac{\log^{\beta(n-1)} \left( 1/r_k \right)}{\log^{\beta(n-1)} \left( 2v_k \right)} \right)^{\frac{n}{n \beta (n-1) - p + 1}} =: I_k
\]

If \( \xi < \frac{1}{\beta} \), then \( I_k \) converges to infinity as \( k \to \infty \). This implies that \( \|u_k - 0\|_{L^{p}(G)} = \|u_k\|_{L^{p}(G)} \to \infty \) as \( k \to \infty \). Thus, inequality (6.2) fails. \( \square \)

References

An embedding into an Orlicz space for $L^1$-functions


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