ON OPERATOR-MULTIPLIERS FOR MIXED-NORM $L^{\bar{p}}$ SPACES

TUOMAS P. HYTÖNEN

Abstract. We use a bootstrapping property of R-boundedness to show that some (known) n-dimensional Fourier multiplier theorems on UMD spaces with property α are immediate corollaries of their one-dimensional versions. The method also gives easy extensions of these results to mixed-norm $L^{\bar{p}}$ spaces.

The purpose of this note is to consider operator-valued Fourier multipliers on the class of UMD Banach spaces having so-called property α of Pisier (call them α-UMD spaces for short), with two principal goals in mind: First, to show that the (known) Mihlin–Lizorkin and Littlewood–Paley multiplier theorems on $L^p(\mathbf{R}^n, X)$, for such spaces X and any $n \in \mathbf{Z}_+$, are in fact almost immediate consequences of the $n = 1$ case. Second, to extend these theorems to the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$, where $\bar{p} = (p_1, \ldots, p_n)$. We start by recalling relevant definitions and earlier results, and then proceed to the above-mentioned contributions.

1. Notation and known results

Let us write $\bar{p} < \bar{q}$ if $p_j < q_j$ for all j, and moreover $\bar{1} := (1, 1, \ldots, 1)$ and $\bar{\infty} := (\infty, \infty, \ldots, \infty)$. The mixed-norm spaces (mainly of our interest for $\bar{1} < \bar{p} < \bar{\infty}$) can be defined inductively as follows: For $n = 1$, of course $L^{(p_1)}(\mathbf{R}, X) := L^{p_1}(\mathbf{R}, X)$. Then $L^{(\bar{p},q)}(\mathbf{R}^{n+1}, X) := L^q(\mathbf{R}, L^{\bar{p}}(\mathbf{R}^n, X))$, and actually this definition already contains the simple idea of our approach, for if X is an α-UMD space, so is $L^{\bar{p}}(\mathbf{R}^n, X)$ for $\bar{1} < \bar{p} < \bar{\infty}$ (use induction on n). Note that for $\bar{p} = p \cdot \bar{1}$, the spaces $L^{\bar{p}}(\mathbf{R}^n, X)$ and $L^p(\mathbf{R}^n, X)$ are canonically isometric by Fubini's theorem.

Denote $\mathbf{R}_* := \mathbf{R} \setminus \{0\}$, and then $\mathbf{R}_*^n := (\mathbf{R}_*)^n$ is the Euclidean n-space minus the coordinate hyperplanes. We say that m is a Y-valued *Mihlin–Lizorkin-multiplier* on $\mathbf{R}^n$ if m is a Y-valued distribution on $\mathbf{R}_*^n$, and all the distributional derivatives $D^\alpha m$, $\alpha \in \{0, 1\}^n$, coincide with locally integrable functions on $\mathbf{R}_*^n$.

Let $\mathscr{I}$ denote the collection of all real intervals of either of the forms $\left[2^j, 2^{j+1}\right[$ or $\left]-2^{j+1}, -2^j\right]$ for $j \in \mathbf{Z}$. $\mathscr{I}^n$ is the set of all n-fold products of such intervals. We say that m is a Y-valued *Littlewood–Paley-multiplier* on $\mathbf{R}^n$ if m is a Y-valued function on $\mathbf{R}_*^n$ which takes a constant value $m(I)$ on every $I \in \mathscr{I}^n$.

If m is an $\mathscr{L}(X)$-valued multiplier of either type, we denote by T_m the operator $f \mapsto \mathscr{F}^{-1}[m\hat{f}]$ acting on $X \otimes \mathscr{F}C_c^\infty(\mathbf{R}_*^n)$. Of course $\mathscr{F}f \equiv \hat{f}$ is the Fourier transform of f. The density of $\mathscr{F}C_c^\infty(\mathbf{R}_*)$ in $L^p(\mathbf{R})$ for $1 < p < \infty$ is easily seen. Taking into account the density of $X \otimes L^p(\mathbf{R})$ in $L^p(\mathbf{R}, X)$, the density of $X \otimes \mathscr{F}C_c^\infty(\mathbf{R}_*^n)$ in $L^{\bar{p}}(\mathbf{R}^n, X)$ is readily established by induction.

Let ε_j be independent random variables with distribution $\mathbf{P}(\varepsilon_j = 1) = \mathbf{P}(\varepsilon_j = -1) = \frac{1}{2}$ (so called Rademacher variables), and $\mathbf{E}$ be the related expectation. An operator collection $\mathscr{T} \subset \mathscr{L}(X)$ is called R-bounded if

$$\mathbf{E}\Big|\sum_{j=1}^n \varepsilon_j T_j x_j\Big|_X \leq C\mathbf{E}\Big|\sum_{j=1}^n \varepsilon_j x_j\Big|_X$$

for all $n \in \mathbf{Z}_+$ and $(x_j) \subset X$, $(T_j) \subset \mathscr{T}$. The smallest C is denoted by $\mathscr{R}[\mathscr{T}|\mathscr{L}(X)]$. We often abbreviate $\mathscr{R}[\{T : \mathscr{C}(T)\}|\mathscr{L}(X)] =: \mathscr{R}[T : \mathscr{C}(T)|\mathscr{L}(X)]$.

2000 *Mathematics Subject Classification.* 42B15.

Recall Kahane's inequality: $\mathbf{E}\,|\sum \varepsilon_j x_j|_X \le \big(\mathbf{E}\,|\sum \varepsilon_j x_j|_X^p\big)^{1/p} \le K_p \mathbf{E}\,|\sum \varepsilon_j x_j|_X$, where $K_p < \infty$ for all $p \in [1,\infty[$.

Let $\mathrm{Rad}(X)$ be the completion of all finitely non-zero sequences $(x_j)_{j=1}^\infty$ in the norm $\|(x_j)_{j=1}^\infty\|_{\mathrm{Rad}(X)} := \mathbf{E}|\sum_{j=1}^\infty \varepsilon_j x_j|_X$. Since $\Big(\mathbf{E}|\sum_{j=1}^\infty \varepsilon_j x_j|^p\Big)^{1/p}$ is an equivalent norm for every $p \in [1,\infty[$, Fubini's theorem gives a canonical isomorphism of $L^p(S, \mathrm{Rad}(X))$ and $\mathrm{Rad}(L^p(S,X))$ for all these p.

For $\mathscr{T} \subset \mathscr{L}(X)$, denote by $\tilde{\mathscr{T}} \subset \mathscr{L}(\mathrm{Rad}(X))$ the collection of all finitely non-zero $\tilde{T} = (T_j)_{j=1}^\infty$, where the action of such a $\tilde{T}$ on $\tilde{x} = (x_j)_{j=1}^\infty \in \mathrm{Rad}(X)$ is defined in the obvious way $\tilde{T}\tilde{x} := (T_j x_j)_{j=1}^\infty$. It is clear that $\mathscr{R}[\mathscr{T}|\mathscr{L}(X)] = \sup_{\tilde{T}\in\tilde{\mathscr{T}}} \|\tilde{T}|\mathscr{L}(\mathrm{Rad}(X))\|$. If X has property α (i.e., if $\mathbf{E}\tilde{\mathbf{E}}|\sum_{i,j=1}^n \varepsilon_i\tilde{\varepsilon}_j\alpha_{ij}x_{ij}|_X \le C\mathbf{E}\tilde{\mathbf{E}}|\sum_{i,j=1}^n \varepsilon_i\tilde{\varepsilon}_j x_{ij}|_X$ for all $n \in \mathbf{N}$, $x_{ij} \in X$ and $|\alpha_{ij}| \le 1$, where ε_i and $\tilde{\varepsilon}_j$ are independent Rademacher variables), then more can be said (also cf. [2]):

1.1. **Proposition** (Clément *et al.* [1]). *If X has property α, then for all $\mathscr{T} \subset \mathscr{L}(X)$*

$$\mathscr{R}[\tilde{\mathscr{T}}|\mathscr{L}(\mathrm{Rad}(X))] \le C\mathscr{R}[\mathscr{T}|\mathscr{L}(X)].$$

Next, we recall the following operator-valued multiplier theorem:

1.2. **Theorem** (Weis [4]). *Let X be a UMD space, and $1 < p < \infty$. If m is a Littlewood–Paley-multiplier on $\mathbf{R}$, then*

$$\|T_m|\mathscr{L}(L^p(\mathbf{R},X))\| \le C\mathscr{R}[m(I) : I \in \mathscr{I}|\mathscr{L}(X)].$$

If m is a Mihlin–Lizorkin-multiplier on $\mathbf{R}$, then

$$\|T_m|\mathscr{L}(L^p(\mathbf{R},X))\| \le C\mathscr{R}[m(\xi), \xi m'(\xi) : \xi \in \mathbf{R}_*|\mathscr{L}(X)].$$

For α-UMD spaces, Prop. 1.1 can be used to bootstrap Theorem 1.2 as follows, and this observation was in fact the principal inspiration of the present note:

1.3. **Theorem** (Girardi, Weis [2]). *Let X be an α-UMD space, and $1 < p < \infty$. If $m \in \mathscr{M}$ are $\mathscr{L}(X)$-valued Mihlin–Lizorkin-multipliers on $\mathbf{R}$, then*

$$\mathscr{R}[T_m : m \in \mathscr{M}|\mathscr{L}(L^p(\mathbf{R},X))] \le C\mathscr{R}[m(\xi), \xi m'(\xi) : \xi \in \mathbf{R}_*, m \in \mathscr{M}|\mathscr{L}(X)].$$

2. New results and proofs

We first prove a result analogous to Theorem 1.3 for Littlewood–Paley-multipliers. The idea of the proof is the same, it is borrowed from Girardi and Weis [2].

2.1. **Proposition.** *Let X be an α-UMD space, and $1 < p < \infty$. If $m \in \mathscr{M}$ are $\mathscr{L}(X)$-valued Littlewood–Paley multipliers on $\mathbf{R}$, then*

$$\mathscr{R}[T_m : m \in \mathscr{M}|\mathscr{L}(L^p(\mathbf{R},X))] \le C\mathscr{R}[m(I) : I \in \mathscr{I}, m \in \mathscr{M}|\mathscr{L}(X)].$$

Proof. If m_j, $j = 1, 2, \ldots$, is a finitely non-zero sequence of $\mathscr{L}(X)$-valued Littlewood–Paley-multipliers on $\mathbf{R}$, then $\tilde{m} := (m_j)_{j=1}^\infty$ is an $\mathscr{L}(\mathrm{Rad}(X))$-valued multiplier of the same type, and we have $(T_{m_j})_{j=1}^\infty = \mathscr{F}^{-1}(m_j)_{j=1}^\infty\mathscr{F} = T_{\tilde{m}}$. Thus

$$\begin{aligned}
&\mathscr{R}[T_m : m \in \mathscr{M}|\mathscr{L}(L^p(\mathbf{R},X))]\\
&= \sup\big\{\|(T_{m_j})_{j=1}^\infty|\mathscr{L}(\mathrm{Rad}(L^p(\mathbf{R},X)))\| : (m_j)_{j=1}^\infty \in \mathscr{M}^\infty \text{ finitely non-zero}\big\}\\
&\le C\sup\{\|T_{\tilde{m}}|\mathscr{L}(L^p(\mathbf{R},\mathrm{Rad}(X)))\| : \tilde{m} \in \mathscr{M}^\infty \text{ finitely non-zero}\}\\
&\le C'\sup\{\mathscr{R}[\tilde{m}(I) : I \in \mathscr{I}|\mathscr{L}(\mathrm{Rad}(X))] : \tilde{m} \in \mathscr{M}^\infty \text{ finitely non-zero}\}\\
&\le C''\sup\big\{\mathscr{R}[m_j(I) : I \in \mathscr{I}, j \in \mathbf{Z}_+|\mathscr{L}(X)] : (m_j)_{j=1}^\infty \in \mathscr{M}^\infty\big\}\\
&\le C''\mathscr{R}[m(I) : I \in \mathscr{I}, m \in \mathscr{M}|\mathscr{L}(X)].
\end{aligned}$$

The first inequality used the isomorphism of $\mathrm{Rad}(L^p(\mathbf{R},X))$ and $L^p(\mathbf{R},\mathrm{Rad}(X))$, the second was an application of Theorem 1.2 (with the UMD-space $\mathrm{Rad}(X)$ in place of X), the third follows from Prop. 1.1, and the fourth is obvious. □

Now we come to the main theorem:

2.2. **Theorem.** *Let X be an α-UMD space, and $\bar{1} < \bar{p} < \bar{\infty}$. If $m \in \mathscr{M}$ are $\mathscr{L}(X)$-valued Littlewood–Paley-multipliers on $\mathbf{R}^n$, then*

$$\mathscr{R}[T_m : m \in \mathscr{M} | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \leq C\mathscr{R}[m(I) : I \in \mathscr{I}^n, m \in \mathscr{M} | \mathscr{L}(X)].$$

If $m \in \mathscr{M}$ are $\mathscr{L}(X)$-valued Mihlin–Lizorkin-multipliers on $\mathbf{R}^n$, then

$$\begin{aligned}\mathscr{R}[T_m &: m \in \mathscr{M} | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ &\leq C\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*, \alpha \in \{0,1\}^n, m \in \mathscr{M} | \mathscr{L}(X)].\end{aligned}$$

Proof. We use induction on n. The case $n = 1$ is just Theorem 1.3 for Mihlin–Lizorkin-multipliers and Prop. 2.1 for Littlewood–Paley-multipliers. Suppose that we have proved the theorem for some $n \geq 1$, and let us prove it for $n+1$. We consider the case of Mihlin–Lizorkin-multipliers, the other one being similar.

The defining equality $L^{(\bar{p},q)}(\mathbf{R}^{n+1}, X) = L^q(\mathbf{R}, Y)$, where $Y := L^{\bar{p}}(\mathbf{R}^n, X)$, gives rise to the identification of an $\mathscr{L}(X)$-valued Mihlin–Lizorkin-multiplier m on $\mathbf{R}^{n+1}$ with an $\mathscr{L}(Y)$-valued Mihlin–Lizorkin-multiplier $\tilde{m}$ on $\mathbf{R}$ such that $\tilde{m}(\eta) = T_{m(\cdot,\eta)}$. Then

$$\begin{aligned}&\mathscr{R}[T_m : m \in \mathscr{M} | \mathscr{L}(L^{(\bar{p},q)}(\mathbf{R}^{n+1}, X))] = \mathscr{R}[T_{\tilde{m}} : m \in \mathscr{M} | \mathscr{L}(L^q(\mathbf{R}, L^{\bar{p}}(\mathbf{R}^n, X)))] \\ &\leq C\mathscr{R}[\tilde{m}(\eta), \eta \cdot \tilde{m}'(\eta) : \eta \in \mathbf{R}_*, m \in \mathscr{M} | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ &= C\mathscr{R}[T_{m(\cdot,\eta)}, T_{\eta D_\eta m(\cdot,\eta)} : \eta \in \mathbf{R}_*, m \in \mathscr{M} | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ &\leq \tilde{C}\mathscr{R}[\zeta^\alpha D^\alpha_\zeta m(\zeta,\eta), \zeta^\alpha \eta D^\alpha_\zeta D_\eta m(\zeta,\eta) : (\zeta,\eta) \in \mathbf{R}^{n+1}_*, \alpha \in \{0,1\}^n, m \in \mathscr{M} | \mathscr{L}(X)] \\ &= \tilde{C}\mathscr{R}[\xi^\beta D^\beta m(\xi) : \xi \in \mathbf{R}^{n+1}_*, \beta \in \{0,1\}^{n+1}, m \in \mathscr{M} | \mathscr{L}(X)].\end{aligned}$$

The first estimate was an application of Theorem 1.3, while the second was the induction assumption. □

Immediate corollaries of Theorem 2.2 for a single multiplier are the following: Let X be an α-UMD space and $\bar{1} < \bar{p} < \bar{\infty}$. If m is an $\mathscr{L}(X)$-valued Littlewood–Paley-multiplier on $\mathbf{R}^n$, then

$$\|T_m | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))\| \leq C\mathscr{R}[m(I) : I \in \mathscr{I}^n | \mathscr{L}(X)].$$

If m is an $\mathscr{L}(X)$-valued Mihlin–Lizorkin-multiplier on $\mathbf{R}^n$, then

$$\|T_m | \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))\| \leq C\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*, \alpha \in \{0,1\}^n | \mathscr{L}(X)].$$

For $\bar{p} = p \cdot \bar{1}$, these are due to Ž. Štrkalj and L. Weis [3], and the bootstrapped Mihlin–Lizorkin part of Theorem 2.2 is due to Girardi and Weis [2]. The case when m is scalar-valued was already proved by F. Zimmermann [5]. We feel that our approach to these theorems is simpler, even when specialized to the situation of [5]. Observe, however, that it is essential for our induction that we have an operator-valued multiplier theorem in one dimension as a starting point: Even if m were a $\mathbf{C}$-valued multiplier on $\mathbf{R}^{n+1}$, the related multiplier $\tilde{m}$ appearing in the proof of Theorem 2.2 would be $\mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))$-valued.

The n-dimensional Littlewood–Paley and Mihlin–Lizorkin multiplier theorems, as formulated above, are not valid in general UMD spaces for $n > 1$, as shown in [5]. There are, however, slightly weaker substitute results. Their proof does not seem to allow such a trivial reduction to the one-dimensional case as in the present situation, but requires a more genuine n-dimensional theory.

Finally, we note that all our results remain valid for $\mathscr{L}(X, Y)$-valued multipliers acting from $L^{\bar{p}}(\mathbf{R}^n, X)$ to $L^{\bar{p}}(\mathbf{R}^n, Y)$, where X and Y are two different α-UMD spaces. In fact, this situation can be easily reduced to the one already investigated by considering the new α-UMD space $X \oplus Y$.

References

[1] Ph. Clément, B. de Pagter, F. A. Sukochev, H. Witvliet. Schauder decomposition and multiplier theorems. *Studia Math.* **138** #2 (2000), 135–163.

[2] M. Girardi, L. Weis. Criteria for R-boundedness of operator families. *Recent Contributions to Evolution Equations*, Lecture Notes in Math., Marcel Dekker, (to appear).

[3] Ž. Štrkalj, L. Weis. On operator-valued Fourier multiplier theorems. Submitted.

[4] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.* **319** # 4 (2001), 735–758.

[5] F. Zimmermann. On vector-valued Fourier-multiplier theorems. *Studia Math.* **93** (1989), 201–222.

Current address (until April 12, 2005): Faculty EWI/TWA, Technische Universiteit Delft, Mekelweg 4, NL-2628 CD Delft, The Netherlands

Permanent address: Institute of Mathematics, Helsinki University of Technology, P.O.B. 1100, FIN-02015 TKK, Finland

E-mail address: tuomas.hytonen@tkk.fi