SINGULAR INTEGRALS ON BESOV SPACES

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Abstract. The boundedness of singular convolution operators $f \mapsto k * f$ is studied on Besov spaces of vector-valued functions, the kernel k taking values in $\mathcal{L}(X, Y)$. The main result is a Hörmander-type theorem giving sufficient conditions for the boundedness of such an operator on these spaces.

1. Introduction

Translation-invariant operators between spaces of vector-valued functions have been studied extensively in recent years, especially because many problems in the theory of evolution equations give rise to such operators. The attention has been concentrated, in particular, on the Lebesgue–Bôchner spaces, where a breakthrough was made in the turn of the millennium, opening the way for theorems concerning operator-valued multipliers between these spaces [2, 12].

However, before the right line of attack to the multiplier theorems on L^p spaces was found, the setting of the Besov spaces $B_q^{s,p}$ was found to be more fertile. Whereas in the X-valued L^p spaces the analogues of the classical multiplier theorems required special geometry of the underlying Banach space X, it was observed (independently) by H. Amann [1] and one of us [11] (see also [5]) in the second half of the 90's that the situation was quite different for the Besov spaces. In fact, even operator-valued multiplier theorems were obtained on $B_q^{s,p}(X)$ (the Besov space of X-valued functions) with no geometric restrictions on the underlying Banach space X. Moreover, norm boundedness conditions on the derivatives of the multiplier function (imitating the classical ones due to Mihlin and Hörmander, and some generalizations) were found to be sufficient to give the boundedness of the associated operator on $B_q^{s,p}(X)$, whereas the recent studies [2, 3, 12] of operator-valued $L^p(X)$-multipliers have revealed the necessity of a strengthened notion of uniform boundedness, the so-called *R-boundedness*, in this connection.

Now that the situation is better understood on both scales of spaces, the results on $L^p(X)$ and $B_q^{s,p}(X)$ are seen to complement each other: Although it is perhaps desirable to work with the more concrete and familiar Bôchner spaces when this is possible, it is not always possible, and one is therefore forced to use substitute results when X is non-reflexive, or more generally, non-UMD. On the other hand, the results on the Besov spaces remain to hold invariant, to a large extent at least, under the geometry of the underlying Banach space X. Continuity results on more classical function spaces can then be derived using sharp embedding theorems by which spaces such as $L^p(X)$ and BUC are related to the Besov scale. Moreover, the Besov spaces include as subscales several "semi-classical" function spaces such as BUC^s $(= B_\infty^{s,\infty})$ and $W^{s,p}$ $(= B_p^{s,p}$, $p \in [1, \infty[)$ for non-integral values of $s > 0$. The reader is refered to [1] for details on these points.

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The philosophy of the present paper is to adopt the convolution-integral point of view to the translation-invariant operators on $B^{s,p}_q(X)$, i.e., instead of thinking $f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ (the Fourier multiplier point of view), we write this directly as $f \mapsto k * f$, where $k = \mathcal{F}^{-1}m$, and the goal is to find sufficient conditions for the boundedness on $B^{s,p}_q(X)$ in terms of the (singular) convolution kernel k.

This approach has several advantages: First of all, operators appear in applications which are naturally given in the convolution form, so that it is desirable to be able to determine the boundedness from the structure of the convolution kernel, without the need to first transform everything to the frequency domain. Second, such an approach helps to decouple the boundedness conditions in the theorems from certain properties of the underlying Banach space X. In fact, when the conditions are expressed in terms of the multiplier m, the minimal order of smoothness required for the boundedness of the associated operator depends on the *Fourier-type* of the underlying Banach space X (see [5]). On the other hand, the Fourier-type does not enter the present results in any way; yet these results are strong enough to be used to rederive many of the multiplier theorems in [5], i.e., the Fourier-type only enters the scene when we want to show that the conditions assumed on the multiplier actually imply the kind of conditions we want to have on the corresponding kernel. (We are not going to consider this point any further, but dedicate this paper to the convolution point-of-view. However, the parallel question in the L^p setting is discussed in full detail in [8], and the computations in the Besov space setting are similar but easier.)

The paper is organized as follows: Sect. 2 collects preliminary results and notation, including the definition of the vector-valued Besov spaces and the operators to be studied. In Sect. 3, we formulate the problems we address and we study the convolution operators $k*$ on $B^{s,p}_q(X)$ in rather general terms. The main result of this section, Theorem 3.15, gives a characterization of the convolutors (see Def. 3.11) on $B^{s,p}_q(X)$ in terms of convolutors on $L^p(X)$. In this way, the original problem of boundedness is reduced to a sequence of subproblems on $L^p(X)$ (related to the "dyadic pieces" of the kernel k obtained from a Littlewood–Paley decomposition). Sect. 4 collects some results for the treatment of the above-mentioned L^p-subproblems, and the results so far combine to give Theorem 4.10, where the sufficient conditions for k to be a $B^{s,p}_q$-convolutor are expressed more explicitly, without reference to L^p-convolutors. This result is used in Sect. 5 to derive our Hörmander-type Theorem 5.7, where the sufficient conditions are expressed in as classical a style as possible. Although this is no longer an exact characterization, partial converse results are proven along the way which show that the assumptions cannot be essentially weakened in general. An application to evolutionary integral equations is considered in Sect. 6. Finally, we comment on similar results for the homogenous Besov spaces.

2. Preliminaries

Spaces of functions and distributions. We are mostly concerned with functions (or distributions) defined on all of $\mathbb{R}^n$, where n is arbitrary but fixed throughout the discussion. Hence the domain $\mathbb{R}^n$ will not be indicated explicitly, and we write, e.g., $L^p(X)$ for the space of Bôchner measurable X-valued functions on $\mathbb{R}^n$, with $\|f\|_p :=$

$\left(\int |f(t)|_X^p \,\mathrm{d}t\right)^{1/p} < \infty$. Here and below an integral always refers to integration over the whole space $\mathbb{R}^n$, unless another domain is specified explicitly.

$\mathcal{S}(X)$ is the Schwartz space of smooth, X-valued, rapidly decreasing functions, and $\mathcal{S} := \mathcal{S}(\mathbb{C})$. $\mathcal{S}(X)$ is endowed with its usual topology generated by the countable collection of seminorms $\|\psi\|_{\alpha,\beta} := \left\|t \mapsto t^\beta D^\alpha \psi(t)\right\|_\infty$, $\alpha, \beta \in \mathbb{N}^n$. $\mathcal{D}(X)$ consists of the compactly supported elements $\mathcal{S}(X)$. The space of X-valued tempered distributions is $\mathcal{S}'(X) := \mathcal{L}(\mathcal{S}, X)$.

As with $\mathcal{S}$, we write more generally $\mathfrak{F} := \mathfrak{F}(\mathbb{C})$ for the scalar-valued version of any function (or distribution) space $\mathfrak{F} \in \{\mathcal{D}, L^p, \mathcal{S}', \ldots\}$. In some rare occations where the vector-valuedness of a fucntion space is immaterial, we may depart from this convention and simply write $\mathfrak{F}$ even for the vector-valued function-space, but this is always indicated explicitly.

Rather than $\mathcal{S}(X)$, our most important test-function class will be the smaller algebraic tensor product $X \otimes \mathcal{S}$, a reason for which will appear below. We note that this is dense in $\mathcal{S}(X)$ w.r.t. its usual topology. A sketch of the proof is as follows: First, it is well known that $\mathcal{D}(X)$ is dense in $\mathcal{S}(X)$; thus it suffices to approximate a compactly supported ψ by functions in $X \otimes \mathcal{S}$. We take a (fine enough) finite partition of unity $(\varphi_j)_{j=1}^m$ of the support of ψ. Let ψ_j be the Taylor expansion of degree N at t_j (a point chosen from the support of φ_j), where N is chosen large enough. Then $\psi_j \phi_j \in X \otimes \mathcal{S}$, and $\sum_{j=1}^m \varphi_j \psi_j$ can be chosen as close to ψ as desired, the closedness being measured in terms of any preassigned finite collection of the seminorms $\|\cdot\|_{\alpha,\beta}$.

Fourier transform and convolutions. The Fourier transform is defined by $\hat{f}(\xi) \equiv \mathcal{F}f(\xi) := \int f(t) e^{-\mathrm{i}2\pi t\cdot\xi} \,\mathrm{d}t$ for $f \in L^1(X)$. It is an isomorphism on $\mathcal{S}(X)$, and on $\mathcal{S}'(X)$ where it is defined by duality: $\langle \mathcal{F}f, \psi\rangle := \langle f, \mathcal{F}\psi\rangle$. The inverse Fourier transform is denoted $\check{f} \equiv \mathcal{F}^{-1} f$. We recall the identity $\mathcal{F}^2 f = \tilde{f}$, where $\tilde{f}(t) = f(-t)$.

The convolution of a tempered distribution $k \in \mathcal{S}'(X)$ and a Schwartz function $\psi \in \mathcal{S}$ is defined pointwise by $k * \psi(t) := \langle k, \psi(t - \cdot)\rangle$. It can be shown, and the vector-valued situation brings no complications at this point, that $k*\psi$ is a smooth, slowly increasing function. It can be identified with a tempered distribution, and satisfies $\langle k * \psi, \varphi\rangle = \left\langle k, \tilde{\psi} * \varphi\right\rangle$

Besov spaces. The Besov spaces $B_q^{s,p}(X)$ can be defined in various ways. For the Fourier analytic definition which we use, we require the following Littlewood–Paley-type decomposition: Let $(\varphi_j)_{j=0}^\infty$ be a resolution of the identity, defined (in terms of the corresponding Fourier transforms) as follows: Let $\hat{\varphi}_0 \in \mathcal{D}$ be radial, equal to unity in $\bar{B}(0,1)$, and supported in $\bar{B}(0,2)$. (The definition of the Besov spaces is [up to equivalence of norms] independent of this choice; in fact, one could allow much more general resolutions of the identity than considered here.) Denote $\hat{\phi} := \hat{\varphi}_0 - \hat{\varphi}_0(2\cdot)$ and $\hat{\varphi}_j := \hat{\phi}(2^{-j}\cdot)$ for $j = 1, 2, \ldots$ We can then decompose $\hat{f} = \sum_{j=0}^\infty \hat{f}\hat{\varphi}_j$, i.e., $f = \sum_{j=0}^\infty f * \varphi_j$, where the series converges in $\mathcal{S}(X)$ for $f \in \mathcal{S}(X)$ and in $\mathcal{S}'(X)$ for $f \in \mathcal{S}'(X)$. Then, for $s \in \mathbb{R}$, $p, q \in [1, \infty]$ the space $B_q^{s,p}(X)$ consists of those $f \in \mathcal{S}'(X)$ for which

$$\|f\|_{s,p;q} := \left\| \left(2^{js} \|f * \varphi_j\|_p\right)_{j=0}^\infty \right\|_{\ell_q}$$

is finite.

We have $\mathcal{S}(X) \hookrightarrow B^{s,p}_q(X) \hookrightarrow \mathcal{S}'(X)$, where $\hookrightarrow$ denotes continuous embedding, and $B^{s,p}_q(X)$ are Banach spaces for all values of the indices as above.

It is convenient to define $\chi_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ (where $\varphi_{-1} := 0$), so that $\hat{\chi}_j = 1$ on the support of $\hat{\varphi}_j$.

The operators of interest. We study convolutions $f \mapsto k * f$, where $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. These are initially defined on the algebraic tensor product $X \otimes \mathcal{S}$ as follows: For $\psi \in \mathcal{S}$ and $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, the convolution $k * \psi$ is defined as above; for every $t \in \mathbb{R}^n$, we have a well-defined pointwise value $k * \psi(t) \in \mathcal{L}(X,Y)$. Then also $[k * \psi(t)]x$ is well-defined for $x \in X$. Thus $(k * f)(t) := [k * \psi(t)]x$ for $f = x \otimes \psi$, and this definition extends to $f \in X \otimes \mathcal{S}$ by linearity. The transformation $f \mapsto k * f$ maps $X \otimes \mathcal{S}$ into the subset of $\mathcal{S}'(Y)$ consisting of smooth, slowly increasing functions.

We note in passing that there is an elaborate method for defining the action of $k*$ on the whole of $\mathcal{S}(X)$ instead of only $X \otimes \mathcal{S}$. An interested reader should consult the paper of Amann [1] for this. However, the more modest approach adopted here suffices for our purposes. In fact, it is shown in [1] that $\mathcal{S}(X)$ is dense in $B^{s,p}_q(X)$ iff both $p < \infty$ and $q < \infty$, and the same argument can be used to show the density of the smaller class $X \otimes \mathcal{S}$ in exactly the same range of Besov spaces. Actually, since $X \otimes \mathcal{S}$ is dense in $\mathcal{S}(X)$ w.r.t. the usual topology of $\mathcal{S}(X)$, which is stronger than the topology of $B^{s,p}_q(X)$, the $B^{s,p}_q(X)$-closures of $X \otimes \mathcal{S}$ and $\mathcal{S}(X)$ always coincide. Thus, to have an *a priori* estimate $\|k * f\|_{s,p;q} \leq C \|f\|_{s,p;q}$ for all $f \in X \otimes \mathcal{S}$ is just as good as the corresponding estimate for all $f \in \mathcal{S}(X)$: When $p, q < \infty$, either one allows us to conclude the existence of a unique extension $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ s.t. $T|_{X \otimes \mathcal{S}} = k*$. When p or q is infinite, we get an extension to the same closed subspace of $B^{s,p}_q(X)$, and to have an extension to the whole space, we require an extra argument anyway.

3. General theory

In this section we investigate general conditions for the boundedness of convolution operators from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$. The task is essentially two-fold: For $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, our operator $k*$ is initially defined on the subspace $X \otimes \mathcal{S}$ of $B^{s,p}_q(X)$. Thus the first problem is

Problem 3.1. When do we have $k * f \in B^{s,p}_q(Y)$ and $\|k * f\|_{s,p;q} \leq C \|f\|_{s,p;q}$ for all $f \in X \otimes \mathcal{S}$, with $C < \infty$ independent of f?

Of course, this is the only problem if $X \otimes \mathcal{S}$ is dense in $B^{s,p}_q(X)$, since a unique operator $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ with the property $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$ is then determined by k, as soon as k satisfies the condition searched in Problem 3.1. However, we know that the density holds iff $p, q < \infty$. Thus, in the general case, we are faced with another problem:

Problem 3.2. When and how can we extend $k*$ to $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ s.t. $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$?

Moreover, it is natural to ask

Problem 3.3. Is the extension T unique? If not, is it possible to choose it in some canonical manner so as to have uniqueness by requiring some additional property? Is the extended operator translation-invariant, like the original operator $T|_{X \otimes \mathcal{S}} = k*$ was?

By translation-invariance, we will mean not only the property $T(f(\cdot - h)) = (Tf)(\cdot - h)$ for $h \in \mathbb{R}^n$, but also $T(\psi * f) = \psi * Tf$ for all $\psi \in \mathcal{S}$. Formally, the latter property is a consequence of the former, but making this precise requires suitably continuity, and it is easier to study the validity of this condition directly. Moreover, it appears that the property $T(\psi * f) = \psi * Tf$ is actually the more useful of the two in applications. Both these properties are easily seen to be satisfied by the operator $k*$ acting on $X \otimes \mathcal{S}$.

We first consider Problem 3.1. To facilitate notation related to this problem, we denote

$$\mathcal{L}^\circ(\mathfrak{F}(X), \mathfrak{F}(Y)) := \{T : X \otimes \mathcal{S} \to \mathfrak{F}(Y) |\ \|Tf\|_{\mathfrak{F}} \leq C \|f\|_{\mathfrak{F}}\ \forall f \in X \otimes \mathcal{S}\},$$

where $\mathfrak{F}$ means either L^p or $B^{s,p}_q$, and $\|T\|_{\mathcal{L}^\circ(\mathfrak{F}(X),\mathfrak{F}(Y))}$ is the smallest possible C, as usual. Moreover, since the vector-valuedness plays no rôle in the proof of the next result, we make even further simplification, and only write $B^{s,p}_q$ instead of $B^{s,p}_q(X)$, and $\mathcal{L}^\circ(B^{s,p}_q)$ instead of $\mathcal{L}^\circ(B^{s,p}_q(X); B^{s,p}_q(Y))$ in the proof.

Since the membership and the norm of a distribution f in the spaces $B^{s,p}_q$ is determined solely in terms of the L^p norm of its dyadic pieces, it is not surprising that the boundedness of a convolution operator $k*$ on $B^{s,p}_q$ depends only on the boundedness on L^p of the convolution operators induced by the dyadic pieces of k. More precisely, the following proposition holds:

Proposition 3.4. *For arbitrary Banach spaces X and Y, there is an equivalence of norms*

$$\|k*\|_{\mathcal{L}^\circ(B^{s,p}_q(X),B^{s,p}_q(Y))} \approx \sup_j \|(k * \varphi_j)*\|_{\mathcal{L}^\circ(L^p(X),L^p(Y))} \approx \sup_j \|(k * \chi_j)*\|_{\mathcal{L}^\circ(L^p(X),L^p(Y))},$$

and the constants of equivalence depend only on s and q.

Proof. The latter comparability is elementary, since

$$\|(k * \chi_j)*\|_{\mathcal{L}^\circ(L^p)} \leq \sum_{i=j-1}^{j+1} \|(k * \varphi_i)*\|_{\mathcal{L}^\circ(L^p)},$$

and to the other direction we have $\|(k * \varphi_j)*\|_{\mathcal{L}^\circ(L^p)} = \|\varphi_j * (k * \chi_j)*\|_{\mathcal{L}^\circ(L^p)} \leq \|\varphi_j\|_1 \|(k * \chi_j)*\|_{\mathcal{L}^\circ(L^p)}$, and $\|\varphi_j\|_1 \leq C$.

In view of the fact that $\hat{\chi}_j = 1$ on $\operatorname{supp} \hat{\varphi}_j$, we have

$$\|(k * f) * \varphi_j\|_p = \|(k * \chi_j) * (f * \varphi_j)\|_p \leq \sup_i \|(k * \chi_i)*\|_{\mathcal{L}^\circ(L^p)} \|f * \varphi_j\|_p,$$

and thus

$$\begin{aligned} \|k * f\|_{B^{s,p}_q} &= \left\| \left(2^{js} \|k * f * \varphi_j\|_p\right)_j \right\|_{\ell_q} \\ &\leq \sup_i \|(k * \chi_i)*\|_{\mathcal{L}^\circ(L^p)} \left\| \left(2^{js} \|f * \varphi_j\|_p\right)_j \right\|_{\ell_q} = \sup_i \|(k * \chi_i)*\|_{\mathcal{L}^\circ(L^p)} \|f\|_{B^{s,p}_q}, \end{aligned}$$

which shows that $\|k*\|_{\mathcal{L}^\circ(B^{s,p}_q)} \leq \sup_i \|(k * \chi_i)*\|_{\mathcal{L}^\circ(L^p)}$.

For the converse inequality, note first that all is clear if $\|(k * \varphi_j)*\|_{\mathcal{L}^\circ(L^p)} = 0$ for all j. Otherwise, we fix an index j_0 with $\|(k * \varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)} > 0$, consider an arbitrary positive $M < \|(k * \varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$, and let $g \in \mathcal{S} \setminus \{0\}$ satisfy

$$\|(k * \varphi_{j_0}) * g\|_p \geq M \|g\|_p. \tag{3.5}$$

(Note that such choices can be made whether $\|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$ is finite or infinite.) Take $f := g*\chi_{j_0} \in \mathcal{S}$. Then $\hat{f} = \hat{g}$ on $\operatorname{supp}\hat{\varphi}_{j_0}$, and hence $(k*\varphi_{j_0})*g = (k*\varphi_{j_0})*f$. This equality in combination with (3.5) shows that f is non-zero.

Moreover, we have

$$\|\varphi_j * f\|_p \leq \|\varphi_j * \chi_{j_0}\|_1 \|g\|_p \leq C\|g\|_p ,$$

and in fact $\varphi_j * f = 0$ for $|j - j_0| > 1$, again by considering the supports of the Fourier transforms. These facts show that

$$\|f\|_{B^{s,p}_q} = \left\| \left(2^{(j_0+i)s} \|f * \varphi_{j_0+i}\|_p \right)_{i=-1}^{1} \right\|_{\ell_q} \leq 2^{j_0 s} C(s,q)\|g\|_p .$$

Finally, we have

$$\begin{aligned}\|k * f\|_{B^{s,p}_q} &= \left\| \left(2^{js} \|(k*f)*\varphi_j\|_p \right)_{j=0}^{\infty} \right\|_{\ell_q} \geq 2^{j_0 s}\|(k*\varphi_{j_0})*f\|_p \\ &= 2^{j_0 s}\|(k*\varphi_{j_0})*g\|_p \geq 2^{j_0 s}M\|g\|_p \geq C^{-1}(s,q)M\|f\|_{B^{s,p}_q} .\end{aligned}$$

Since this holds for arbitrary j_0 and any $M < \|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$, with some non-zero $f \in \mathcal{S}$, we conclude that $\|k*\|_{\mathcal{L}^\circ(B^{s,p}_q)} \geq C^{-1}(s,q)\sup_j \|(k*\varphi_j)*\|_{\mathcal{L}^\circ(L^p)}$. □

The proposition shows that the question of boundedness of the convolution operator $k*$ in the $B^{s,p}_q$-norm reduces to the problem of L^p-boundedness of the convolution operators $(k*\varphi_j)*$, which will be studied in detail in the subsequent section. For the while, we turn to Problems 3.2 and 3.3. As mentioned above, these only require consideration if either p or q is infinite. The rest of this section will be concerned with developing a theory applicable to these cases. Thus, a reader mainly interested in the case $p, q < \infty$ might wish to move immediately to the beginning of the next section. For those who stay, we are next going to give a preliminary result for the solution of Problem 3.3 when $p = \infty$; it has also some use in understanding Problem 3.2, which is the reason for taking up this consideration at this early stage.

Lemma 3.6. *Let T be a linear and $\sigma(L^p(X), L^{p'}(X'))$-to-$\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$-continuous operator from $L^p(X)$ to $\mathcal{S}'(Y)$, such that $T|_{X\otimes\mathcal{S}} = k*$, $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. Then $\psi * Tg = T(\psi * g)$ and $(Tg)(\cdot - h) = T[g(\cdot - h)]$ for all $\psi \in \mathcal{S}$, $h \in \mathbb{R}^n$.*

Proof. Suppose $g \in X\otimes\mathcal{S}$. Then $\mathcal{F}[\psi*Tg] = \mathcal{F}[\psi*(k*g)] = \hat{\psi}\hat{k}\hat{g}$, and $\mathcal{F}[T(\psi*g)] = \mathcal{F}[k*(\psi*g)] = \hat{k}\hat{\psi}\hat{g}$, so everything is clear.

For arbitrary $g \in L^p(X)$, we consider functions $g_n \in X\otimes\mathcal{S}$ which converge to g in $\sigma(L^p(X), L^{p'}(X'))$. Observe that $X\otimes\mathcal{S}$ is $\sigma(L^p(X), L^{p'}(X'))$-dense in $L^p(X)$; for $p \in [1,\infty[$ it is even norm-dense, as is well known, and for $p = \infty$ the verification of this assertion is an exercise in vector-valued integration.

Now $Tg_n \to Tg$ in $\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$, i.e., $y'(\langle Tg_n, \phi\rangle) \to y'(\langle Tg, \phi\rangle)$ for all $y' \in Y'$, $\phi \in \mathcal{S}$. With $\tilde{\psi}*\phi$ in place of ϕ this gives $y'(\langle \psi * Tg_n, \phi\rangle) \to y'(\langle \psi * Tg, \phi\rangle)$.

From $g_n \to g$ in $\sigma(L^p(X), L^{p'}(X'))$ we easily have $\psi * g_n \to \psi * g$ in the same topology. By assumption then, $y'(\langle T(\psi * g_n), \phi\rangle) \to y'(\langle T(\psi * g), \phi\rangle)$ for all y' and ϕ as above.

Since the assertion was shown for the g_n and the limit is unique, we conclude that $y'(\langle \psi * Tg, \phi\rangle) = y'(\langle T(\psi * g), \phi\rangle)$ for all $y' \in Y'$, $g \in L^p(X)$, $\psi, \phi \in \mathcal{S}$, and this implies $\psi * Tg = T(\psi * g)$ as tempered distributions, thus a.e. (since both sides are locally integrable functions), and this is the assertion for convolutions. The proof for the translations is similar. □

Remark 3.7. For $p \in [1,\infty[$, the same conclusion would follow from the continuity assumption $T \in \mathcal{L}(L^p(X), L^p(Y))$; indeed, for these p, the class $X \otimes \mathcal{S}$ is norm-dense in $L^p(X)$, and we could have simply argued that $T(\psi * g) = \lim T(\psi * g_n) = \lim \psi * Tg_n = \psi * Tg$, and similarly for translations.

However, the same is not true for $p = \infty$ (counterexamples can be constructed with the help of Banach limits), and this is the reason for establishing the result for weak-to-weak-type continuity.

We are now in a position to present the extension procedure to obtain from the original convolution operator $k* \in \mathcal{L}^\circ(B^{s,p}_q(X), B^{s,p}_q(Y))$ an operator $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$. The idea of the method comes from Amann [1] and Girardi and Weis [5]. The previous Lemma 3.6 will play a rôle in establishing that the equivalence of the two slightly differing extensions used by these authors do agree under mild weak-to-weak-continuity assumptions. We note that we always obtain *an* extension, as soon as $k* \in \mathcal{L}^\circ(B^{s,p}_q(X), B^{s,p}_q(Y))$; however, under the additional assumptions, as illustrated in the subsequent results, we are more justified to call it *the* extension.

Proposition 3.8. *Let $T_j|_{X\otimes\mathcal{S}} = (k * \chi_j)*$. Suppose that $\|T_j\|_{\mathcal{L}(L^p(X),L^p(Y))} \le \kappa < \infty$ for all $j \in \mathbb{N}$. Then, for every $f \in B^{s,p}_q(X)$, the formal series*

$$Tf := \sum_{j=0}^{\infty} \chi_j * T_j(\varphi_j * f)$$

converges in $B^{s,p}_q(Y)$ if $q < \infty$ and always in $\mathcal{S}'(Y)$ to an element of $B^{s,p}_q(Y)$ of norm at most $C\kappa$. We have $T|_{X\otimes\mathcal{S}} = k*$.

If, moreover, either $p < \infty$, or $p = \infty$ and each T_j is $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuous, the above series agrees, term by term, with

$$\tilde{T}f := \sum_{j=0}^{\infty} T_j(\varphi_j * f),$$

and hence the same assertions hold for $\tilde{T}f$.

Proof. That $Tf = k*f$ for $f \in X\otimes\mathcal{S}$ is clear from $\mathcal{F}[\chi_j * T_j(\varphi_j * f)] = \hat{\chi}_j(\hat{k}\hat{\chi}_j)(\hat{\varphi}_j\hat{f}) = \hat{\varphi}_j(\hat{k}\hat{f}) = \mathcal{F}[\varphi_j * (k * f)]$.

For $p = \infty$ and under the $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuity assumption, we have $\chi_j * T_j(\varphi_j * f) = T_j((\chi_j * \varphi_j) * f) = T_j(\varphi_j * f)$ by Lemma 3.6. When $p < \infty$, this is clear from Remark 3.7. Thus it remains to establish the assertions for Tf.

Convergence in $B^{s,p}_q(Y)$, $q < \infty$. We have

$$\varphi_i * \sum_{j=M}^{N} \chi_j * T_j(\varphi_j * f) = \sum_{j=M}^{N} (\varphi_i * \chi_j) * T_j(\varphi_j * f),$$

and $\varphi_i * \chi_j = 0$ for $|i-j|>1$. Thus, denoting by $T_M^N f$ the truncated series of Tf above, we have

$$\left\|\varphi_i * T_M^N f\right\|_p \leq \sum_{j=i-2\vee M}^{i+2\wedge N} \|\varphi_i * \chi_j\|_1 \|T_j(\varphi_j * f)\|_p \leq C\kappa \sum_{j=i-2\vee M}^{i+2\wedge N} \|\varphi_j * k\|_p ,$$

and then, for $q<\infty$,

$$\sum_{i=0}^{\infty} 2^{isq} \left\|\varphi_i * T_M^N f\right\|_p^q \leq C\kappa^q \sum_{j=M}^{N} 2^{jsq} \|\varphi_j * f\|_p^q \to 0 \qquad \text{as } M, N \to \infty.$$

Thus $T_0^N f$ is a Cauchy sequence in $B_q^{s,p}(Y)$. Once we know that the formal series has a meaning, we can set in the above equations $M=0$, $N=\infty$, and we deduce that $\|Tf\|_{s,p;q} \leq C\kappa \|f\|_{s,p;q}$.

Convergence in $\mathcal{S}'(Y)$. For $\psi \in \mathcal{S}$ we have

$$\begin{aligned}\sum_{j=0}^{\infty} |\langle \chi_j * T_j(\varphi_j * f), \psi\rangle|_Y &= \sum_{j=0}^{\infty} |\langle T_j(\varphi_j * f), \chi_j * \psi\rangle|_Y \leq \sum_{j=0}^{\infty} \|T_j(\varphi_j * f)\|_p \|\chi_j * \psi\|_{p'} \\ &\leq \sum_{j=0}^{\infty} \kappa 2^{js} \|\varphi_j * f\|_p 2^{-js} \|\chi_j * \psi\|_{p'} \leq C\kappa \|f\|_{s,p;q} \|\psi\|_{-s,p';q'} ,\end{aligned}$$

which is finite, since $\psi \in \mathcal{S} \subset B_{q'}^{-s,p'}$, and this gives the convergence. Then we can evaluate $\varphi_i * Tf$ just as above, and we get that $Tf \in B_q^{s,p}(Y)$, with a norm estimate of the same form as before. □

Remark 3.9. Amann [1] uses [somewhat implicitly, with an intermediate notion of sequence-spaces denoted $B_q^s(L^p(X))$] the series Tf, whereas Girardi and Weis [5] use $\tilde{T}f$. The operators of the latter authors are always even $\sigma(L^p(X), L^{p'}(X'))$-to-$\sigma(L^p(Y), L^{p'}(Y'))$-continuous, so that the definitions agree. The convergence of the series Tf in $\mathcal{S}'(Y)$ was shown in [5] under this stronger continuity assumption.

Proposition 3.10. *Under (all) the assumptions of Prop. 3.8, the operator T also has the following properties:*

- **Translation-invariance:** $\varphi * Tf = T(\varphi * f)$ *and* $(Tf)(\cdot - h) = T(f(\cdot - h))$ *for all* $\varphi \in \mathcal{S}$, $h \in \mathbb{R}^n$, *and*
- **Compact-to-weak continuity:** *If* $p=\infty$, *whenever* $\operatorname{supp} \hat{f}_m, \operatorname{supp} \hat{f} \subset K$, *a compact set, and* $f_m \to f$ *in* $\sigma(L^\infty(X), L^1(X'))$, *then* $Tf_m \to Tf$ *in* $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$.

Moreover, the T in Prop. 3.8 is the only operator in $\mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ with these properties.

Proof. Translation-invariance follows from the corresponding property of the operators T_j (Lemma 3.6 and Remark 3.7), and of $\chi_j *$ and $\varphi_j *$, from the $\mathcal{S}'(Y)$-convergence of the series defining Tf, and from the the continuity of $f \mapsto \varphi * f$ and $f \mapsto f(\cdot - h)$ on $\mathcal{S}'(Y)$.

For f_m and f as in the continuity assertion, we have $\sum_{i=0}^{M} \varphi_i \equiv 1$ on K for some large enough M. It is then clear from the definition of the Besov norm that the norms $\|f\|_p$ and $\|f\|_{s,p;q}$ are equivalent for all $f \in \mathcal{S}'(X)$ with $\operatorname{supp}\hat{f} \subset K$. In particular, $f_m, f \in B_q^{s,\infty}(X)$, so that Tf_m, Tf make sense. Moreover,

$$Tf_m = \sum_{j=0}^{M} \chi_j * T_j(\varphi_j * f_m) \to \sum_{j=0}^{M} \chi_j * T_j(\varphi_j * f) = Tf,$$

where the convergence is in $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$. Indeed, $\varphi_j * f_m \to \varphi_j * f$ in $\sigma(L^\infty(X), L^1(X'))$ when $f_m \to f$ in the same topology, and T_j is $\sigma(L^\infty(X), L^1(X))$-to-$\sigma(\mathcal{S}'(Y), Y \otimes \mathcal{S})$-continuous by assumption.

Uniqueness of T. To establish the last assertion, let $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ now be any operator which extends $k*$ and satisfies the translation-invariance and compact-to-weak continuity assertions of Prop. 3.10. For any $f \in B_q^{s,p}(X)$, we have $\varphi_j * Tf = T(\varphi_j * f)$, where $\varphi_j * f \in L^p(X)$. By density, we can find a sequence $g_m \in X \otimes \mathcal{S}$ s.t. $g_m \to \varphi_j * f$ in $L^p(X)$ if $p < \infty$ and in $\sigma(L^\infty(X), L^1(X'))$ if $p = \infty$. Then also $\chi_j * g_m \to \chi_j * (\varphi_j * f) = \varphi_j * f$ in the same topology, and it is clear that the Fourier transforms of $\chi_j * g_m$ and of $\varphi_j * f$ are supported on a fixed compact set K. Thus, when $p = \infty$, the compact-to-weak continuity guarantees that

$$T(\varphi_j * f) = \sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})\text{-}\lim_{m\to\infty} T(\chi_j * g_m) = \lim_{m\to\infty} k * (\chi_j * g_m).$$

When $p < \infty$, we have $\chi_j * g_m \to \varphi_j * f$ in $L^p(X)$, and then also in $B_q^{s,p}(X)$, because of the support condition on the Fourier transforms. Since $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$, this guarantees that $\varphi_j * Tf = T(\varphi_j * f) = \lim T(\chi_j * g_m) = \lim k * (\chi_j * g_m)$, the limit now taken in the norm-topology of $B_q^{s,p}(X)$.

Thus, in either case, $\varphi_j * Tf = T(\varphi_j * f)$ is uniquely determined by k. Since $Tf = \mathcal{S}'(Y)\text{-}\sum_{j=0}^{\infty} \varphi_j * Tf$, the same is true of Tf. □

Prop. 3.10 at hand, the following definition seems justified:

Definition 3.11. Let $k \in \mathcal{S}(\mathcal{L}(X,Y))$. We say that k is a *convolutor* from $B_q^{s,p}(X)$ to $B_q^{s,p}(Y)$ if there exists a $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ with the following properties:

- $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$, and
- T is translation-invariant and compact-to-weak continuous in the sense of Prop. 3.10.

T is called the *operator associated with* k.

Remark 3.12. It follows from Prop. 3.10 that the operator associated with k is unique. For $p = \infty$ or $q = \infty$, this would not be the case if we only required $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ s.t. $T|_{X\otimes\mathcal{S}} = k*$.

Our definition of a convolutor could be contrasted with that of a *Fourier multiplier* from $B_q^{s,p}(X)$ to $B_q^{s,p}(Y)$ used in [5]. The uniqueness question of the associated operator is there settled by requiring that T be $\sigma(B_q^{s,p}(X), B_{q'}^{-s,p'}(X'))$-to-$\sigma(B_q^{s,p}(Y), B_{q'}^{-s,p'}(Y'))$ continuous. This is a stronger requirement than that in Def. 3.11. Indeed, the translation-invariance of such an operator can be derived by continuity from the fact that it holds for the restriction of T to $X \otimes \mathcal{S}$, a dense subspace of $B_q^{s,p}(X)$ w.r.t. the topology $\sigma(B_q^{s,p}(X), B_{q'}^{-s,p'}(X'))$. The compact-to-weak continuity of T also follows; in fact, if $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$ and $\operatorname{supp}\hat{f}_m, f \subset K$, then, since $f_m = \sum_{i=0}^{M} \varphi_i * f_m$ for some fixed finite M, it follows easily that also $f_m \to f$ in $\sigma(B_q^{s,\infty}(X), B_{q'}^{-s,1}(X'))$ (see [5] for the definition of the duality pairing in this context), and then $Tf_m \to Tf$ in $\sigma(B^{s,\infty}_q(Y), B^{-s,1}_{q'}(Y'))$ and hence in $\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$. In particular, if m is a Fourier multiplier from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ in the sense of the definition in [5], then $\check{m}$ is a convolutor in the sense of Def. 3.11, and the associated operators agree.

Because of the intimate connection of convolution operators on $B^{s,p}_q(X)$ and those on $L^p(X)$, which we already discovered in Prop. 3.4 and will describe in even more detail below, we also give a parallel definition on $L^p(X)$:

Definition 3.13. Let $k \in \mathcal{S}(\mathcal{L}(X,Y))$. We say that k is a *convolutor* from $L^p(X)$ to $L^p(Y)$ if there exists a $T \in \mathcal{L}(L^p(X), L^p(Y))$ with the following properties:

- $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$, and
- if $p=\infty$, T is $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$-continuous.

Again, T is called the *operator associated with* k.

Remark 3.14. Again, the operator T associated with k is unique. This follows from the density of $X\otimes\mathcal{S}$ in the norm-topology of $L^p(X)$ when $p<\infty$ and in the $\sigma(L^\infty(X), L^1(X'))$-topology when $p=\infty$. Lemma 3.6 and Remark 3.7 show that T is translation-invariant. The compact-to-weak continuity required in the Besov space setting holds now rather trivially.

The following theorem completes the general description of convolutors from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$.

Theorem 3.15. *Let $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. Then k is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ if and only if*

- *$\chi_i * k$ is a convolutor from $L^p(X)$ to $L^p(Y)$ for all $i\in\mathbb{N}$, and*
- *$\sup_{i\in\mathbb{N}} \|T_i\|_{p\to p} < \infty$, where T_i is the operator associated with $\chi_i * k$.*

When this is the case, we have

$$Tf = \sum_{i=0}^{\infty} T_i(\varphi_i * f)$$

for all $f\in B^{s,p}_q(X)$, with convergence in $B^{s,p}_q(X)$ if $q<\infty$ and always in $\mathcal{S}'(Y)$.

Proof. The implication "$\Leftarrow$" is the content of Prop.'s 3.8 and 3.10. Let us establish "$\Rightarrow$".

Suppose k is a convolutor, and let $T\in\mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ be the associated operator. Define $T_j f := T(\chi_j * f)$ for $f\in L^p(X)$ (then $\chi_j * f\in B^{s,p}_q(X)$, so this makes sense). Now we need to observe certain properties of the operators T_j:

- For $f\in X\otimes\mathcal{S}$, also $\chi_j * f \in X\otimes\mathcal{S}$, and $T_j f = T(\chi_j * f) = k*(\chi_j * f) = (\chi_j * k) * f$; thus $T_j|_{X\otimes\mathcal{S}} = (\chi_j * k)*$.
- Denoting $\kappa := \|T\|_{\mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))}$, we have the norm estimate

$$\begin{aligned}\|T_j f\|_p &\leq 4^{|s|}2^{-js}\sum_{i=-2}^{2} 2^{(j+i)s}\|\varphi_{j+i} * T(\chi_j * f)\|_p \leq 4^{|s|}2^{-js}5^{1/q'}\left\|\left(2^{is}\|\varphi_i * T(\chi_j * f)\|_p\right)_{i=0}^{\infty}\right\|_{\ell_q}\\ &= 4^{|s|}5^{1/q'}2^{-js}\|T(\chi_j * f)\|_{s,p;q} \leq 4^{|s|}5^{1/q'}2^{-js}\kappa\|\chi_j * f\|_{s,p;q} \leq C(s,q)2^{-js}\kappa 2^{js}\|f\|_p;\end{aligned}$$

hence $\sup_{j\in\mathbb{N}}\|T_j\|_{p\to p} \leq C(s,q)\kappa < \infty$.

- Suppose $p = \infty$, and $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$. Then $\chi_j * f_m \to \chi_j * f$ in the same topology, and obviously the supports of the Fourier transforms of these functions are contained in a fixed compact set K. Thus $T(\chi_j * f_m) \to T(\chi_j * f)$ in $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$, i.e., $T_j f_m \to T_j f$ in the same topology.

Thus every $\chi_j * k$ is a convolutor from $L^p(X)$ to $L^p(Y)$, and the T_j's defined above are the associated operators.

If we now *define* $\tilde{T} f := \sum_{j=0}^\infty T_j(\varphi_j * f)$ for all $f \in B_q^{s,p}(X)$, then Prop.'s 3.8 and 3.10 show that $\tilde{T}$ is the operator associated with k, and by uniqueness we have $\tilde{T} f = T f$. □

4. Auxiliary results for L^p-convolutors

The previous section culminated in Theorem 3.15, which completely reduced the problem of boundedness of convolution operators on $B_q^{s,p}(X)$ to a related problem in $L^p(X)$. It is worth observing that this related problem is *not* the question of boundedness of general convolution operators on $L^p(X)$; rather, it deals with the convolution kernels $k*\chi_j$ having a very special structure: they are $\mathcal{C}^\infty$ and moreover have Fourier transforms supported on dyadic annuli. Nevertheless, it is convenient first to collect some general criteria for the L^p-boundedness of convolution operators with an operator-valued kernel. These results are mostly taken from [6] (where also more general versions are contained), and hence stated here without proof.

Proposition 4.1 ([6])**.** *Let $k : \mathbb{R}^n \to \mathcal{L}(X,Y)$ be a function s.t. $k(\cdot)$ and $k(\cdot)'$ are strongly measurable, and*

$$\int_{\mathbb{R}^n} |k(t)x|_Y \, \mathrm{d}t \le \kappa_1 |x|_X , \qquad \int_{\mathbb{R}^n} |k(t)'y'|_{X'} \, \mathrm{d}t \le \kappa_\infty |y'|_{Y'} . \tag{4.2}$$

Then $k*$, initially defined on $X \otimes [L^1 \cap L^\infty]$, say, extends to a bounded mapping from $L^p(X)$ to $L^p(Y)$ for all $p \in [1, \infty[$, of norm at most $\kappa_1^{1/p} \kappa_\infty^{1/p'}$.

In order to ensure the existence of an extension from $L^\infty(X)$ to $L^\infty(Y)$, further assumptions are required. For this, we introduce the following notion:

Definition 4.3. We say that an operator-valued function $k(\cdot) : \mathbb{R}^n \to \mathcal{L}(X,Y)$ is **uniformly strongly integrable**, for short $k \in L_u^1(\mathcal{L}(X,Y))$, if it is strongly integrable and the following property holds: For all measurable sets $E_m, E \subset \mathbb{R}^n$,

$$\sup_{|x|_X \le 1} \int_{E_m} |k(t)x|_Y \, \mathrm{d}t \underset{m\to\infty}{\longrightarrow} 0 \qquad \text{whenever } E_m \downarrow E,\ |E| = 0.$$

Remark 4.4. It is easy to see that the condition of uniform strong integrability of k can be separated (equivalently) into the following two parts concerning bounded and unbounded sets:

$$\sup_{|x|_X \le 1} \int_{E_m} |k(t)x|_Y \, \mathrm{d}t \underset{m\to\infty}{\longrightarrow} 0 \qquad \text{whenever } K \text{ (a compact set)} \supset E_m \downarrow E \text{ and } |E_m| \to 0$$

and

$$\sup_{|x|_X \le 1} \int_{|t|>r} |k(t)x|_Y \, \mathrm{d}t \underset{r\to\infty}{\longrightarrow} 0. \tag{4.5}$$

It is clear that the norm integrability condition $k \in L^1(\mathcal{L}(X,Y))$ implies uniform strong integrability $k \in L^1_u(\mathcal{L}(X,Y))$; the point of introducing this notion is exactly to avoid the rather strong notion of norm integrability.

A more reasonable condition which occurs in practise and implies both (4.2) and $k \in L^1_u(\mathcal{L}(X,Y))$ was pointed to us by the referee: k and k' be strongly measurable, and there exist a scalar-valued $w \in L^1$ such that, for all $x \in X$, $|k(t)x|_Y \leq w(t)\,|x|_X$ for a.e. t.

Now we state the result:

Proposition 4.6 ([6]). *Assume the second condition in* (4.2), *and define the integrals (which exist for all variables as below)*

$$\langle Kf(t), y'\rangle := \int_{\mathbb{R}^n} \langle y', k(t-s)f(s)\rangle \,\mathrm{d}s \tag{4.7}$$

for all $t \in \mathbb{R}^n$ and all $f \in L^\infty(X)$. Then $Kf(t) \in Y''$ and in fact $\|Kf(t)\|_{Y''} \leq \kappa_\infty \|f\|_\infty$.

If moreover $k(\cdot)' \in L^1_u(\mathcal{L}(Y',X'))$, then $Kf(t) \in Y$ for all $t \in \mathbb{R}^n$ and $t \mapsto Kf(t)$ is strongly measurable; thus, by the norm estimate, $Kf \in L^\infty(Y)$ and $\|Kf\|_\infty \leq \kappa_\infty \|f\|_\infty$. Moreover,

$$\tilde{K}g(t) := \int k(s-t)'g(s)\,\mathrm{d}s,$$

initially defined on $Y' \otimes [L^1 \cap L^\infty]$, extends to a bounded operator $\tilde{K}$ from $L^1(Y')$ to $L^1(X')$), and $\tilde{K} = K'|_{L^1(Y')}$, where K' is the adjoint of K.

Remark 4.8. It is clear that $Kf = k * f$ for $f \in X \otimes [L^1 \cap L^\infty]$.

It is also shown in [6] that the assertion of Prop. 4.6 remains valid even without $k(\cdot)' \in L^1_u(\mathcal{L}(Y',X'))$ provided that the Banach space Y *does not contain* c_0.

Corollary 4.9. *Suppose k satisfies the conditions* (4.2), *and $k(\cdot)' \in L^1_u(\mathcal{L}(Y',X'))$. Then k is a convolutor from $L^p(X)$ to $L^p(Y)$ for all $p \in [1,\infty]$, and the associated operator $K_p \in \mathcal{L}(L^p(X), L^p(Y))$ has norm at most $\kappa_1^{1/p}\kappa_\infty^{1/p'}$.*

Conversely, suppose $k \in \mathcal{S}'(\mathcal{L}(X,Y))$ coincides with a strongly locally integrable function, and that $k(\cdot)'$ is also strongly locally integrable. If k is a convolutor from $L^1(X)$ to $L^1(Y)$ [resp. from $L^\infty(X)$ to $L^\infty(Y)$], then it satisfies the first [resp. second] condition in (4.2).

Proof. Concerning the first assertion, everything else is contained in Prop.'s 4.1 and 4.6, except for the $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuity of $K = K_\infty$. However, even more follows easily from Prop. 4.6: Suppose $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$, and let $g \in L^1(Y')$. Then

$$\langle g, Kf_m\rangle = \langle K'g, f_m\rangle \to \langle K'g, f\rangle = \langle g, Kf\rangle\,,$$

where the convergence follows from the assumption and the fact that $K'g = \tilde{K}g \in L^1(X')$ by Prop. 4.6. Thus K_∞ is even $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(L^\infty(Y), L^1(Y'))$ continuous.

The necessary conditions. Suppose now that $\|k*\|_{1\to 1} = \kappa_1 < \infty$. We fix a non-negative $\psi \in \mathcal{S}$ with $\hat{\psi}(0) = \int \psi = 1$, and denote $\psi_\epsilon := \epsilon^{-n}\psi(\epsilon^{-1}\cdot)$. Then $\|\psi_\epsilon\|_1 = \|\psi\|_1 = 1$ for all $\epsilon > 0$, and hence $\|k * \psi_\epsilon(\cdot)x\|_1 \leq \kappa_1 |x|_X$.

It is known that $k * \psi_\epsilon \to k$ as $\epsilon \to 0$ in the sense of distributions, thus in particular $\langle k * \psi_\epsilon(\cdot)x, \Phi\rangle \to \langle k(\cdot)x, \Phi\rangle$ for all $x \in X$ and $\Phi \in Y' \otimes \mathcal{S}$. By the norm estimate above, we conclude that

$$\sup_{\Phi \in Y'\otimes\mathcal{S},\, \|\Phi\|_\infty \le 1} |\langle k(\cdot)x, \Phi\rangle| \le \kappa_1 |x|_X ,$$

and this gives the estimate $\|k(\cdot)x\|_1 \le \kappa |x|_X$ for the L^1-norm of the locally integrable function $k(\cdot)x$.

Finally, if $\|k * f\|_\infty \le \kappa_\infty \|f\|_\infty$ for all $f \in X \otimes \mathcal{S}$, then

$$\|k(\cdot)'y'\|_{L^1(X')} = \|k(t-\cdot)'y'\|_{L^1(X')} = \sup_{f\in X\otimes\mathcal{D}, \|f\|_\infty \le 1} |\langle k(t-\cdot)'y', f\rangle|$$

$$= \sup_f \left|\left\langle y', \int k(t-s)f(s)\,\mathrm{d}s\right\rangle\right| = \sup_f |\langle y', (k*f)(t)\rangle| \le \sup_f \|k*f\|_\infty |y'|_Y \le \kappa_\infty |y'|_Y .$$

Now all the assertions have been verified. □

The following result is now an immediate consequence of the previous ones:

Theorem 4.10. *Let $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, and suppose that we have the estimates*

$$\|\varphi_j * k(\cdot)x\|_1 \le \kappa |x|_X , \qquad \|\varphi_j * k(\cdot)'y'\|_1 \le \kappa |y'|_{Y'} , \tag{4.11}$$

and moreover that $\varphi_j * k(\cdot)'$ is uniformly strongly integrable. Then k is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$.

Conversely, the estimates (4.11) *are also necessary.*

Proof. Cor. 4.9 shows that every $\varphi_j * k$ (and then every $\chi_j * k = \sum_{i=-1}^{i} \varphi_{j+i} * k$) is a convolutor from $L^p(X)$ to $L^p(Y)$, and the associated operators are uniformly bounded. Then Theorem 3.15 shows that k is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$. The converse statement is obtained from the converse assertions of these same results. □

Remark 4.12. The uniform strong integrability can be dropped if Y does not contain c_0, or else if only the exponents $p < \infty$ are considered.

5. A Hörmander-type condition for singular integrals

We are now approaching our main goal of giving sufficient criteria for $B^{s,p}_q$-convolutors in terms of conditions with the flavour of L. Hörmander's classical theorem. In particular, we want to express our conditions more explicitly in terms of the kernel k itself, rather than using the auxiliary kernels $\varphi_j * k$ or $\chi_j * k$ appearing in Theorems 3.15 and 4.10.

In the context of the reflexive L^p spaces of scalar-valued functions, it is well-known (cf. e.g. [4]) that a sufficient condition for the boundedness of $k*$, where $k \in \mathcal{S}'$ coincides with a locally integrable function outside the origin, is obtained by requiring $\hat{k} \in L^\infty$ and, in addition, the Hörmander condition (see Def. 5.1 below). As we will see, in the context of the Besov spaces, it is necessary to strengthen these assumptions by imposing a stronger integrability condition in a neighbourhood of the infinity. This arises from the inhomogeneity of the Besov spaces, more precisely, the requirement that we should have $k * \varphi_0 \in L^1$, where $\hat{\varphi}_0(0) = \int \varphi_0 \ne 0$.

We first formulate several conditions that will play a rôle in our Hörmander-type convolution theorem.

Definition 5.1. Let $k \in L^{1,\mathrm{loc}}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X,Y)) \cap \mathcal{S}'(\mathcal{L}(X,Y))$. We define the following conditions, which k may or may not satisfy:

Hörmander's conditions: This holds if, for some $b > 1$, $\kappa < \infty$,

$$\int_{|t|>b|s|} |(k(t-s) - k(t))x|_Y \, \mathrm{d}t \le \kappa \, |x|_X, \qquad \text{for all } x \in X,\ s \in \mathbb{R}^n \setminus \{0\},$$

and we write for short (following Hörmander's original notation from [7]) $k \in K^1(X,Y)$ in this case. We write $k \in K^1_0(X,Y)$ if $\kappa_0 \ge \kappa(b,s) \to 0$ as $b_0 \le b \to \infty$. (Only the bound κ_0, but not the rate of convergence, is required to be uniform in s.)

Principal value conditions: We say that k satisfies the strong (resp. weak resp. weak*) principal value condition, and write $k \in PV(X,Y)$ (resp. $k \in w\text{-}PV(X,Y)$ resp. $k \in w^*\text{-}PV(X,Y)$) provided

$$\int_{r<|t|<2r} |k(t)x|_Y \, \mathrm{d}t \le \kappa \, |x|_X \qquad \text{for all } r > 0,$$

$$\left| \int_{r<|t|<R} k(t)x \, \mathrm{d}t \right|_Y \le \kappa \, |x|_X \qquad \text{for all } R > r > 0,$$

and moreover the limit

$$\lim_{r \downarrow 0} \int_{r<|t|<1} k(t)x \, \mathrm{d}t \tag{5.2}$$

exists in the norm (resp. weak resp. weak*) topology of Y for every $x \in X$. (It is assumed that Y is a dual space when dealing with the condition $w^*\text{-}PV(X,Y)$.)

Strong integrability and strong vanishing at infinity: By the first one we mean that

$$\int_{|t|>r} |k(t)x|_Y \, \mathrm{d}t \le \kappa \, |x|_X$$

for some $r \in \,]0,\infty[$ and $\kappa = \kappa(r) < \infty$. For the latter we even require that $\kappa(r) \to 0$ as $r \to \infty$, i.e., this is the second half of uniform strong integrability, (4.5).

Remark 5.3. If k satisfies the strong (resp. weak resp. weak*) principal value condition, then the limit

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{|t|>\epsilon} k(t)x \, \phi(t) \, \mathrm{d}t \\ &= \lim_{\epsilon \downarrow 0} \int_{\epsilon<|t|\le 1} k(t)x(\phi(t) - \phi(0)) \, \mathrm{d}t + \lim_{\epsilon \downarrow 0} \int_{\epsilon<|t|\le 1} k(t)x \, \mathrm{d}t \, \phi(0) + \int_{|t|>1} k(t)x \, \phi(t) \, \mathrm{d}t \end{aligned}$$

exists in the norm (resp. weak resp. weak*) topology of Y for every $x \in X$ and $\phi \in \mathcal{S}$, which explains the name. (Note that integral over $|t| > 1$ is convergent due to the boundedness of the integrals of k over annuli $r < |t| < 2r$, and the rapid decrease of ϕ.) The above mentioned limit defines the action of the tempered distribution p. v. - k, or just k, on the Schwartz function ϕ, and we also have the estimate

$$|\langle k, \phi \rangle \, x|_Y \le 2\kappa(\|\nabla\phi\|_\infty + |\phi(0)| + \||s| \, \phi(s)\|_\infty) \, |x|_X \, .$$

All this follows from the assumed principal value condition by a direct adaptation of the scalar-valued calculations in [4], pp. 193–4.

Remark 5.4. If the limit (5.2) exists in $\sigma(Y, Y')$, then the limit

$$\lim_{r\downarrow 0}\int_{r<|t|<1} k(t)'y'\,\mathrm{d}t$$

exists in $\sigma(X', X)$ for every $y' \in Y'$. In this way, the weak* principal value condition arises naturally in connection with the adjoint kernel $k(\cdot)'$.

Of course, whenever the sets are well-defined, $PV(X,Y) \subset w\text{-}PV(X,Y) \subset w^*\text{-}PV(X,Y)$.

Assuming conditions like those in the previous definition, we now wish to derive good estimates for the dyadic pieces $\varphi_i * k$. This is naturally divided into two cases: the inhomogeneous term $i = 0$, and the homogeneous terms $i = 1, 2, \ldots$ For $i > 0$, we can exploit the fact that φ_i then has a vanishing integral. But we need a uniform estimate for all such i, since otherwise the conditions of Theorem 4.10 will not be fulfilled.

In the following, we examine $k*\phi$, where k satisfies some of the conditions above, and $\phi \in \mathcal{S}$ is assumed to have a vanishing integral, so that it serves as a prototype of the functions φ_i, $i > 0$.

Lemma 5.5. *Suppose that $k \in K^1(X,Y)$ (with constant κ) and either satisfies any one of the principal value conditions (const. κ), or $\hat{k} \in L^\infty(\mathcal{L}(X,Y))$ with $\|\hat{k}\|_\infty \le \kappa$. Let $\phi \in \mathcal{S}$ with $\int \phi = 0$. Then $\|k*\phi(\cdot)x\|_1 \le C(\phi)\kappa\,|x|_X$ for all $x \in X$.*

If, moreover, $k \in K_0^1(X,Y)$, then $k*\phi$ is uniformly strongly integrable.

Proof. Assuming one of the principal-value conditions, from Rem. 5.3 we have the estimate

$$\begin{aligned}|k*\phi(t)x|_Y = |\langle k(\cdot)x, \phi(t-\cdot)\rangle|_Y \le 2\kappa(\|\nabla\phi(t-\cdot)\|_\infty + |\phi(t)| + \||t-\cdot|\,\phi(t-\cdot)\|_\infty \\ + |t|\,\|\phi(t-\cdot)\|_\infty)\,|x|_X \le C(\phi)\kappa(1+|t|)\,|x|_X\,.\end{aligned}$$

On the other hand, the Fourier condition gives

$$|(k*\phi)(t)x|_Y = \left|\mathcal{F}^{-1}[\hat{k}(\cdot)x\,\hat{\phi}](t)\right|_Y \le \left\|\hat{k}(\cdot)x\,\hat{\phi}\right\|_1 \le \kappa\,|x|_X\,\|\hat{\phi}\|_1.$$

Thus in either case we can say that $|k*\phi(t)x|_Y \le \kappa C(\phi)(1+|t|)\,|x|_X$. From this it is already clear that $k*\phi$ satisfies the first half of the condition of uniform strong integrability, cf. Remark 4.4.

Estimation of the L^1-norm. We invoke the following decomposition lemma from [8]: For $\phi \in \mathcal{S}$ with $\int\phi = 0$, there exists a decomposition $\phi = \sum_{m=0}^\infty \psi_m$ s.t. $\operatorname{supp}\psi_m \subset \bar{B}(0, 2^m) =: \bar{B}_m$, $\int\psi_m = 0$, and finally for any fixed $\alpha, \beta \in \mathbb{N}^n$ and $M > 0$, the sequence of Schwartz norms $\|\psi_m\|_{\alpha,\beta}$ is $\mathcal{O}(2^{-mM})$. The same is true for $\|\hat{\psi}_m\|_{\alpha,\beta}$ as well as for $\|\psi_m\|_p$, $\|\hat{\psi}_m\|_p$ for all $p \in [1, \infty]$.

Outside $b\bar{B}_m$, we estimate $k*\psi_m$ by the Hörmander condition:

$$\begin{aligned}\int_{b\bar{B}_m^c} |k*\psi_m(t)x|_Y\,\mathrm{d}t &= \int_{b\bar{B}_m^c}\left|\int_{\bar{B}_m}(k(t-s)x - k(t)x)\psi_m(s)\,\mathrm{d}s\right|_Y\mathrm{d}t \\ &\le \int_{\bar{B}_m}\mathrm{d}s\,|\psi_m(s)|\int_{|t|>b|s|}|(k(t-s)-k(t))x|_Y\,\mathrm{d}t \le \|\psi_m\|_1\,\kappa\,|x|_X\,.\end{aligned}$$

Inside $b\bar{B}_m$ we invoke the estimate $|\psi_m * k(t)x|_Y \leq \kappa C(\psi_m)(1+|t|)$, which gives

$$\int_{b\bar{B}_m} |\psi_m * k(t)x|_Y \, \mathrm{d}t \leq \kappa C(\psi_m) c_n b^{n+1} 2^{m(n+1)},$$

after integrating $1+|t|$ in polar coordinates and recalling that $\bar{B}_m$ has radius 2^m. The two estimates combine to give $\|\psi_m * k(\cdot)x\|_1 \leq \kappa C_{n,b}(\psi_m) 2^{m(n+1)} |x|_X$.

Recalling the estimates in which the size of ψ_m entered in the constant $C_{n,b}(\psi_m)$, as well as the properties of the sequence $(\psi_m)_{m=0}^{\infty}$ from the decomposition lemma, it follows that $C_{n,b}(\psi_m)$ is $\mathcal{O}(2^{-mM})$ for any preassigned $M > 0$ as $m \to \infty$. It suffices to take $M > n+1$ to conclude that $\sum_{m=0}^{\infty} \|\psi_m * k(\cdot)x\|_1$ converges, and thus we obtain $\phi * k(\cdot)x \in L^1(Y)$ with a norm estimate of the desired form.

Uniform integrability at ∞. Concerning the strong uniform integrability of $k * \phi$, only the estimate at infinity (cf. Remark 4.4) remains to be established. We estimate

$$\int_{|t|>r} \sum_{m=0}^{\infty} |k * \psi_m(t)x|_Y \, \mathrm{d}t \leq \sum_{m:2^m \leq \sqrt{r}} \int_{|t|>r} |k * \psi_m(t)x|_Y \, \mathrm{d}t + \sum_{m:2^m > \sqrt{r}} \|k * \psi_m(\cdot)x\|_1 \, ,$$

In the sum with large m's, $\|k * \psi_m(\cdot)x\|_1$ is $\mathcal{O}(2^{-mM})$ uniformly in $|x|_X \leq 1$, and this shows that the entire sum is $\mathcal{O}(r^{-M/2})$, $M > 0$.

The sum with small m's will be dealt with as follows, in analogy with the estimate of the L^1 norm above:

$$\int_{|t|>r} |k * \psi_m(t)x|_Y \, \mathrm{d}t \leq \int_{\bar{B}_m} \mathrm{d}s \, |\psi_m(s)| \int_{|t|>\sqrt{r}|s|} |(k(t-s) - k(t))x|_Y \, \mathrm{d}t.$$

Thus the sum over $2^m \leq \sqrt{r}$ is dominated by $\int \sum_{m=0}^{\infty} |\psi_m(s)| \, \kappa(\sqrt{r}, s) \, \mathrm{d}s$, where $\kappa(\sqrt{r}, s)$ is as in the definition of the condition $K_0^1(X,Y)$. Since $\sum_{m=0}^{\infty} |\psi_m(\cdot)|$ is integrable, and $\kappa(\sqrt{r}, s)$ converges boundedly to zero as $r \to \infty$, the convergence of the whole integral to zero follows from Lebesgue's theorem. This completes the proof. □

The previous Lemma 5.5 is essentially all we need to handle the homogeneous terms $k * \varphi_i$ with $i > 0$. However, it clearly fails to apply directly to the inhomogeneity $k * \varphi_0$, since $\int \varphi_0 = \hat{\varphi}_0(0) = 1 \neq 0$. The following result shows that, in order to get a similar estimate for this term, it is necessary and sufficient to add the condition of strong integrability at ∞.

Lemma 5.6. *The following conditions are equivalent for any* $k \in \mathcal{S}'(\mathcal{L}(X,Y))$*:*

- $\|k * \varphi(\cdot)x\|_1 \leq C |x|_X$ *for some* $\varphi \in \mathcal{S}$ *with* $\int \varphi \neq 0$.
- *In a neighbourhood of the origin,* $\hat{k}$ *coincides with some* $\hat{f}$ *s.t.* $\|f(\cdot)x\|_1 \leq \tilde{C} |x|_X$.

If $k(\cdot) \in K^1(X,Y)$*, and* k *satisfies one of the principal-value conditions or* $\hat{k} \in L^{\infty}(\mathcal{L}(X,Y))$*, then these are further equivalent to either of the following:*

- $\|k * \varphi(\cdot)x\|_1 \leq C(\varphi) |x|_X$ *for all* $\varphi \in \mathcal{S}$.
- k *is strongly integrable at infinity.*

Proof. Let us first establish the equivalence of the two properties valid for general k. Let φ be as in the first condition. Since $\hat{\varphi}(0) = \int \varphi \neq 0$ and $\hat{\varphi}$ is continuous, there are $\epsilon, r > 0$ such that $|\hat{\varphi}(\xi)| > \epsilon$ for $|\xi| < 2r$. Let $\eta \in \mathcal{D}$ have support in $\bar{B}(0,2r)$ and equal to unity in $\bar{B}(0,r)$. Then $\hat{\psi} := \eta \cdot \hat{\varphi}^{-1} \in \mathcal{D}$, and $\psi \in \mathcal{S} \subset L^1$. Then, since $k*\varphi(\cdot)x \in L^1(Y)$, we also have $(k*\varphi)(\cdot)x*\psi \in L^1(Y)$ and $\|k*\varphi*\psi(\cdot)x\|_1 \leq \|\psi\|_1 \|k*\varphi(\cdot)x\|_1 \leq C\|\psi\|_1 |x|_X$. But the Fourier transform of $k*\varphi*\psi$ is $\hat{k}\hat{\varphi}\hat{\psi}$, and in $\bar{B}(0,r)$, this agrees with $\hat{k}$.

Conversely, if $\hat{k} = \hat{f}$ in $\bar{B}(0,r)$ [in the sense that $\left\langle \hat{k}-\hat{f},\psi\right\rangle = 0$ for $\psi \in \mathcal{S}$ supported in $\bar{B}(0,r)$], where $\|f(\cdot)x\|_1 \leq C|x|_X$, let $\hat{\varphi} \in \mathcal{D}$ be 1 at the origin and supported in $\bar{B}(0,r)$. Then $\hat{k}\hat{\varphi} = \hat{f}\hat{\varphi}$, i.e., $k*\varphi = f*\varphi$, so $\|k*\varphi(\cdot)x\|_1 \leq \|\varphi\|_1\|f(\cdot)x\|_1 \leq C\|\varphi\|_1|x|_X$, and $\int\varphi = \hat{\varphi}(0) = 1 \neq 0$.

To show that, with the additional conditions on k, the estimate $\|k*\varphi_0(\cdot)x\|_1 \leq C_0|x|_X$ for some $\varphi_0 \in \mathcal{S}$ with non-vanishing integral implies the same property for $k*\varphi$ and any $\varphi \in \mathcal{S}$, it suffices to observe that any $\varphi \in \mathcal{S}$ is [uniquely] decomposed as $\varphi = \lambda\varphi_0 + \psi$, where $\lambda \in \mathbb{C}$ and $\int\psi = 0$. Then $\|k*\lambda\varphi_0(\cdot)x\|_1 \leq C|\lambda||x|_X$ by assumption, and the fact that $\|k*\psi(\cdot)x\|_1 \leq C(\psi)|x|_X$, whenever k has the properties assumed and $\psi \in \mathcal{S}$ a vanishing integral, was shown in Lemma 5.5.

Next, let us assume $\|k*\varphi(\cdot)x\|_1 \leq C(\varphi)|x|_X$ for all $\varphi \in \mathcal{S}$ and show that k is strongly integrable in a neighbourhood of the inifinity. To this end, fix a non-negative $\varphi \in \mathcal{D}$, supported in $\bar{B} := \bar{B}(0,r)$ and with $\int\varphi = 1$. We then have

$$C(\varphi)|x|_X \geq \|k*\varphi(\cdot)x\|_1 \geq \int_{(b\bar{B})^c} |k*\varphi(t)x|_Y \,\mathrm{d}t = \int_{(b\bar{B})^c} \left|\int_{\bar{B}} k(t-s)\varphi(s)x\,\mathrm{d}s\right|_Y \mathrm{d}t,$$

where $b\bar{B} := \bar{B}(0,br)$. On the other hand, we have

$$\begin{aligned}&\int_{(b\bar{B})^c}\int_{\bar{B}} |(k(t-s)-k(t))\varphi(s)x|_Y \,\mathrm{d}s\,\mathrm{d}t \\ &\quad\leq \int_{\bar{B}}\mathrm{d}s\int_{|t|>b|s|} |(k(t-s)-k(t))\varphi(s)x|_Y \,\mathrm{d}t \leq \kappa\int_{\bar{B}}|\varphi(s)|\,\mathrm{d}s\,|x|_X = \kappa|x|_X,\end{aligned}$$

by Hörmander's condition. Estimating by the triangle inequality, we then obtain

$$\int_{(2\bar{B})^c} |k(t)x|_Y \,\mathrm{d}t = \int_{(2\bar{B})^c}\left|\int_{\bar{B}} k(t)\varphi(s)\,\mathrm{d}s\right|_Y \mathrm{d}t \leq (C(\varphi)+\kappa)|x|_X$$

but this means exactly the integrability of k in a neighbourhood of the infinity.

Finally, we show that the estimate $\int_{(b-1)\bar{B}^c}|k(t)x|_Y\,\mathrm{d}t \leq \kappa|x|_X$ implies the inequality $\|k*\varphi(\cdot)x\|_1 \leq C|x|_X$ for all $\varphi \in \mathcal{D}$, supported in $\bar{B} := \bar{B}(0,r)$. Indeed, we have

$$\int_{b\bar{B}} |k*\varphi(t)x|_Y \,\mathrm{d}t \leq \int_{b\bar{B}} \kappa C(\phi)(1+|t|)|x|_X \,\mathrm{d}t = C(\phi, b\bar{B})|x|_X,$$

where the estimate was shown in the first part of the proof of Lemma 5.5. Moreover,

$$\begin{aligned}\int_{(b\bar{B})^c}\left|\int_{\bar{B}} k(t-s)\varphi(s)x\,\mathrm{d}s\right|_Y \mathrm{d}t &\leq \int_{\bar{B}}\mathrm{d}s\int_{(b\bar{B})^c}|k(t-s)\varphi(s)x|_Y\,\mathrm{d}t \\ &\leq \int_{\bar{B}}|\varphi(s)|\,\mathrm{d}s\int_{(b-1)\bar{B}^c}|k(t)x|_Y\,\mathrm{d}t \leq \|\varphi\|_1\kappa|x|_X.\end{aligned}$$

This completes the proof. □

Theorem 5.7. *Let $k \in \mathcal{S}'(\mathcal{L}(X;Y))$ satisfy the following conditions:*

- $k(\cdot) \in K^1(X,Y)$ *and* $k(\cdot)' \in K^1(Y',X')$,
- $\hat{k} \in L^\infty(\mathcal{L}(X,Y))$, *or both* $k(\cdot)$ *and* $k(\cdot)'$ *satisfy a principal value condition,*
- $k(\cdot)$ *and* $k(\cdot)'$ *are strongly integrable at infinity.*

Then $k*$ *is a convolutor from* $B_q^{s,p}(X)$ *to* $B_q^{s,p}(Y)$ *for all* $s \in \mathbb{R}$, $p \in [1,\infty[$ *and* $q \in [1,\infty]$.

The assertion remains true for $p = \infty$ *under either of the following additional assumptions:*

- $k(\cdot)'$ *vanishes strongly at infinity, or*
- Y *does not contain* c_0.

Proof. The plan is to verify the conditions in Theorem 4.10 for the $\varphi_i * k$, $i = 0,1,2,\ldots$.

Case $i > 0$. First of all, we observe that if k satisfies Hörmander's conditions resp. the principal value condition resp. $\|\hat{k}\|_\infty \leq \kappa$, then then the same holds for $2^{-in}k(2^{-i}\cdot)$ with the same constant κ. Moreover, $k * \varphi_i = k * 2^{in}\phi(2^i\cdot) = 2^{in}(2^{-in}k(2^{-i}\cdot) * \phi)(2^i\cdot)$, and then by the dilation-invariance of the L^1-norm, we have

$$\|k * \varphi_i(\cdot)x\|_1 = \left\|2^{-in}k(2^{-i}\cdot) * \phi(\cdot)x\right\|_1 \leq \kappa C(\phi)\,|x|_X$$

by the assumptions, Lemma 5.5 and the above-mentioned invariance of the conditions on k under dilation. Now this estimate is uniform in $i = 1,2,\ldots$ The same argument with $k(\cdot)'y'$ in place of $k(\cdot)x$ clearly yields $\|k(\cdot)' * \varphi_i(\cdot)y'\|_1 \leq \kappa C(\phi)\,|x|_X$.

Under the condition that $k(\cdot)' \in K_0^1(Y',X')$ (which is implied by the assumptions that $k(\cdot)' \in K^1(Y',X')$ and $k(\cdot)'$ vanish strongly at ∞), Lemma 5.5 shows that $\varphi_i * k(\cdot)'$ is uniformly strongly integrable.

Case $i = 0$. According to Lemma 5.6, we have $\|\varphi_0 * k(\cdot)x\|_1 \leq C(\varphi_0)\,|x|_X$ and $\|\varphi_0 * k(\cdot)'y'\|_1 \leq C(\varphi_0)\,|y'|_{Y'}$.

As for the uniform strong integrability of $\varphi_0 * k(\cdot)$ (under the additional assumption of strong vanishing of $k(\cdot)$ at ∞), we write $\varphi_0 = \varphi + \psi$, where $\varphi \in \mathcal{D}$ is supported in $\bar{B}(0,\epsilon)$ and $\int \psi = 0$. Then $\psi * k(\cdot)' \in L_u^1(\mathcal{L}(Y',X'))$ by Lemma 5.5, and moreover

$$\int_{|t|>r} |\varphi * k'(t)y'|_{X'}\,\mathrm{d}t \leq \int_{|t|>r} \mathrm{d}t \int_{|s|\leq\epsilon} |\varphi(s)k(t-s)'y'|_{X'}\,\mathrm{d}s \leq \|\varphi\|_1 \int_{|t|>r-\epsilon} |k(t)'y'|_{X'}\,\mathrm{d}t.$$

Thus also $\varphi * k(\cdot)'$ satisfies the second half of the uniform strong integrability, and the first half (cf. Rem. 4.4) is proven just like in the first part of the proof of Lemma 5.5. (This part of the proof did not require the vanishing integral of the test function, as is easily seen.)

Now all the conditions required for Theorem 4.10 (and Remark 4.12) have been verified. □

6. An application

We will here apply our results to kernels arising from solution formulae for certain evolutionary integral equations considered in [9]. It is there shown (see § 7.4 of [9]) that a related maximal regularity problem leads one to investigate the boundedness on $B^{s,p}_q([0,t_0];X)$ of the operator $f \mapsto u$ given by

$$u(t) = f(t) + \int_0^t \dot{S}_0(t-\tau) f(\tau)\,\mathrm{d}\tau, \tag{6.1}$$

where the *resolvent* or *solution operator* $S_0 \in \mathcal{C}^1(]0,\infty[;\mathcal{L}(X))$ is strongly continuous at the origin and satisfies the estimates

$$\|S_0(t)\|_{\mathcal{L}(X)} + \left\|t\dot{S}_0(t)\right\|_{\mathcal{L}(X)} \le \kappa, \qquad 0 < t < t_0 \tag{6.2}$$

and

$$\left\|\dot{S}_0(t) - \dot{S}_0(t-s)\right\|_{\mathcal{L}(X)} \le \kappa \frac{s}{t(t-s)}\left(1 + \log\frac{t}{s}\right), \qquad 0 < s < t < t_0. \tag{6.3}$$

It is clear from (6.1) that the values of $u(t)$ for $t \in [0,t_0]$ remain unchanged if we truncate the kernel at t_0, so that we are lead to consider the convolution operator with the kernel $k(t) := \dot{S}_0(t)\chi_{]0,t_0[}(t)$. Let us check the conditions of Theorem 5.7 for this kernel.

Hörmander's condition. For a kernel supported on the positive half-line, it is easily seen that it suffices to consider the case $s > 0$. If $2s \ge t_0 + s$, the condition is trivial; if $t_0 \le 2s < t_0 + s$, i.e., $t_0/2 \le s < t_0$, then

$$\begin{aligned}\int_{t>2s} \|k(t) - k(t-s)\|_{\mathcal{L}(X)}\,\mathrm{d}t &= \int_{2s}^{t_0+s} \left\|\dot{S}_0(t-s)\right\|_{\mathcal{L}(X)}\,\mathrm{d}t \\ &\le \kappa \int_{2s}^{t_0+s} \frac{\mathrm{d}t}{t-s} = \kappa \log\frac{t_0}{s} \le \kappa \log 2.\end{aligned}$$

Finally, let $0 < 2s < t_0$. For $2s < t < t_0$, (6.3) gives

$$\|k(t) - k(t-s)\|_{\mathcal{L}(X)} \le 2\kappa s t^{-2}(1 + \log(t/s)),$$

and for $t_0 \le t < t_0 + s$ we have

$$\|k(t) - k(t-s)\|_{\mathcal{L}(X)} = \left\|\dot{S}_0(t-s)\right\|_{\mathcal{L}(X)} \le \kappa(t-s)^{-1}$$

by (6.2). Hence

$$\begin{aligned}\int_{t>2s} \|k(t) - k(t-s)\|_{\mathcal{L}(X)}\,\mathrm{d}t &\le 2\kappa \int_{2s}^{t_0} \frac{s}{t^2}(1 + \log(t/s))\,\mathrm{d}t + \int_{t_0}^{t_0+s} \frac{\kappa}{t-s}\,\mathrm{d}t \\ &\le 2\kappa \int_2^\infty \frac{1}{u^2}(1 + \log u)\,\mathrm{d}u + \kappa \log\frac{t_0}{t_0 - s} \le (1 + 2\log 2)\kappa,\end{aligned}$$

since $t_0/(t_0 - s) < 2$. The norm version of the condition established implies in particular the corresponding strong estimates as well as their dual versions.

Principal value condition. Using the assumption (6.2) only, we have

$$\begin{aligned}\int_r^{2r} \|k(t)\|_{\mathcal{L}(X)}\,\mathrm{d}t &\le \int_r^{2r} \frac{\kappa}{t}\,\mathrm{d}t = \kappa\log 2, \\ \left\|\int_r^R k(t)\,\mathrm{d}t\right\|_{\mathcal{L}(X)} &= \left\|\int_r^{R\wedge t_0} \dot{S}_0(t)\,\mathrm{d}t\right\|_{\mathcal{L}(X)} = \|S(R\wedge t_0) - S(r)\|_{\mathcal{L}(X)} \le 2\kappa.\end{aligned}$$

These norm estimates imply the first two conditions in the definition of the principal value conditions. Finally, from the strong continuity we have

$$\int_{\epsilon}^{1} k(t)x\,\mathrm{d}t = \int_{\epsilon}^{1\wedge t_0} \dot{S}_0(t)x\,\mathrm{d}t = S_0(1\wedge t_0)x - S_0(\epsilon)x \underset{\epsilon\downarrow 0}{\longrightarrow} S_0(1\wedge t_0)x - S_0(0)x,$$

which shows that $k \in PV(X)$. Then in particular $k \in w\text{-}PV(X)$, and thus $k(\cdot)' \in w^*\text{-}PV(X')$ (cf. Rem. 5.4).

Conclusion. Having verified all the conditions of Theorem 5.7 (those at infinity being trivially satisfied, since k vanishes outside a compact set!), we conclude that the solution map $f \mapsto u$ defined in (6.1) is indeed bounded on $B^{s,p}_q([0,t_0];X)$. This we knew, of course, from [9] already; but the ease with which the conditions of our Theorem 5.7 were verified for this operator illustrates the applicability of this general theorem in concrete situations.

7. On homogeneous Besov spaces

It was pointed to us by the referee that very much the same proofs as above also yield analogous results on the *homogeneous Besov spaces* $\dot{B}^{s,p}_q(X)$. In this connection, the relevant test function space is $\mathcal{Z} := \{\psi \in \mathcal{S} : D^\alpha\hat{\psi}(0) = 0\ \forall\alpha \in \mathbb{N}^n\}$, which is a closed subspace of $\mathcal{S}$, while the relevant distribution space is its dual $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the space of polynomials. See Triebel [10], 5.1.2, for this. In the X-valued situation we simply take $\mathcal{Z}'(X) := \mathcal{S}'(X)/\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the space of polynomials with X-coefficient. As in the scalar-valued case, this definition coincides with $\mathcal{Z}'(X) = \mathcal{L}(\mathcal{Z}, X)$. To see this, one can essentially copy the argument from [10] in the scalar case. The only non-trivial point is the fact that a distribution $u \in \mathcal{S}'(X)$ with $\operatorname{supp}\hat{u} \subset \{0\}$ is a polynomial, or equivalently, that $\operatorname{supp} u \subset \{0\}$ iff u is a linear combination (with X-coefficient) of δ_0 and its derivatives. We give the following

Sketch of proof. Let $u \in \mathcal{S}'(X)$ have $\operatorname{supp} u \subset \{0\}$. Then, for all $x' \in X'$, the scalar-distributions $x'u$ given by $\langle x'u, \phi\rangle := x'(\langle u, \phi\rangle)$ satisfy the same support condition, so by the scalar-valued result we have $x'u = \sum_{|\alpha|\leq N} c_\alpha(x')D^\alpha\delta_0$, where N is the (finite) order of the distribution u. Letting $\psi_\alpha \in \mathcal{S}$ coincide with $t^\alpha/\alpha!$ in a neighbourhood of 0, we find that $c_\alpha(x') = x'(\langle u, \psi_\alpha\rangle)$. By the Hahn–Banach theorem we conclude that $u = \sum_{|\alpha|\leq N} \langle u, \psi_\alpha\rangle\, D^\alpha\delta_0$. □

Our resolution of the identity now consists solely of dilates of a single function in $\mathcal{Z}$: take $\hat{\phi}_j := \hat{\phi}(2^{-j}\cdot)$, where $\hat{\phi} = \hat{\varphi}_0 - \hat{\varphi}_0(2\cdot)$ and $j = 0, \pm1, \pm2, \ldots$. Then $\dot{B}^{s,p}_q(X)$ consists of all $f \in \mathcal{Z}'(X)$ for which the norm

$$\|f\|^{\cdot}_{s,p;q} := \left\| \left(2^{js}\,\|f * \phi_j\|_p\right)_{j=-\infty}^{\infty} \right\|_{\ell_q}$$

is finite. This is, again, a Banach space.

On these spaces, we can pose exactly the same problems that we addressed above in the context of the inhomogeneous Besov spaces. As far as Section 3 is concerned, analogous theory in the new setting is obtained by essentially typographical modifications: Replace $B^{s,p}_q$ by $\dot{B}^{s,p}_q$, $\mathcal{S}$ by $\mathcal{Z}$, $\mathcal{S}'$ by $\mathcal{Z}'$, $\sum_{j=0}^{\infty}$ by $\sum_{j=-\infty}^{\infty}$, and χ_j should of course be defined by $\chi_j := \phi_{j-1} + \phi_j + \phi_{j+1}$ now. In the definition of compact-to-weak continuity, the set K is taken to be a compact subset of $\mathbb{R}^n \setminus \{0\}$.

More interesting is the modification of Section 5. Everything works again, but in fact even better than before! In proving an analogue of Theorem 5.7, all $k*\phi_i$, $i \in \mathbb{Z}$, can now be handled like the case $i > 0$ in the inhomogeneous situation. As a result, the integrablility conditions at ∞, which were forced upon us by the inhomogeneity of φ_0, are no longer required. These observations lead to the following result, the proof of which is essentially a subset of the proof of Theorem 5.7:

Theorem 7.1. *Let $k \in \mathcal{S}'(\mathcal{L}(X,Y))$ satisfy the conditions*

- *$k(\cdot) \in K^1(X,Y)$, $k(\cdot)' \in K^1(Y',X')$, and*
- *at least one of the following: $\hat{k} \in L^\infty(\mathcal{L}(X,Y))$, or both $k(\cdot)$ and $k(\cdot)'$ satisfy a principal value condition.*

Then $k*$ is a convolutor from $\dot{B}^{s,p}_q(X)$ to $\dot{B}^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p \in [1,\infty[$ and $q \in [1,\infty]$. The assertion remains true for $p = \infty$ if in addition $k(\cdot)' \in K^1_0(Y',X')$, or else Y does not contain c_0.

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