

# **CONFORMAL GEOMETRY AND QUASIREGULAR MAPPINGS**

Matti Vuorinen  
University of Helsinki  
Helsinki, Finland

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## Preface

This book is based on my lectures on quasiregular mappings in the euclidean  $n$ -space  $\mathbf{R}^n$  given at the University of Helsinki in 1986. It is assumed that the reader is familiar with basic real analysis or with some basic facts about quasiconformal mappings (an excellent reference is pp. 1–50 in J. Väisälä's book [V7]), but otherwise I have tried to make the text as self-contained and easily accessible as possible. For the reader's convenience and for the sake of easy reference I have included without proof most of those results from [V7] which will be exploited here. I have also included a brief review of those properties of Möbius transformations in  $\mathbf{R}^n$  which will be used throughout.

In order to make the text more useful for students I have included nearly a hundred exercises, which are scattered throughout the book. They are of varying difficulty, with hints for solution provided for some. For specialists in the field I have included a list of open problems at the end of the book. The bibliography contains, besides references, additional items which are closely related to the subject matter of this book.

From its beginning twenty years ago the subject of quasiregular mappings in  $n$ -space has developed into an extensive mathematical theory having connections with PDE theory, calculus of variations, non-linear potential theory, and especially geometric function theory and quasiconformal mapping theory. Excellent contributions to this subject have been made, in particular, by the following five mathematicians:

F. W. Gehring, O. Martio, Yu. G. Reshetnyak, S. Rickman, and J. Väisälä.

The subject matter of this book relies heavily on their work. I am indebted to them not only for their scientific contributions but also for the help and advice they have given me during the various stages of my work. It was O. Martio who suggested I start writing this book. The writing was made possible by a research fellowship of the Academy of Finland, which I held in 1979–85. A draft for the text was finished in the

fall of 1982 during my stay at the Mittag–Leffler Institute in Sweden.

The following mathematicians have provided their generous help by checking various versions of the manuscript, pointing out errors, and contributing corrections: J. Heinonen, G. D. Anderson, and M. K. Vamanamurthy. Useful remarks were also made by J. Ferrand and P. Järvi. At the final stage I have had the good fortune to work with J. Kankaanpää, who prepared the final version of the text using the  $\text{T}_{\text{E}}\text{X}$  system of D. E. Knuth and improved the text in various ways. The previewer program for  $\text{T}_{\text{E}}\text{X}$  written by A. Hohti was very helpful in the course of this project. The work of Kankaanpää was supported by a grant of the Academy of Finland. Hohti and O. Kanerva have provided their generous assistance in the use of the  $\text{T}_{\text{E}}\text{X}$  system.

Helsinki

October 1987

Matti Vuorinen

## Introduction

Quasiconformal and quasiregular mappings in  $\mathbf{R}^n$  are natural generalizations of conformal and analytic functions of one complex variable, respectively. In the two-dimensional case these mappings were introduced by H. Grötzsch [GRÖ] in 1928 and the higher-dimensional case was first studied by M. A. Lavrent'ev [LAV] in 1938. Far-reaching results were obtained also by O. Teichmüller [TE] and L. V. Ahlfors [A1]. The systematic study of quasiconformal mappings in  $\mathbf{R}^n$  was begun by F. W. Gehring [G1] and J. Väisälä [V1] in 1961, and the study of quasiregular mappings by Yu. G. Reshetnyak in 1966 [R1]. In a highly significant series of papers published in 1966–69 Reshetnyak proved the fundamental properties of quasiregular mappings by exploiting tools from differential geometry, non-linear PDE theory, and the theory of Sobolev spaces.

In 1969–72 O. Martio, S. Rickman and J. Väisälä ([MRV1]–[MRV3], [V8]) gave a second approach to the theory of quasiregular mappings which was based on some results of Reshetnyak, most notably on the fact that a non-constant quasiregular mapping is discrete and open. On the other hand, their approach made use of tools from the theory of quasiconformal mappings, such as curve families and moduli of curve families. The extremal length and modulus of a curve family were introduced by L. V. Ahlfors and A. Beurling in their celebrated paper [AB] on conformal invariants in 1950.

A third approach was suggested by B. Bojarski and T. Iwaniec [BI2] in 1983. Their methods are real analytic in nature and largely independent of Reshetnyak's work.

In this book a fourth approach is suggested, which is a ramification of the curve family method in [MRV1]–[MRV3] and in which conformal invariants play a central role. Each of the above three approaches yields a theory covering the whole spectrum of results of the theory of quasiregular mappings. So far the fourth approach of this book, introduced by the author in [VU10]–[VU13] has been applied mainly to distortion theory. This work has been continued in [AVV1], [AVV2], [FV], [LEVU], where some

quantitative distortion theorems were discovered. These papers also include results which are sharp as the maximal dilatation  $K$  approaches 1. Perhaps surprisingly it also turned out in [AVV1] that to a considerable degree a distortion theory can be developed independently of the dimension  $n$ .

In short, this fourth approach consists of the following. In a domain  $G$  in  $\mathbf{R}^n$  one studies two conformal invariants  $\lambda_G(x, y)$  and  $\mu_G(x, y)$  associated with a pair of points  $x$  and  $y$  in  $G$ . These invariants were apparently first introduced by J. Ferrand [LF2] in 1973 and I. S. Gál [GÁL] in 1960, respectively. The systematic application of these invariants was begun by the author in a recent series of papers [VU10]–[VU13]. By their definitions,  $\lambda_G(x, y)$  and  $\mu_G(x, y)$  are solutions of certain extremal problems associated with the moduli of some curve families. To derive distortion theorems exploiting  $\lambda_G$  and  $\mu_G$  we require two things:

- (a) the quasiinvariance of moduli of curve families under quasiconformal and quasiregular mappings ([MRV1]–[MRV3]),
- (b) quantitative estimates for  $\lambda_G$  and  $\mu_G$  in terms of “geometric quantities”.

For a general domain  $G$  in  $\mathbf{R}^n$  these invariants have no explicit expression. In the particular case  $G = \mathbf{B}^n$  such an expression is known for both  $\lambda_G$  and  $\mu_G$ , and for  $G = \mathbf{R}^n \setminus \{0\}$  good two-sided estimates for the invariant  $\lambda_G$  will be obtained. We then generalize these results for a wider class of domains. In the two-dimensional case we can obtain the exact value of  $\lambda_{\mathbf{R}^2 \setminus \{0\}}(x, y)$  if we use the solution of a classical extremal problem of geometric function theory, the modulus problem of O. Teichmüller [KU, Ch. V].

This book is divided into four chapters. Chapter I deals with geometric preliminaries, including a discussion of Möbius transformations. In Chapter II we study certain conformal invariants and apply these results in Chapter III to obtain distortion theorems, the main theme of this book. The final part, Chapter IV, is a brief discussion of some boundary properties of quasiconformal mappings.

## A survey of quasiregular mappings

The goal of this survey is to give the reader a brief overview of the theory of quasiconformal (qc) and quasiregular (qr) mappings and of some related topics. We shall also try to indicate the many ways in which the classical function theory of one complex variable (CFT) is related to quasiregular mapping theory (QRT) in  $\mathbf{R}^n$  as well as to point out some differences between CFT and QRT. This survey deals chiefly with results not discussed elsewhere in the book.

For a general orientation the reader is urged to read some of the existing excellent surveys [A4], [L1], [L2], [BAM], [G4], [G8]–[G10], [I], and [V10], of which the first three deal with the two-dimensional case and the others the multidimensional case. Several open problems are listed in the surveys of A. Baernstein and J. Manfredi [BAM], F. W. Gehring [G9], and J. Väisälä [V10].

**1. Foundations.** In his pioneering papers [R1]–[R10], in which were laid the foundations of QRT, Yu. G. Reshetnyak successfully combined the powerful analytic machinery of PDE's in the sense of Sobolev with some geometric ideas from CFT. Reshetnyak showed that the basic properties of qr mappings can be derived from the properties of the function  $u_f(x) = \log |f(x)|$ , where  $f$  is qr. He proved that  $u_f$  satisfies a non-linear elliptic PDE which for  $n = 2$  is linear and coincides with the Laplace equation. It follows from the work of J. Moser [MOS], F. John – L. Nirenberg, and J. Serrin [SE] that the solutions of this equation satisfy the Harnack inequality in  $\{z : u_f(z) > 0\}$ . Note that if  $f$  is analytic, then  $\log |f(z)|$  has a similar role in CFT. Obviously only a part of CFT can be carried over to QRT: for instance power series expansions and the Riemann mapping theorem have no  $n$ -dimensional counterpart.

**2. Quasiconformal balls.** By Riemann's mapping theorem a simply-connected plane domain with more than one boundary point can be mapped conformally onto the unit disk  $\mathbf{B}^2$ . Liouville's theorem says that the only conformal mappings in  $\mathbf{R}^n$ ,  $n \geq 3$ , are the Möbius transformations. Thus Riemann's mapping theorem has

no counterpart in  $\mathbf{R}^n$  when  $n \geq 3$ : since Möbius transformations preserve spheres, the unit ball  $\mathbf{B}^n$  in  $\mathbf{R}^n$  can be mapped conformally only onto another ball or a half-space. A quasiconformal counterpart of the Riemann mapping theorem is also false: for  $n \geq 3$  there are Jordan domains in  $\mathbf{R}^n$  homeomorphic to  $\mathbf{B}^n$  which cannot be mapped quasiconformally onto  $\mathbf{B}^n$  although their complements can be so mapped. Also, the unit ball  $\mathbf{B}^n$ ,  $n \geq 3$ , can be mapped quasiconformally onto a domain with non-accessible boundary points, as shown by Gehring and Väisälä in [GV1]. This fact shows that for each  $n \geq 3$  the quasiconformal mappings in  $\mathbf{R}^n$  constitute a class of mappings substantially larger than the class of Möbius transformations.

**3. Topological properties.** A basic fact from CFT is that a non-constant analytic function is discrete (i.e. point-inverses  $f^{-1}(y)$  are discrete sets if  $f$  analytic) and open (i.e.  $fA$  is open whenever  $f$  is analytic and  $A$  is open). By Reshetnyak's fundamental work a similar result holds in QRT. Next let  $B_f$  denote the set of all points where  $f$  fails to be a local homeomorphism. In CFT it is a basic fact that  $B_f$  is a discrete set if  $f$  is non-constant and analytic. A topological difference between the cases  $n = 2$  and  $n \geq 3$  is that  $B_f$  is never discrete if  $f$  is qr in  $\mathbf{R}^n$ ,  $n \geq 3$ , and  $B_f \neq \emptyset$ . By a result of A. V. Chernavskii  $\dim B_f = \dim fB_f \leq n - 2$  if  $f: G \rightarrow \mathbf{R}^n$  ( $G$  a domain in  $\mathbf{R}^n$ ) is discrete and open ([CHE1], [CHE2], [V5]). Also the metric properties are different: if  $n = 2$  and  $f$  is analytic, then  $\text{cap } B_f = 0$ , while if  $n \geq 3$  and  $f$  is qr in  $\mathbf{R}^n$ , then either  $B_f = \emptyset$  or  $\text{cap } B_f > 0$  (for the definition of the capacity see Section 7; see also [R10], [MR2], [S2]).

By a result of S. Stoilow a qr mapping  $f$  of  $\mathbf{B}^2$  onto a domain  $D$  can be represented as  $f = g \circ h$ , where  $h$  is a qc mapping of  $\mathbf{B}^2$  onto itself and  $g$  is an analytic function ([LV2]). Thus the powerful two-dimensional arsenal of CFT is applicable to the "analytic part" of  $f$ , greatly facilitating the study of two-dimensional qr mappings. No such result is known for the multidimensional case.

Another result which is known only for the dimension  $n = 2$  is the powerful existence theorem for plane quasiconformal mappings (cf. [LV2]). In the multidimensional case there is no general existence theorem and all examples of qc and qr mappings known to the author are based on direct constructions. In the qc case several examples are given in [GV1]. In the qr case a basic mapping is the winding mapping, given in the cylindrical coordinates  $(r, \varphi, z)$  by  $(r, \varphi, z) \mapsto (r, k\varphi, z)$ ,  $k$  a positive integer [MRV1]. An important example of a qr mapping is the so called Zorich mapping ([ZO1], [MSR1]) and its various generalizations due to Rickman (cf. e.g. [RI1]).

Additional examples are given in [R12, pp. 27–32], [MSR2], and [MSR3]. One can also construct new qc (qr) mappings by composing qc (qr) mappings.

**4. Quasiconformality versus Lipschitz and Hölder maps.** A homeomorphism  $f: G \rightarrow fG$ ,  $G \subset \mathbb{R}^n$ , is said to be  $K$ -qc if

$$(*) \quad M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)$$

for all curve families  $\Gamma$  in  $G$  where  $M(\Gamma)$  is the modulus of  $\Gamma$  (see Section 5 below). This definition is somewhat implicit because the concept of modulus is rather complicated. To clarify the geometric consequences of (\*) let us point out that

$$H(x, f) = \limsup_{r \rightarrow 0} \left\{ \frac{|f(x) - f(z)|}{|f(x) - f(y)|} : |z - x| = r = |y - x| \right\} \leq d(n, K)$$

for all  $x \in G$ , where  $d(n, K) < \infty$  depends only on  $n$  and  $K$ . A well-known property of conformal mappings can be expressed by stating that  $H(x, f) = 1$  for  $K = 1$  (while, unfortunately,  $d(n, K) \not\rightarrow 1$  as  $K \rightarrow 1$  for  $n \geq 3$ , cf. p. 193).

A homeomorphism  $f: G \rightarrow fG$  satisfying

$$|x - y|/L \leq |f(x) - f(y)| \leq L|x - y|$$

for all  $x, y \in G$ , is called  $L$ -bilipschitz. It is easy to show that  $L$ -bilipschitz maps are  $L^{2(n-1)}$ -qc. But the converse is false. The standard counterexample is the qc radial stretching  $x \mapsto |x|^{\alpha-1}x$ ,  $x \in \mathbb{B}^n$ ,  $\alpha \in (0, 1)$ , which is not bilipschitz. All qc mappings are, however, locally Hölder continuous; e.g., if  $f: \mathbb{B}^n \rightarrow \mathbb{B}^n$  is  $K$ -qc, then for  $|x|, |y| \leq \frac{1}{2}$

$$|f(x) - f(y)| \leq A(n, K) |x - y|^\alpha, \quad \alpha = K^{1/(1-n)},$$

where  $A(n, K)$  depends only on  $n$  and  $K$ . For details see Section 11 below.

Let  $\mathcal{B}$ ,  $\mathcal{QC}$ , and  $\mathcal{H}$  denote the classes of all bilipschitz, qc, and locally Hölder continuous mappings. By what was said above the inclusions  $\mathcal{B} \subset \mathcal{QC} \subset \mathcal{H}$  hold, where the first inclusion is strict. Simple examples can be constructed to show that also the second inclusion is strict.

Many fundamental features of qc mappings are related to the strictness of the inclusion  $\mathcal{B} \subset \mathcal{QC}$ . For instance, one can construct qc mappings such that the image of a segment is not even locally rectifiable and such that the Hausdorff dimension of a set is different from the Hausdorff dimension of its image ([GV2]).

The Hölder continuity of qc mappings on the boundary of the domain of definition has been thoroughly investigated by R. Näkki and B. Palka in a series of papers (see e.g. [NP]).

**5.  $L^p$ -integrability.** A  $K$ -qc mapping has the property that its partial derivatives are locally  $L^n$ -integrable. Moreover, these partial derivatives are even locally  $L^p$ -integrable for some  $p = p(n, K) > n$ . This was proved by B. Bojarski for  $n = 2$  and generalized to the multidimensional case by F. W. Gehring [G5]. The method of proof in [G5], which makes use of so-called reverse Hölder inequalities, has found several applications to the calculus of variations and to PDE theory ([GIA], [STR1], [STR2]). Some estimates dealing with the case  $K \rightarrow 1$  were given by Yu. G. Reshetnyak in [R13] (see also [GUR]). In connection with qr mappings the integrability has also been discussed by B. Bojarski and T. Iwaniec [BI2] and O. Martio [M2].

**6. Stability theory.** The stability theory of  $K$ -qc and  $K$ -qr mappings in  $\mathbf{R}^n$  in the sense of this book deals with the quantitative description of the behavior of these mappings when  $K \rightarrow 1$ . Roughly speaking, the expectation is that the mapping should become more or less like a conformal mapping under this passage to the limit. By Liouville's classical theorem the two cases  $n \geq 3$  and  $n \geq 2$  are substantially different, and we shall therefore consider them separately.

*Case A.  $n \geq 3$ .* Liouville's classical theorem, which was mentioned above in connection with quasiconformal balls, requires that the mappings be sufficiently smooth ( $C^3$  is enough). By deep results of F. W. Gehring [G2] and Yu. G. Reshetnyak ([R3], [R13]) the differentiability assumption can be replaced by the requirement that the mapping be 1-qr or even 1-qr. Recently a different proof was given by B. Bojarski and T. Iwaniec [BI1]. Next, as shown by Reshetnyak ([R3], [R11], [R13]), one can show that as  $K \rightarrow 1$  any  $K$ -qr mapping must approach a Möbius transformation. For the exact statement of these results the reader is referred to [R13]. The methods of [R13] involve normal family arguments. Unfortunately the "speed" with which the convergence to Möbius transformations takes place as  $K \rightarrow 1$  is usually only qualitatively defined and no quantitative estimate for the "speed" in terms of  $K$  and  $n$  are known. Additional results have been proved by A. P. Kopylov [KO], J. Sarvas [S3], V. I. Semenov [SEM1], D. A. Trotsenko [TR], and others.

*Case B.  $n \geq 2$ .* The paucity of such distortion theorems for  $K$ -qc or  $K$ -qr mappings in  $\mathbf{R}^n$ , which are asymptotically sharp as  $K \rightarrow 1$  and provide quantitative distortion estimates, may be startling when compared to the rich qualitative theory described above in Case A. This state of affairs is due partly to the fact that to prove such results one needs to find sharp estimates for certain little-known special functions. Several results with explicit bounds dealing with the case  $K \rightarrow 1$  have

been proved by V. I. Semenov in several papers (e.g. [SEM1], [SEM2]). Some other distortion theorems of this kind together with associated estimates of special functions were developed in [VU10], [VU11], [AVV1]–[AVV3], [FV].

A survey including some two-dimensional results of this kind is given in [HEL]. See also the important paper [AG] of S. Agard.

**7. Dirichlet integral minimizing property.** Let  $G$  be a domain in  $\mathbf{R}^2$  and  $v: G \rightarrow \mathbf{R}$  harmonic. For a domain  $D \subset G$  with  $\bar{D} \subset G$  let

$$\mathcal{F}_v(D) = \{ u: G \rightarrow \mathbf{R} : u|_{\partial D} = v|_{\partial D}, u \in C^2(G) \}.$$

A well-known extremal property of the class of harmonic functions, the Dirichlet principle, states that they minimize the Dirichlet integral [T, pp. 9–14]. In the above notation this means that

$$\int_D |\nabla v|^2 dm = \inf_{u \in \mathcal{F}_v(D)} \int_D |\nabla u|^2 dm.$$

Analogous Dirichlet integral minimizing properties hold as well for the solutions of the non-linear elliptic PDE's which arise in connection with qr mappings. This important fact was proved by Yu. G. Reshetnyak [R5]. In [MIK3] V. M. Miklyukov continued this research and studied subsolutions of these PDE's.

In a series of papers S. Granlund, P. Lindqvist, and O. Martio have considerably extended these results ([GLM1]–[GLM3], [LI1], [LIM], [M6]). They have also found a unified approach to some function-theoretic parts of QRT including, in particular, the harmonic measure. See also [HMA]. Further results were obtained by J. Heinonen and T. Kilpeläinen.

**8. Value distribution theory.** In 1967 V. A. Zorich [ZO1] asked whether Picard's theorem holds for spatial qr mappings and whether the value distribution theory of Nevanlinna [NE] has a counterpart in this context. These questions have been answered by S. Rickman in a series of papers [RI3]–[RI11], the main results being reviewed in [RI6] and [RI9]. Additional results appear in [MATR] as well as in [PE1]. An analogue of Picard's theorem was published in [RI4]. One of the methods used in [RI4] is a two-constants theorem for qr mappings (analogous to the two-constants theorem of CFT [NE]), which Rickman derives from an estimate for the solutions of certain non-linear elliptic PDE's due to V. G. Maz'ya [MAZ1]. An alternative proof which only makes use of curve family methods is given in [RI9].

**9. Special classes of domains.** The standard domain, in which most of the CFT is developed, is the unit disk. During the past ten years an increasing number of papers have been published in which function-theory on a more general domain arises in a natural way. In the early 1960's two highly significant studies of this kind appeared in quite different contexts authored by L. V. Ahlfors and F. John, respectively. Ahlfors studied domains bounded by quasicircles, i. e. images of the usual circle under a qc mapping of  $\mathbf{R}^2$ , and found remarkable properties of these domains. In a paper related to elasticity properties of materials John introduced a class of domains, nowadays known as John domains.

The importance of John domains was pointed out by Yu. G. Reshetnyak [R11] in connection with injectivity studies of qr mappings. This direction of research was then continued by O. Martio and J. Sarvas [MS2], who also introduced the important class of uniform domains. Uniform domains have found applications in the study of extension operators of function spaces, e. g. in P. Jones' work ([J1], [J2]) as well as elsewhere ([GO], [GM1], [TR], [V12]). Other related classes of domains are QED domains [GM1] and  $\varphi$ -uniform domains ([VU10], [HVU]). The interrelation between some of these classes of domains has been studied by F. W. Gehring in [G8] and [G10], where also several characterizations of quasidisks are given.

Important results dealing with function spaces and their extension to a larger domain have been proved by S. K. Vodop'yanov, V. M. Gol'dstein, and Yu. G. Reshetnyak in [VGR], where additional references can be found.

**10. Concluding remarks.** The above remarks cover only a part of the existing QRT, and a wider overview can be obtained from the surveys of A. Baernstein and J. Manfredi [BAM] and F. W. Gehring [G9]. We shall conclude this survey by mentioning some directions of active research close to QRT.

Recently qc and qr mappings have appeared in stochastic analysis in B. Øksendal's work [ØK1] and in the theory of manifolds (M. Gromov [GROM]). P. Pansu [PA] has studied quasiconformality in connection with Heisenberg groups, in which he has exploited among other methods the conformal invariant  $\lambda_G$  of J. Ferrand [LF2]. Qc mappings also arise in a natural way in the study of BMO functions (H. M. Reimann-T. Rychener [REIR], K. Astala-F. W. Gehring [ASTG], M. Zinsmeister [ZI]).

In a series of papers V. M. Miklyukov [MIK4] has shown how the extremal length method can be used to study minimal surfaces. Extremely important are the partly topological results connecting geometric topology and quasiconformality, which were

proved by D. Sullivan, P. Tukia, J. Väisälä, J. Luukkainen, and others. Discrete groups and quasiconformality have been studied in an important series of papers by P. Tukia ([TU1], [TU2]) and B. N. Apanasov, O. Martio and U. Srebro ([MSR1]–[MSR3]), F. W. Gehring and G. Martin [GMA]. Let us point out that we have confined ourselves here (and also elsewhere in this book) to the case of  $n$ -space,  $n \geq 2$ . For  $n = 2$  the reader may consult the excellent surveys of O. Lehto [L1] and [L2] as well as his new book [L3]. The standard references for  $n = 2$  are the books by L. V. Ahlfors [A2], H. P. Künzi [KÜ], and O. Lehto and K. I. Virtanen [LV2].

The variety of these results indicates the many ways in which qc and qr mappings arise in mathematics. Many fascinating connections between QRT and other parts of mathematics remain yet to be discovered.

## Notation and terminology

The standard unit vectors in the euclidean space  $\mathbf{R}^n$ ,  $n \geq 2$ , are denoted by  $e_1, \dots, e_n$ . A point  $x$  in  $\mathbf{R}^n$  can be represented as a vector  $(x_1, \dots, x_n)$  or as a sum of vectors  $x = x_1 e_1 + \dots + x_n e_n$ . For  $x, y \in \mathbf{R}^n$  the inner product is defined by  $x \cdot y = \sum_{i=1}^n x_i y_i$ . The length (norm) of  $x \in \mathbf{R}^n$  is  $|x| = (x \cdot x)^{1/2}$ . The ball centered at  $x \in \mathbf{R}^n$  with radius  $r > 0$  is  $B^n(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$  and the sphere with the same center and radius is  $S^{n-1}(x, r) = \{y \in \mathbf{R}^n : |x - y| = r\}$ . We employ the abbreviations

$$\begin{aligned} B^n(r) &= B^n(0, r), & \mathbf{B}^n &= B^n(1), \\ S^{n-1}(r) &= S^{n-1}(0, r), & S^{n-1} &= S^{n-1}(1). \end{aligned}$$

The  $n$ -dimensional volume of  $\mathbf{B}^n$  is denoted by  $\Omega_n$  and the  $(n-1)$ -dimensional surface area of  $S^{n-1}$  by  $\omega_{n-1}$ . For  $x, y \in \mathbf{R}^n$  let  $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$  and for  $x \in \mathbf{R}^n \setminus \{0\}$  let  $[x, \infty] = \{sx : s \geq 1\} \cup \{\infty\}$ . The Möbius space  $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$  is the one-point compactification of  $\mathbf{R}^n$ . The Möbius space, equipped with the spherical chordal distance  $q$ , is a metric space. In addition to  $(\mathbf{R}^n, | \cdot |)$  and  $(\overline{\mathbf{R}}^n, q)$  we shall require some other metric spaces such as the hyperbolic spaces  $(\mathbf{B}^n, \rho_{\mathbf{B}^n})$  and  $(\mathbf{H}^n, \rho_{\mathbf{H}^n})$  as well as  $(G, k_G)$  where  $G \subset \mathbf{R}^n$  is a domain and  $k_G$  is the quasihyperbolic metric on  $G$ .

For a metric space  $(X, d)$  let  $B_X(y, r) = \{x \in X : d(x, y) < r\}$ . If  $A, B \subset X$  are non-empty let  $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$  and  $d(A) = \sup\{d(x, y) : x, y \in A\}$ . For  $x \in X$  set  $d(x, A) = d(\{x\}, A)$ .

The set of natural numbers  $0, 1, 2, \dots$  is denoted by  $\mathbf{N}$  and the set of all integers by  $\mathbf{Z}$ . The set of complex numbers is denoted by  $\mathbf{C}$ . We often identify  $\mathbf{C} = \mathbf{R}^2$ .

For a set  $A$  in  $\mathbf{R}^n$  or  $\overline{\mathbf{R}}^n$  the topological operations  $\overline{A}$  (closure),  $\partial A$  (boundary),  $\overline{\mathbf{R}}^n \setminus A$  (complement) are always taken with respect to  $\overline{\mathbf{R}}^n$ . Thus the domain  $\mathbf{R}^n \setminus \{0\}$  has two boundary points,  $0$  and  $\infty$ , and the half-space  $\mathbf{H}^n = \{x \in \mathbf{R}^n : x_n > 0\}$  has  $\infty$  as a boundary point. A domain is an open connected non-empty set. A neighborhood of a point is a domain containing it. The notation  $f: D \rightarrow D'$  usually includes the assumption that  $D$  and  $D'$  are domains in  $\overline{\mathbf{R}}^n$ .

Let  $G$  be an open set in  $\mathbf{R}^n$ . A mapping  $f: G \rightarrow \mathbf{R}^m$  is differentiable at  $x \in G$  if there exists a linear mapping  $f'(x): \mathbf{R}^n \rightarrow \mathbf{R}^m$ , called the derivative of  $f$  at  $x$ , such that

$$f(x+h) = f(x) + f'(x)h + |h|\epsilon(x,h)$$

where  $\epsilon(x,h) \rightarrow 0$  as  $h \rightarrow 0$ . The Jacobian determinant of  $f$  at  $x$  is denoted by  $J_f(x)$ . Assume next that  $n = m$  and that all the partial derivatives exist at  $x \in G$  (thus  $f$  need not be differentiable at  $x$ ). In this case one defines the formal derivative of  $f = (f_1, \dots, f_n)$  at  $x$  as the linear map defined by

$$f'(x)e_i = \nabla f_i(x) = \left( \frac{\partial f_i}{\partial x_1}(x), \dots, \frac{\partial f_i}{\partial x_n}(x) \right), \quad i = 1, \dots, n.$$

For an open set  $D \subset \mathbf{R}^n$  and for  $k \in \mathbf{N}$ ,  $C^k(D)$  denotes the set of all those continuous real-valued functions of  $D$  whose partial derivatives of order  $p \leq k$  exist and are continuous.

The  $n$ -dimensional volume of the unit ball  $m_n(\mathbf{B}^n)$  is denoted by  $\Omega_n$  and the  $(n-1)$ -dimensional surface area of  $S^{n-1}$  by  $\omega_{n-1}$ . Then  $\omega_{n-1} = n\Omega_n$  and

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2}n)}$$

for all  $n = 2, 3, \dots$  where  $\Gamma$  stands for Euler's gamma function. For  $k = 1, 2, \dots$  we have by the well-known properties of the gamma function [AS, 6.1]

$$\omega_{2k-1} = \frac{2\pi^k}{(k-1)!}; \quad \omega_{2k} = \frac{2^{k+1}\pi^k}{1 \cdot 3 \cdots (2k-1)}.$$

Algorithms suitable for numerical computation of  $\Gamma(s)$  are given in [AS, Ch. 6] and in [PFTV, Ch. 6].

We next give a list of the additional notation used.

$\mathbf{H}_+^n = \mathbf{R}_+^n$	the Poincaré half-space	1
$P(a, t)$	an $(n-1)$ -dimensional hyperplane	2
$\mathcal{GM}$	the group of Möbius transformations	3
$\mathcal{O}(n)$	the group of orthogonal mappings	3
$\mathcal{M}$	the group of sense-preserving Möbius transformations	3
$\tilde{x}, \tilde{f}$	a generic point of $\{x \in \mathbf{R}^{n+1} : x_{n+1} = 0\}$	4
$\pi(x), \pi_2(x)$	the stereographic projection	4, 6

$q(x, y)$	the spherical (chordal) distance between $x$ and $y$	4, 5
$\tilde{x}$	the antipodal (diametrically opposite) point	5
$Q(x, r)$	the spherical ball	7
$ a, b, c, d $	the absolute (cross) ratio	9
$a^*$	the image of a point $a$ under an inversion in $S^{n-1}$	10
$T_a$	a hyperbolic isometry with $T_a(a) = 0$	11
$\text{Lip}(f)$	the Lipschitz constant of $f$	11
$t_x$	a spherical isometry with $t_x(x) = 0$	14
$\rho(x, y)$	the hyperbolic distance between $x$ and $y$	20, 23
$J[x, y]$	the geodesic segment joining $x$ and $y$ in $\mathbb{R}_+^n$	21
$D(x, M)$	the hyperbolic ball with center $x$ and radius $M$	22, 24
$j_D(x, y)$	a point-pair function (metric)	28
$k_D(x, y)$	the quasihyperbolic distance between $x$ and $y$	33
$D_G(x, M)$	the quasihyperbolic ball with center $x$ and radius $M$	35
$s_G(x, y)$	a point-pair function	39
$p_X(A, t)$	the number of balls in a covering of the set $A$	46
$ \gamma $	the locus of a path	49
$\ell(\gamma)$	the length of a curve $\gamma$	49
$M_p(\Gamma), M(\Gamma)$	the ( $p$ -)modulus of a curve family $\Gamma$	49
$\Delta(E, F; G)$	the family of all closed non-constant curves joining $E$ and $F$ in $G$	51
$\Delta(E, F)$		52
$c_n$	the constant in the spherical cap inequality	59
$R_{G,n}(s)$	the Grötzsch ring	65
$R_{T,n}(s)$	the Teichmüller ring	65
$\gamma_n(s) = \gamma(s)$	the capacity of $R_{G,n}(s)$	66
$\tau_n(s) = \tau(s)$	the capacity of $R_{T,n}(s)$	66
$\mu(r)$	a function related to the complete elliptic integrals	67
$\varphi_{K,n}(r)$	a special function related to the Schwarz lemma	68, 97
$c(E)$	a set function related to the modulus	74

$p\text{-cap } E, \text{ cap } E$	the ( $p$ -)capacity of a condenser 82
$\Lambda_\alpha(F)$	the $\alpha$ -dimensional Hausdorff measure of $F$ 86
$\Phi_n(s)$	the modulus of the Grötzsch ring 88
$\Psi_n(s)$	the modulus of the Teichmüller ring 88
$\lambda_n$	the Grötzsch ring constant 88
$r_G(x, y)$	a point-pair function 102
$\lambda_G(x, y)$	a conformal invariant (introduced by J. Ferrand) 103, 118
$\mu_G(x, y)$	the modulus (conformal) metric 103
$p(x)$	a function related to an extremal problem 106
$m_G(x, y)$	a point-pair invariant 116
$\mu(y, f, D), \mu(f, D)$	the topological degree 121, 123
$B_f$	the branch set of a mapping $f$ 122
$\dim E$	the topological dimension of a set $E$ 123
$J(G)$	the collection of all relatively compact subdomains of a domain $G$ 123
$i(x, f)$	the local (topological) index of $f$ at $x$ 123
$U(x, f, r)$	a normal neighborhood of $x$ 124
$N(f, A)$	the maximal multiplicity of $f$ in $A$ 125
$K(f), K_O(f), K_I(f)$	the maximal, outer, and inner dilatations of $f$ 128
$H(x, f)$	the linear dilatation of a mapping $f$ at $x$ 134
$\lambda(K)$	a special function related to the linear dilatation 136
$C(f, b)$	the cluster set of a mapping $f$ at $b$ 174
$\text{cap dens}(E, 0)$	the lower capacity density of $E$ at 0 178
$\text{cap } \overline{\text{dens}}(E, 0)$	the upper capacity density of $E$ at 0 178
$\text{rad dens}(E, 0)$	the lower radial density of $E$ at 0 178
$\text{rad } \overline{\text{dens}}(E, 0)$	the upper radial density of $E$ at 0 178
$\text{Dir}(u)$	the Dirichlet integral of $u$ 187