WORKSHOP ON MODERN TRENDS IN CLASSICAL ANALYSIS AND APPLICATIONS

The 2nd Chinese-Finnish Seminar

August 14, 2013
University of Turku, Finland

Program

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Wednesday, August 14

09:00	Matti Vuorinen	University Botanical Garden, Opening remarks
09:05-09:30	Toshiyuki Sugawa	Schwarz-Christoffel formula for the exterior of a quadrangle
09:30-09:50	Tanran Zhang	Uniformisation and description of a once-punctured annulus
09:50-10:00		**Coffee**
10:00-10:25	Riku Klén	Examples of fractals satisfying the quasihyperbolic boundary condition
10:30-10:55	Antti Rasila	Numerical computation of conformal modulus on bad domains
11:00-11:25	Vesna Manojlović	On harmonic quasiregular mappings
11:30-11:55	Xiaohui Zhang	On λ-Lipschitz mappings
12:00-13:00		**Lunch**
13:00-13:25	Parisa Hariri	Distortion Theorems for triangular ratio metric
13:30-13:55	Markus Glader	Subdomain geometry of quasihyperbolic metric and j-metric
14:00-14:25	Chenyan Yang	Inequalities for zero-balanced Hypergeometric functions and p-trigonometric Functions

Subdomain geometry of quasihyperbolic metric and j-metric

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Abstract. In the article [KLV] R. Klén, Y. Li and M. Vuorinen introduce the following problem:

Let D_1 and D_2 be two proper subdomains of $\mathbb{R}^n$ such that $D_1 \subset D_2$ and that $\partial D_1 \cap \partial D_2$ is either empty or discrete set. Does there exist a constant $c > 1$ such that

$$m_{D_1}(z_1, z_2) \geq c m_{D_2}(z_1, z_2) \quad \forall z_1, z_2 \in D_1, \tag{1}$$

where $m_{D_i} \in \{j_{D_i}, k_{D_i}\}$ for $i \in \{1, 2\}$?

The following talk will deal with this problem in the cases of infinite strips, sectors, regular polygons and corresponding circumcircles. Also a connected problem about finding a subregion that fills the above condition will be discussed.

References

[KLV] R. Klén, Y. Li and M. Vuorinen: *Subdomain Geometry of Hyperbolic Type Metrics*, Transactions of Institute of Mathematics of the National Academy of Sciences of Ukraine, Vol. 10, p. 177-194, 2013.

Distortion Theorems for triangular ratio metric

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Abstract.
In this paper we study so called triangular ratio metric and its quasi-invariance under quasi-conformal and quasi-regular maps in punctured space and in half space of $\mathbb{R}^n$.

References

[1] G. D. Anderson, M. K. Vamanamurthy, And M. Vuorinen: Conformal invariants, inequalities and quasiconformal maps. J. Wiley, 1997.

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[0]*represents the speaker in the case of multiple authors.

Examples of fractals satisfying the quasihyperbolic boundary condition

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Abstract. Domains that satisfies the quasihyperbolic boundary condition with a constant $\beta \in (0, 1]$ were introduced in 1985 by F. W. Gehring and O. Martio and after that they have been studied intensively. The constant β has a crucial role in these studies and many properties have been proved in the terms of it. For example in [2] Koskela and Rohde showed that the Minkoski dimension of the boundary is at most $d - c\beta^{d-1}$, where d is the dimension and the constant c depends only on the dimension d. However, there seems to be very few examples where the exact value for β is known.

John domains form a proper subclass of domains that satisfies the quasihyperbolic boundary condition. They were originally introduced in 1961 by F. John but the systematical studies started from [3]. Nowadays John domains are known as the widest class of irregular domains where the classical results still hold. Thus it is surprising that only for trivial examples the value of the parameter is known.

We give explicit examples of bounded domains that satisfy the quasihyperbolic boundary condition and calculate the values for the constants. These domains are also John domains and we calculate John constants as well. The talk is based on manuscript [1].

References

[1] P. Harjulehto and R. Klén: Examples of fractals satisfying the quasihyperbolic boundary condition, preprint, arXiv:1205.1873.

[2] P. Koskela and S. Rohde: Hausdorff dimension and mean porosity, *Math. Ann.* **309** (1997), no. 4, 593–609.

[3] O. Martio and J. Sarvas: Injective theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1978/1979), 383–401.

On harmonic quasiregular mappings

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Abstract. This talk is based on joint results by M. Vuorinen and V. Manojlovic, including also results on moduli of continuity on the boundary of domain an inside the domain of harmonic quasiconformal/quasiregular mappings.

References

[VeM] V. Manojović *Harmonic Quasiconformal Mappings in Domains in R^n* J. Analysis 18 (2010) 297-316.

[AMV] M. Arsenović, V. Manojlović, M. Vuorinen *Hölder continuity of harmonic quasiconformal mappings* Journal of Inequalities and Applications, (2011): 37

[AAMS] A. Abaob, M. Arsenović, M. Mateljević, A. Shkheam *Moduli of Continuity of Harmonic Quasiregular Mappings on Bounded Domains*

Numerical computation of conformal modulus on bad domains

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Abstract. Conformal moduli of quadrilaterals and rings are fundamental tools in geometric function theory, and these quantities also have applications in computational science and engineering. It is usally not possible to compute the values of conformal modulus analytically, but several algorithms for their numerical approximation has been developed (see e.g. [DT, K, LV, PS] and references therein).

In this talk numerical computation the conformal modulus of rings and quadrilaterals with strong singularities and cusps on their boundary is discussed. We reduce this problem to the numerical solution of the associated Dirichlet and Dirichlet-Neumann type boundary values problems for the Laplace equation (see [HRV1, HRV2]). In addition, it is possible to reconstruct the canonical conformal mapping from the potential function [HQR].

Several experimental results, with error estimates, are reported. In particular, we consider domains with dendrite like boundaries, where an analytic formula for the conformal modulus can be derived. The numerical method makes use of an *hp*-FEM algorithm [BS, HRV1], written for this very complicated geometry with strong singularities [HRV3]. This presentation is based on joint research with H. Hakula and M. Vuorinen.

References

[BS] I. Babuška and M. Suri, *The p and h-p versions of the finite element method, basic principles and properties.* SIAM Review 36 (1994), 578–632.

[DT] T.A. Driscoll and L.N. Trefethen, *Schwarz-Christoffel mapping.* Cambridge Monographs on Applied and Computational Mathematics, 8. Cambridge University Press, Cambridge, 2002.

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[HRV3] H. Hakula, A. Rasila, and M. Vuorinen, *Conformal modulus on domains with strong singularities and cusps.* Manuscript.

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[LV] O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, 2nd edition, Springer, Berlin, 1973.

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Schwarz-Christoffel formula for the exterior of a quadrangle

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Abstract. It is an interesting but difficult problem to express explicitly a conformal mapping of the upper half-plane onto the exterior of a prescribed quadrangle. Such a mapping is indeed given by a sort of Schwarz-Christoffel formula. The difficulty lies in the determination of the four points on the real line, which correspond to the vertices. This is equivalent to determine the accessary parameters. It is possible, however, to do so when the quadrangle has a certain symmetry. We will treat the case when the quadrangle is symmetric in a line; namely, it is an isosceles trapezium. In this case, we can express the for points in terms of the Appel hypergeometric function F_1 of two variables.

Acknowledgements. The present research has emerged in the discussion with Matti Vuorinen.

Inequalities for zero-balanced Hypergeometric and p-trigonometric Functions

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Abstract. It is well known that the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ has perhaps the widest range of applications among special functions. Many other important special functions are special cases of ${}_2F_1(a,b;c;z)$. The so called zero-balanced hypergeometric functions ${}_2F_1(a,b;a+b;z), \quad a,b>0$ contain, as a special case, the complete elliptic integral of the first kind. Recently, many authors have studied generalized trigonometric and hyperbolic functions [KVZ, N]. Here I proved two inequalities for zero-balanced hypergeometric functions and inequalities for p-trigonometric functions.

References

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[KVZ] R. Klén,M. Vuorinen,and X.-H.Zhang: *Inequalities for the generalized trigonometric and hyperbolic functionds.* J.Math.Anal.Appl(to appear),available at http://dx.doi.org/10.1016/j.jmaa.2013.07.021

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Uniformisation and description of a once-punctured annulus

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Abstract. The Uniformisation Theorem indicates that if $\widetilde{X}$ is homeomorphic to a proper subset of $\mathbb{C}$ with at least three boundary points, then $\widetilde{X}$ is conformally equivalent to a quotient space $\mathbb{D}/G$, where G is a torsion-free Fuchsian group that acts (discontinuously) on $\mathbb{D}$ (or $\mathbb{H}$). The group G is isomorphic to $\Pi_1(X)$. Hempel and Smith studied the hyperbolic Riemann surface model of the twice-punctured disk $\mathbb{D}\backslash\{p1, p2\}$ in 1980s. They estimated the hyperbolic density on it near aone puncture and considered the coalescing of the two punctures. Later on Beardon gave five different ways to uniformize $\mathbb{D}\backslash\{p1, p2\}$ in 2012. He investigated several conformal invariants to characterize $\mathbb{D}\backslash\{p1, p2\}$ considering the fundamental domain, symmetric collars and extremal length. We extend his work to the once-punctured annulus $A := \{z : 1/R < |z| < R\}\backslash\{a\}$, $R > 1$, $1/R < a < R$. We provide several parameter pairs to uniformize and characterize A. We give the explicit formulas for the largest non-overlapped collars and the extremal lengths. The main tools we use are Möbius transformations, covering space, homotopy classes and elliptic integrals.

Acknowledgements. I would like to thank Prof. Toshiyoki Sugawa for his proposal for this topic and his suggestions on the proofs.

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On λ-Lipschitz mappings

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Abstract. The conformal invariant λ was introduced in [F1] in the general framework of non-compact Riemannian manifolds. In the more elementary case of a domain D in $\overline{\mathbb{R}^n}$ we can define the function $\lambda_D(x,y)$ for $x,y \in D$, $x \neq y$, by

$$\lambda_D(x,y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; D)),$$

where C_z is any curve in D joining z and the boundary of D. As shown in [F2], $\lambda_D(x,y)^{1/(1-n)}$ is always a metric on an arbitrary subdomain D of $\mathbb{R}^n$ with $\mathrm{Card}(\mathbb{R}^n \setminus D) \geq 2$.

We say that a mapping $f : D \to f(D) \subset H$ is λ-Lipschitz if there exists a constant K such that

$$\lambda_D(x,y) \leq K\lambda_H(f(x), f(y))$$

for all $x, y \in D$. Since that $(D, \lambda_D^{1/(1-n)})$ is a metric space, this terminology makes sense. J. Ferrand, G.J. Martin and M. Vuorinen [FMV] showed that λ-Lipschitz mappings are Hölder continuous in the Euclidean metric and that they contain quasiconformal mappings as a proper subclass. Moreover, they proved that many important properties of the quasiconformal mappings are also enjoyed by these more general mappings. By the Riemann mapping theorem, however, it is easy to see that a $\lambda-$isometry is conformal for $n = 2$ when the domain is simply connected and for $n \geq 3$ when the domain is the unit ball. It is conjectured that a $\lambda-$isometry is always conformal for general proper subdomains of $\mathbb{R}^n$. As an attempt to establish the general case we shall prove that λ-isometries are at least quasiconformal.

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