A $T1$ THEOREM FOR INTEGRAL TRANSFORMATIONS WITH OPERATOR-VALUED KERNEL

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Abstract. We consider generalized Calderón–Zygmund operators whose kernel $K(x,y)$ takes values in $\mathscr{L}(X)$ (continuous linear operators on the Banach space X) and satisfies variants of the classical standard estimates involving R-boundedness, which has recently become a crucial notion in connection with operator-valued singular integrals. Boundedness criteria in the spirit of the $T1$ theorem of David and Journé are proved for such operators on the Bôchner spaces $L^p(\mathbf{R}^n, X)$, where $1 < p < \infty$ and X is a UMD-space. For some results, X is also required to have Pisier's property (α).

In the special case $T1 = T'1 = 0$, we obtain an essentially complete analogue of the scalar-valued theorem. We also provide sufficient conditions for the general case, but they are stronger in general than the necessary "$T1, T'1 \in$ BMO"-type conditions, although they reduce to them in the classical situation. A counterexample is given to show that the natural necessary conditions are not sufficient in general.

1. Introduction

Historical background. The classical Calderón–Zygmund theory of singular integrals was substantially generalized into various directions during the 80's. On the one hand, J. Bourgain [2] and D. L. Burkholder [4] identified the class of those Banach spaces X for which the classical L^p continuity results for the Hilbert transform (and many other singular convolution operators [3]) remain valid on the Bôchner space $L^p(\mathbf{R}, X)$ of X-valued functions:

Theorem 1.1 ([2, 4])**.** *The Hilbert transform* $Hf(x) := \text{p.v.-}\frac{1}{\pi}\int_{-\infty}^{\infty}(x-y)^{-1}f(y)\,\mathrm{d}y$ *defines a bounded operator on* $L^p(\mathbf{R}, X)$ *if* [4] *and only if* [2] X *is a* UMD-space (*see* [4] *for definition*).

On the other hand, there was the celebrated $T1$ theorem of G. David and J.-L. Journé [6] which settled the boundedness question (on the scalar-valued L^p spaces) of a wide class of generalized Calderón–Zygmund operators:

Theorem 1.2 (The $T1$ theorem, [6])**.** *Denote* $\mathbf{R}^{2n}_{x\neq y} := \mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,x) : x \in \mathbf{R}^n\}$. *Let* $K : \mathbf{R}^{2n}_{x\neq y} \to \mathbf{C}$ *verify the* standard estimates

$$\sup\{|x-y|^n\,|K(x,y)| :\ x \neq y\} < \infty \tag{1.3}$$

$$\sup\{|x-y|^{n+\gamma}\,\frac{|K(x,y)-K(x,y_0)|}{|y-y_0|^{\gamma}} :\ |x-y| > 2\,|y-y_0| > 0\} < \infty \tag{1.4}$$

$$\sup\{|x-y|^{n+\gamma}\,\frac{|K(x,y)-K(x_0,y)|}{|x-x_0|^{\gamma}} :\ |x-y| > 2\,|x-x_0| > 0\} < \infty \tag{1.5}$$

for some $\gamma \in\,]0,1]$.

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Let $T : \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ be an operator such that, for all $f \in \mathscr{D}(\mathbf{R}^n)$ and a.e. $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y) f(y)\,\mathrm{d}y. \tag{1.6}$$

Then $T \in \mathscr{L}(L^2(\mathbf{R}^n))$ if and only if the following conditions hold:

$$\begin{cases} T \textit{ satisfies the } \text{weak boundedness property } (\textit{Def. } 1.8); \\ T1,\ T'1 \in \mathrm{BMO}(\mathbf{R}^n). \end{cases} \tag{1.7}$$

In this case, one also has $T \in \mathscr{L}(L^p(\mathbf{R}^n))$ for all $p \in \,]1,\infty[$, and moreover $T \in \mathscr{L}(H^1(\mathbf{R}^n), L^1(\mathbf{R}^n))$ and $T \in \mathscr{L}(L^\infty(\mathbf{R}^n), \mathrm{BMO}(\mathbf{R}^n))$. The inclusions $T \in \mathscr{L}(H^1(\mathbf{R}^n))$ and $T \in \mathscr{L}(\mathrm{BMO}(\mathbf{R}^n))$ are valid if and only if $T1 = T'1 = 0$.

The adjoint operator T' is defined, as usual, by $\langle \psi, T'\phi\rangle := \langle \phi, T\psi\rangle$, where $\phi, \psi \in \mathscr{S}(\mathbf{R}^n)$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathscr{S}(\mathbf{R}^n)$ and $\mathscr{S}'(\mathbf{R}^n)$. For the precise definition of $T1$ and $T'1$, see the beginning of Sec. 2 below.

The following notion also appeared in the conditions of the $T1$ theorem:

Definition 1.8 (Weak boundedness property, [6]). The operator T is said to have the *weak boundedness property* provided that, for every pair of normalized bump functions (Def. 1.9) φ, ϕ associated with any ball $\bar{B}(x,r)$, we have $|\langle \phi, T\varphi\rangle| \leq Cr^{-n}$.

Such estimate follows, provided that $|\langle \phi, T\varphi\rangle| \leq C\,\|\phi\|_p\,\|\varphi\|_{p'}$ for all bump functions ϕ, φ and some $p \in \,]1,\infty[$.

Definition 1.9 (Bump functions, [26]). We say that φ is a *normalized bump function* associated with the unit ball if $\varphi \in \mathscr{D}(\mathbf{R}^n) := C_c^\infty(\mathbf{R}^n)$ with $\operatorname{supp}\varphi \subset \bar{B}(0,1)$ and $\|D^\alpha\varphi\|_\infty \leq 1$ for all $|\alpha| \leq N$, where N is a large fixed number. ϕ is a normalized bump function associated with the ball $\bar{B}(x_0,r)$ if $\phi(\cdot) = r^{-n}\varphi(r^{-1}(\cdot - x_0))$, where φ is a normalized bump function associated with the unit ball.

Finally, the work of Bourgain and Burkholder towards more general spaces, and that of David and Journé towards more general kernels, were combined in the results of T. Figiel [10] (see also [11] for the proof of an intermediate estimate omitted in [10]) who showed that not only the boundedness of the Hilbert transform (and wide classes of other convolution transformations [3]), but in fact all of the $T1$ theorem remains valid in the setting of the UMD-spaces: (Figiel in fact proved a "dyadic" version of this theorem, covering a class of kernels slightly larger than those verifying (1.3) through (1.5), but we here restrict the considerations to the standard kernels.)

Theorem 1.10 ([10, 11]). *Let $K : \mathbf{R}^{2n}_{x\neq y} \to \mathbf{C}$ verify the standard estimates, and T be as in* (1.6)*. Then, for any UMD-space X and any $p \in \,]1,\infty[$, the conditions* (1.7) *are equivalent to $T \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.*

Operator-valued kernels, and R-boundedness. In the vector-valued situation, one also encounters (e.g., in the theory of abstract evolution equations) integral transformations whose kernel is operator-valued, and the description of their continuity presents some new phenomena in comparison to the scalar-valued kernels. In fact, the new notion of *R-boundedness* (see Def. 1.12 below) of operator families has proved to be crucial in this setting, as is evident from the following result, which is due to one of us:

Theorem 1.11 ([28]). *Let X, Y be UMD spaces, and $m \in C^1(\mathbf{R}\setminus\{0\}, \mathscr{L}(X,Y))$ satisfy*

$$\mathscr{R}(\{m(x), xm'(x) : x \in \mathbf{R}\setminus\{0\}\}) < \infty.$$

Then $f \mapsto \mathscr{F}^{-1}(m\hat{f})$ (where $\hat{f} = \mathscr{F}f$ is the Fourier transform of f) is bounded from $L^p(\mathbf{R}, X)$ to $L^p(\mathbf{R}, Y)$ for all $p \in]1, \infty[$.

It is also known that R-boundedness assumptions on the multiplier function cannot be avoided in this theorem. In fact, it was shown by Ph. Clément and J. Prüss [5] that if $m \in L^1_{\text{loc}}(\mathbf{R}^n, \mathscr{L}(X,Y))$ and $f \mapsto \mathscr{F}^{-1}(m\hat{f})$ is bounded from $L^p(\mathbf{R}^n, X)$ to $L^p(\mathbf{R}^n, Y)$ for some $p \in]1, \infty[$, then $\{m(x) : x \text{ Lebesgue-point of } m\}$ is R-bounded.

Note that the sufficient condition of Theorem 1.11 for the L^p-continuity of $f \mapsto \mathscr{F}^{-1}(m\hat{f})$ simply replaces the uniform boundedness of $m(x)$ and $xm'(x)$ as in (the one-dimensional case of) the classical multiplier theorem of S. G. Mihlin by R-boundedness, defined as follows:

Definition 1.12 (R-boundedness)**.** A set $\mathscr{T} \subset \mathscr{L}(X,Y)$ is called R-bounded, with R-bound $\mathscr{R}(\mathscr{T}) \le C$, provided that

$$\mathbf{E}\Big|\sum_{i=1}^{N} \varepsilon_i T_i \xi_i\Big|_Y \le C\mathbf{E}\Big|\sum_{i=1}^{N} \varepsilon_i \xi_i\Big|_X$$

for all $N \in \mathbf{Z}_+$, all $\xi_i \in X$ and all $T_i \in \mathscr{T}$, where ε_i denote independent random variables on some probability space Ω with distribution $\mathbf{P}(\varepsilon_i = +1) = \mathbf{P}(\varepsilon_i = -1) = \frac{1}{2}$, and $\mathbf{E}$ is the corresponding expectation.

An often useful way of thinking of R-boundedness (exploited in [12, 13]) is in terms of the following notion well known from Banach space theory:

Definition 1.13 (The space $\operatorname{Rad} X$)**.** For a Banach space X, the Rademacher space $\operatorname{Rad} X$ is the closure in $L^p(\Omega, X)$ (equipped with its norm) of the subspace of all finite linear combinations $\sum \varepsilon_j x_j$, where $x_j \in X$ and ε_j are as in Def. 1.12. (Any $p \in [1, \infty[$ yields an equivalent definition, by a well-known inequality of J.-P. Kahane.)

Remark 1.14. $\mathscr{T} \subset \mathscr{L}(X,Y)$ is R-bounded if and only if the operators

$$\mathfrak{T} : \operatorname{Rad} X \to \operatorname{Rad} Y, \quad \sum \varepsilon_j x_j \mapsto \sum_{j=1}^{N} \varepsilon_j T_j x_j$$

are uniformly bounded for all $(T_j)_{j=1}^N \subset \mathscr{T}$.

We also recall that the notion of R-boundedness is genuinely stronger than that of uniform boundedness, as shown by the following folklore result due to G. Pisier, written up in [1]:

Proposition 1.15. *Let X, Y be Banach spaces. Then every bounded subset of $\mathscr{L}(X,Y)$ is R-bounded if and only if X has cotype 2 and Y has type 2. In particular, every bounded subset of $\mathscr{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space.*

On the other hand, it is well known by now that large classes of classical operators – such as Fourier multiplier operators, singular integrals, semigroups, and resolvents of many partial differential operators – do form R-bounded sets (see [7, 13, 16, 18]), and R-boundedness has become a useful tool in the theory of evolution equations. Several continuity results, exploiting R-boundedness, have recently been proved for translation-invariant integral transformations with an operator-valued kernel, both in the convolution and multiplier representations (see [1, 5, 12, 13, 16, 27, 28], and the references in these papers). All these developments suggest a rather general principle: *The generalization of the classical results on singular integrals to the operator-valued setting requires R-boundedness in place of boundedness.* Although this principle often leads to the right statement of the general result, it is usually not possible to "generalize" the proof in a similar straightforward manner; often a new approach based on the techniques of J. Bourgain [3] has to be developed. This is also the case in the present paper which, in its second part, also relies on the methods of T. Figiel [10].

Motivation of assumptions for the operator $T1$ theorem. The purpose of the present paper is to go beyond the translation-invariant case and to extend the $T1$ Theorem 1.2 to Calderón–Zygmund operators with operator-valued kernels. We start from a continuous linear operator

$$T : \mathscr{S}(\mathbf{R}^n) \mapsto \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y)),$$

where the vector-valued distributions are defined by $\mathscr{S}'(\mathbf{R}^n, Z) := \mathscr{L}(\mathscr{S}(\mathbf{R}^n), Z)$ for any Banach space Z (now $Z = \mathscr{L}(X,Y)$). Note that such an operator T can be identified with the continuous bilinear form

$$\mathfrak{t} : \mathscr{S}(\mathbf{R}^n) \times \mathscr{S}(\mathbf{R}^n) \to \mathscr{L}(X,Y), \qquad \mathfrak{t}(\phi,\psi) = (T\phi)(\psi).$$

We also use the more suggestive notation $\langle \psi, T\phi \rangle$ in place of $(T\phi)(\psi)$.

To such a T we assign an "adjoint" operator

$$T' : \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(Y',X')), \qquad \langle \psi, T'\phi \rangle := \langle \phi, T\psi \rangle',$$

where the latter $'$ designates the usual Banach adjoint of an operator in $\mathscr{L}(X,Y)$.

From T we derive a linear mapping $\tilde{T} : X \otimes \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n, Y)$: For $\xi \in X$ and $\phi, \psi \in \mathscr{S}(\mathbf{R}^n)$, we let $\left\langle \psi, \tilde{T}[\phi(\cdot)\xi] \right\rangle := \langle \psi, T\phi \rangle \xi \in Y$. This makes sense, since $\langle \psi, T\phi \rangle \in \mathscr{L}(X,Y)$. In this way, $\tilde{T}[\phi(\cdot)\xi]$ defines a Y-valued tempered distribution. This action of $\tilde{T}$ on $X \times \mathscr{S}(\mathbf{R}^n)$ is extended to $X \otimes \mathscr{S}(\mathbf{R}^n)$ by linearity. The target space $\mathscr{S}'(\mathbf{R}^n, Y)$ can be paired with $Y' \otimes \mathscr{S}(\mathbf{R}^n)$ in a natural way so that $\widetilde{T'} : Y' \otimes \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n, X')$ is in duality with $\tilde{T}$. The operator $\tilde{T}$ is the one for which, under the assumptions of the $T1$ Theorem, we seek an extension $\tilde{T} : L^p(\mathbf{R}^n, X) \to L^p(\mathbf{R}^n, Y)$. By a slight abuse of notation, we will usually denote $\tilde{T}$ and $\widetilde{T'}$ again by just T and T'.

The sense in which a kernel $K : \mathbf{R}^{2n}_{x \neq y} \to \mathscr{L}(X,Y)$ is now assumed to be associated to the operator T, is to require that (1.6) hold for all $f \in X \otimes \mathscr{D}(\mathbf{R}^n)$ and a.e. $x \notin \operatorname{supp} f$. By pairing this equation with a $g \in Y' \otimes \mathscr{D}(\mathbf{R}^n)$ having disjoint support with f, we readily find that the kernel K' associated to the adjoint operator T' is given by $K'(x,y) = K(y,x)'$, where the last $'$, again, is the Banach adjoint.

From our "principle" it is quite plain what the right substitute of the conditions (1.3) through (1.5) in the $T1$ theorem should be, and in place of the weak boundedness condition 1.8 one might expect the R-boundedness of the sets

$$\{r^n \langle \psi, T\phi \rangle : \phi, \psi \text{ normalized bump functions on } \bar{B}(x_0, r)\} \subset \mathscr{L}(X,Y).$$

Actually, a somewhat weaker (but for us sufficient) condition can be expressed in terms of the translated and dilated operators T^r_x (where $r > 0$, $x \in \mathbf{R}^n$), which we will find very useful. They are defined by

$$\langle \phi, T^r_x \varphi \rangle := r^{-n} \left\langle \phi(r^{-1}(\cdot - x)), T(\varphi(r^{-1}(\cdot - x)) \right\rangle, \qquad \forall\ \phi,\ \varphi \in \mathscr{S}(\mathbf{R}^n).$$

An equivalent definition without reference to duality is

$$T^r_x \varphi(u) := (T[\varphi(r^{-1}(\cdot - x))])(ru + x).$$

It is straightforward to see that T^r_x has an associated kernel

$$K^r_x(u,v) = r^n K(x + ru, x + rv)$$

in the same sense as T has the associated kernel K. Moreover, the kernels K^r_x satisfy uniformly (in fact, with the same constant) any of the conditions (1.3) through (1.5) provided that K satisfies them, and similarly the asumptions (1.7) for T imply the same conditions uniformly for all T^r_x. It then follows from the $T1$ theorem, or also directly, that the operators T^r_x are in fact uniformly bounded on $L^2(\mathbf{R}^n)$, and this being a Hilbert space, the family $\{T^r_x : r > 0, x \in \mathbf{R}^n\} \subset \mathscr{L}(L^2(\mathbf{R}^n))$ is R-bounded.

Motivated by this observation, it turns out to be fruitful not just to consider the boundedness of the single operator T but the R-boundedness of a whole family of operators T^r_x, where we allow arbitrary translations $x \in \mathbf{R}^n$ and dyadic dilations $r = 2^j$ with $j \in \mathbf{Z}$.

Recall that the "scalar-valued" weak boundedness property in Def. 1.8 can be understood as follows: The L^q-norm of a normalized bump function associated with $\bar{B}(x_0, r)$ is $\leq Cr^{-n/q'}$; thus, should the operator T be bounded on $L^q(\mathbf{R}^n)$ for some $q \in \,]1, \infty[$, then

$$|\langle \phi, T\psi \rangle| \leq \|\phi\|_{L^{q'}} \|T\psi\|_{L^q} \leq C \|\phi\|_{L^{q'}} \|\psi\|_{L^q} \leq Cr^{-n/q} r^{-n/q'} = Cr^{-n}.$$

Similarly, assuming the R-boundedness of the family $T^{2^j}_x$, we get the following:

Definition 1.16 (Weak R-boundedness property)**.** The operator $T : \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X, Y))$ is said to have the *weak R-boundedness property* provided there is a constant $C < \infty$ so that, for every pair of normalized bump functions ϕ, φ associated with the unit ball, we have

$$\mathscr{R}(\left\langle \phi, T^{2^j}_x \varphi \right\rangle : (x, j) \in \mathbf{R}^n \times \mathbf{Z}\} \leq C. \tag{1.17}$$

In fact, if ϕ and φ are some normalized bump functions associated with the unit ball and if F is some finite subset of $\mathbf{R}^n \times \mathbf{Z}$ and $\xi_{x,j} \in X$ for every $(x, j) \in F$, then

$$\begin{aligned}
\mathbf{E}\Big| \sum_{(x,j)\in F} \varepsilon_{x,j} \left\langle \phi, T^{2^j}_x \varphi \right\rangle \xi_{x,j} \Big|_Y &= \mathbf{E}\Big| \Big\langle \phi, \sum_{(x,j)\in F} \varepsilon_{x,j} T^{2^j}_x (\varphi(\cdot)\xi_{x,j}) \Big\rangle \Big|_Y \\
&\leq \|\phi\|_{L^{p'}(\mathbf{R}^n)} \mathbf{E}\Big\| \sum_{(x,j)\in F} \varepsilon_{x,j} T^{2^j}_x (\varphi(\cdot)\xi_{x,j}) \Big\|_{L^p(\mathbf{R}^n, Y)} \\
&\leq \|\phi\|_{p'} \mathscr{R}(\{T^{2^j}_x : (x, j) \in F\}) \mathbf{E}\Big\| \sum_{(x,j)\in F} \varepsilon_{x,j} \varphi(\cdot)\xi_{x,j} \Big\|_{L^p(\mathbf{R}^n, X)} \\
&= \|\phi\|_{p'} \mathscr{R}(\{T^{2^j}_x : (x, j) \in F\}) \|\varphi\|_p \mathbf{E}\Big| \sum_{(x,j)\in F} \varepsilon_{x,j} \xi_{x,j} \Big|_X .
\end{aligned}$$

(In the weak R-boundedness property, there is no need to require the condition for general bump functions, since the translations and dilations are already incorporated into the operators $T^{2^j}_x$.)

A special $T1$ theorem. With the above definitions at hand, we are ready to formulate the following special case of the $T1$ theorem, in which we have an essentially complete analogy to the classical situation:

Theorem 1.18. *Let X, Y be UMD-spaces, and suppose that $K : \mathbf{R}^{2n}_{x\neq y} \to \mathscr{L}(X, Y)$ is a kernel which satisfies the* standard R-estimates*:*

$$\mathscr{R}(|u - v|^n K(u, v) : u \neq v) < \infty, \tag{1.19}$$

$$\mathscr{R}(|u - v|^{n+\gamma} \frac{K(u, v) - K(u, v_0)}{|v - v_0|^\gamma} : |u - v| > 2 |v - v_0| > 0) < \infty, \tag{1.20}$$

$$\mathscr{R}(|u - v|^{n+\gamma} \frac{K(u, v) - K(u_0, v)}{|u - u_0|^\gamma} : |u - v| > 2 |u - u_0| > 0) < \infty. \tag{1.21}$$

Let K be associated to $T \in \mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y)))$.

Then the following implications hold:

- *If T satisfies the weak R-boundedness property and $T1 = T'1 = 0$ (see below), then*
$$T \in \mathscr{L}(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$$
for all $p \in]1,\infty[$, and also
$$T \in \mathscr{L}(H^1(\mathbf{R}^n, X), H^1(\mathbf{R}^n, Y)) \cap \mathscr{L}(\mathrm{BMO}(\mathbf{R}^n, X), \mathrm{BMO}(\mathbf{R}^n, Y)). \tag{1.22}$$
- *If, in addition, the spaces X and Y have property (α) (Def. 1.27), then*
$$\{T_x^{2^j} : j \in \mathbf{Z}, x \in \mathbf{R}^n\} \subset \mathscr{L}(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y)) \text{ is } R\text{-bounded.} \tag{1.23}$$
for all $p \in]1,\infty[$.
- *Conversely, (1.22) implies that $T1 = T'1 = 0$, and (1.23) for some $p \in]1,\infty[$ implies the weak R-boundedness property for T.*

Remark 1.24. If $X = Y$ is a Hilbert space, the estimates (1.17), (1.19), (1.20) and (1.21) turn into simple boundedness conditions. Hence, assuming the kernel conditions (1.19), (1.20) and (1.21), the boundedness of T on $H^1(\mathbf{R}^n, X)$, $L^p(\mathbf{R}^n, X)$ $(1 < p < \infty)$ and $\mathrm{BMO}(\mathbf{R}^n, X)$ is simply equivalent to $T1 = T'1 = 0$ and the simplified weak boundedness property (1.17).

We also note that, because of $\mathbf{E}\left|\sum \varepsilon_j x_j\right|_X \approx \left(\sum |x_j|_X^2\right)^{1/2}$, the proof of Theorem 1.18 greatly simplifies in the Hilbert space case. Moreover, since we do not use the Cotlar–Stein lemma, we obtain an alternative proof for the special $T1$ theorem in Hilbert spaces, which seems to be new even in the scalar case.

Remark 1.25. We are going to give a general definition of $T1$ and $T'1$ in Sec. 2, but for the purposes of the above Theorem it is possible to give a meaning to the condition "$T1 = T'1 = 0$" as follows: For any $\phi \in \mathscr{D}(\mathbf{R}^n)$ with vanishing integral and any $\psi \in \mathscr{D}(\mathbf{R}^n)$ which equals 1 in a neighbourhood of $\operatorname{supp}\phi$, we require that
$$\langle \psi, T\phi \rangle + \langle (1-\psi), T\phi \rangle = 0,$$
with a similar condition for T' in place of T. The latter pairing has a meaning as a usual integral, since $T\phi$ coincides with an integrable function outside the support of ϕ as a result of the standard estimates, see Sec. 2.

Remark 1.26. Once we prove the L^p estimate for T, the assertion (1.22) follows from this and the standard estimates for the kernel of T, together with the conditions $T1 = T'1 = 0$ by a similar argument as in the scalar-valued situation which is explained in [20], Ch. 7. Similarly, the necessity of $T1 = T'1 = 0$ for the boundedness (1.22) can be seen as in [20], the vector-valued situation playing a completely unimportant rôle in the reasoning.

The following notion also appeared above:

Definition 1.27 (Pisier's property (α))**.** We say that the Banach space X has *property* (α) provided that there exists a constant $\alpha(X) < \infty$ such that
$$\mathbf{E}\,\mathbf{E}'\Big|\sum_{i,j=1}^N \varepsilon_i\varepsilon_j'\lambda_{ij}\xi_{ij}\Big|_X \leq \alpha(X)\mathbf{E}\,\mathbf{E}'\Big|\sum_{i,j=1}^N \varepsilon_i\varepsilon_j'\xi_{ij}\Big|_X$$
for all $N \in \mathbf{Z}_+$, $\xi_{ij} \in X$ and scalars $|\lambda_{ij}| \leq 1$, where the ε_j' are independent copies of the ε_i on another probability space Ω', and $\mathbf{E}'$ is the corresponding expectation.

Recall that every q-concave Banach function space with $q < \infty$, in particular the spaces $L^q(\mu)$ with $1 \leq q < \infty$, do have property (α).

In addition to providing a partial converse to the $T1$ theorem, property (α) also allows to "bootstrap" Theorem 1.18 to obtain R-bounded classes of Calderón–Zygmund operators. The following corollary is easily deduced from the Theorem by using a trick from [13]; we indicate the details at the end of Sec. 2.

Corollary 1.28. *Let* X, Y *be UMD-spaces with property* (α) *and* $\mathscr{T}$ *be an* R*-bounded set in* $\mathscr{L}(X,Y)$*. Assume that* $\mathscr{K}$ *is a set of kernels* $K : \mathbf{R}^{2n}_{x\neq y} \to \mathscr{L}(X,Y)$ *such that all the operator families appearing in* (1.19)*,* (1.20) *and* (1.21) *are subsets of this fixed set* $\mathscr{T}$*. Assume further that each* $K \in \mathscr{K}$ *is associated to an operator* $T_K \in \mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y)))$ *such that* $T_K 1 = T_K' 1 = 0$*, and whenever* ϕ, ψ *are normalized bump functions associated with the unit ball, we have* $\left\langle \phi, (T_K)^{2^j}_x \right\rangle \in \mathscr{T}$ *for all* $(x,j) \in \mathbf{R}^n \times \mathbf{Z}$ *and* $K \in \mathscr{K}$*.*

Then the set $\{T_K : K \in \mathscr{K}\}$ *is* R*-bounded in* $\mathscr{L}(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, X))$ *for* $1 < p < \infty$ *and in* $\mathscr{L}(H^1(\mathbf{R}^n, X), H^1(\mathbf{R}^n, Y))$*.*

In particular, the Corollary implies R-boundedness – or, what is equivalent for the scalar-valued L^p-spaces, square-function estimates – for operators verifying uniformly the conditions of the original (special) $T1$ theorem. This result could of course be proved by more elementary means.

Counterexamples for paraproduct operators. The general case of the $T1$ theorem (i.e., $T1, T'1 \in \mathrm{BMO}$ instead of $T1 = T'1 = 0$) turns out to be far more subtle in the infinite-dimensional setting. In fact, counterexamples due to F. Nazarov, G. Pisier, S. Treil and A. Volberg [21, 22] show that the so-called *paraproducts*, which provide prototype examples of operators failing the condition $T1 = 0$, are not bounded in general on $L^2(\mathbf{R}, \ell^2(\mathbf{N}))$ if only the natural (necessary) BMO-condition is assumed. These authors consider the *dyadic paraproduct* operator $f \mapsto P_\Delta(g, f)$, for which the following results are shown:

Theorem 1.29 ([21, 22])**.** *There is a constant* $c > 0$ *such that for every* $n = 1, 2, \ldots$*, there exists a non-zero* $g \in \mathrm{BMO}^\Delta_s(\mathbf{R}, \mathscr{L}(\ell^2(n)))$ *such that*

$$\frac{\|P_\Delta(g,\cdot)\|_{\mathscr{L}(L^2(\mathbf{R},\ell^2(n)))}}{\|g\|_{\mathrm{BMO}^\Delta_s(\mathbf{R},\mathscr{L}(\ell^2(n)))}} \geq c \begin{cases} \log^{1/2} n & [21] \\ \log n & [22] \end{cases}$$

Consequently, if X *is an infinite dimensional Hilbert space, then there exists a* $g \in \mathrm{BMO}^\Delta_s(\mathbf{R}, \mathscr{L}(X))$ *for which* $P_\Delta(g, \cdot) \notin \mathscr{L}(L^2(\mathbf{R}, X))$*.*

For the definition of the dyadic paraproduct P_Δ and the strong dyadic BMO-space BMO^Δ_s, we refer to the articles just mentioned. The estimate $\log n$ is in fact sharp; the converse inequality also holds for all $g \in \mathrm{BMO}^\Delta_s(\mathbf{R}, \mathscr{L}(\ell^2(n)))$ as shown in [21, 22]. Paper [21], where the weaker $\log^{1/2} n$ estimate is shown, has the virtue of providing an explicit example of a critical function g.

Although Theorem 1.29 suggests some difficulties for the ∞-dimensional general $T1$ theorem, it is not, stricly speaking, a counterexample to the boundedness of the Calderón–Zygmund operators of our interest, since the kernel of the dyadic paraproduct does not verify the standard estimates (although "dyadic analogues" of these conditions do hold). Nevertheless, using a smooth modification of the counterexample in [21], we can show that an analogue of Theorem 1.29 does hold for the smooth paraproduct

$$P(g,f) := \sum_{j=-\infty}^{\infty} (\Psi_{j+3} * g)(\Phi_j * f), \tag{1.30}$$

where the Ψ_j constitute a dyadic (smooth) resolution of unity in the frequency domain and the Φ_j are the corresponding "partial sum" functions, as explained in detail in Section 3.

As we shall see, the $P(g,\cdot)$ so defined is, for appropriate g, a Calderón–Zygmund operator with a standard kernel, and one which verifies the weak R-boundedness property but fails to be bounded in general. We in fact show the same dimensional lower bound $\log^{1/2} n$ as in [21], and to get this we observe that it actually suffices that the domain of our operators is $L^2(\mathbf{R},\ell^2(n))$ but the target space may just as well be the usual $L^2(\mathbf{R})$ space of scalar-valued functions:

Theorem 1.31. *There is a constant $c>0$ such that for every $n=1,2,\ldots$, there is a non-zero $g\in\mathrm{BMO}_w(\mathbf{R},\ell^2(n))$ such that*

$$\|P(g,\cdot)\|_{\mathscr{L}(L^2(\mathbf{R},\ell^2(n)),L^2(\mathbf{R}))}\geq c\,\|g\|_{\mathrm{BMO}_w(\mathbf{R},\ell^2(n))}\log^{1/2} n.$$

Thus, if X is an infinite-dimensional Hilbert space, there exists $g\in\mathrm{BMO}_w(\mathbf{R},X)$ for which $P(g,\cdot)\notin\mathscr{L}(L^2(\mathbf{R},X),L^2(\mathbf{R}))$.

[Here we identify $X\approx X'=\mathscr{L}(X,\mathbf{C})$, and the subscript w of BMO_w refers to the weak Hilbert space topology of X which coincides, of course, with the strong operator topology of $\mathscr{L}(X,\mathbf{C})$; thus

$$\|g\|_{\mathrm{BMO}_w(\mathbf{R},X)}:=\sup_{\|e\|_X=1}\|(g(\cdot),e)_X\|_{\mathrm{BMO}(\mathbf{R})}=\|g\|_{\mathrm{BMO}_s(\mathbf{R},\mathscr{L}(X,\mathbf{C}))}\,.] \tag{1.32}$$

Positive results for paraproducts, and a general $T1$ theorem. We also consider paraproducts in the general UMD-valued situation, and provide some sufficient conditions for their boundedness which do recover the classical results in the case of scalar-valued kernels, although their formulation becomes somewhat more cumbersome in the general operator-valued situation. Still, they allow us to give a $T1$ theorem for operator-valued kernels k, which has the "right $T1\in\mathrm{BMO}$ -condition" if k takes its values in a space of operators with the UMD-property, such as Hilbert–Schmidt operators and some of their generalizations.

To find the "right" formulation of the assumptions outside the Hilbert space framework, note that the $(L^\infty(\mathbf{R}^n,X),\mathrm{BMO}(\mathbf{R}^n,Y))$ boundedness, which we want to establish for T and T', would imply that $T(x\otimes 1)\in\mathrm{BMO}(\mathbf{R}^n,Y)$ and $T'(y'\otimes 1)\in\mathrm{BMO}(\mathbf{R}^n,X')$, uniformly in $x\in B_X$ (the unit ball of X) and $y'\in B_{Y'}$. That is, $T1\in\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))$ and $T'1\in\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(Y',X'))$, where the subscript s refers to the strong operator topology, is a necessary condition for the sought boundedness of T.

Since we shall also need the R-boundedness of the operators $T_x^{2^j}$ discussed earlier, Remark 1.14 leads us to the conditions

$$(T_x^{2^j}1)_{(x,j)\in F}\in\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(\mathrm{Rad}(X),\mathrm{Rad}(Y))) \tag{1.33}$$

for all finite subsets $F\subset\mathbf{R}^n\times\mathbf{Z}$. Since formally $(T_x^{2^j}1)(u)=(T1)(2^ju+x)$, we use the notation $g_x^{2^j}(u)=g(2^ju+x)$ and introduce the following space, which we use to formalize the requirement that (1.33) should hold uniformly in $F\subset\mathbf{R}^n\times X$ by requiring that $T1\in\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$:

Definition 1.34 (The space $\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$)**.** Let g be an $\mathscr{L}(X,Y)$-valued distribution on $\mathbf{R}^n$, such that each $g(\cdot)\xi$, $\xi\in X$, is locally integrable. We say that $g\in\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ if $(g_x^{2^j}(\cdot)\xi_{xj})\in\mathrm{BMO}(\mathbf{R}^n,\mathrm{Rad}\,Y)$ for all finitely non-zero $(\xi_{xj})\in\mathrm{Rad}(X)$, with the norm estimate

$$\Big(\frac{1}{|\bar B|}\int_{\bar B}\mathbf{E}\Big|\sum_{x,j}\varepsilon_{xj}\left(g(2^ju+x)-g_{2^j\bar B+x}\right)\xi_{xj}\Big|_Y^q\,\mathrm{d}u\Big)^{1/q}\leq C\,\|\xi\|_{\mathrm{Rad}\,X} \tag{1.35}$$

for all balls $\bar B\subset\mathbf{R}^n$ and summations over finite sets of pairs $(x,j)\in\mathbf{R}^n\times\mathbf{Z}$.

As usual, $g_{\bar{B}}$ denotes the average of g on $\bar{B}$, now taken in the strong sense.

It follows from a well-known result of F. John and L. Nirenberg [17] (which also works, with the same proof, in the vector-valued setting) that any $q \in [1,\infty[$ above yields an equivalent definition.

The membership of g in the space $\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ will provide a sufficient condition for the weak boundedness of the paraproduct $P(g,\cdot)$. In order to be able to prove weak R-boundedness, we need the following subspace defined with double random series:

Definition 1.36 (The space $\mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y))$.)**.** For g as in the previous definition, we say that $g \in \mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ if

$$\Big(\frac{1}{|\bar{B}|}\int_{\bar{B}}\mathbf{E}\mathbf{E}'\Big|\sum_{x,j,y,i}\varepsilon_{xj}\varepsilon'_{yi}\big(g(2^j(2^i u+y)+x)-g_{2^j(2^i\bar{B}+y)+x}\big)\lambda_{yi}\xi_{xj}\Big|_Y^q\,\mathrm{d}u\Big)^{1/q} \\ \leq C\,\|\lambda\|_{\mathrm{Rad}\,\mathbf{C}}\,\|\xi\|_{\mathrm{Rad}\,X}$$

for all finitely non-zero $\xi = (\xi_{xj}) \in \mathrm{Rad}\,X$ and $\lambda = (\lambda_{yi}) \in \mathrm{Rad}\,\mathbf{C}$.

A couple of remarks concerning the relation of these spaces are in order. First, $\mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ is a subspace of $\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$, which is seen by taken $\lambda_{0,1} := 1$ and other components of λ equal to zero. If X and Y are Hilbert spaces, taking $q = 2$ in the definitions shows that both $\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ and $\mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y))$ reduce to the strong operator topology BMO space $\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))$.

If the space Y has property (α) and X has finite cotype (as every UMD-space does), then the two new BMO spaces coincide:

Lemma 1.37. *If Y has property (α) and X has finite cotype, then*

$$\mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y)) = \mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y)).$$

Let us also point out that our treatment does cover the scalar-valued BMO-functions (interpreted as operator-valued functions taking values in the linear span of the identity operator):

Lemma 1.38. *If X has finite cotype, then* $\mathrm{BMO}(\mathbf{R}^n) \hookrightarrow \mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X))$.

The proofs of these two lemmata will be given at the end of Sec. 3.

For the study of the paraproduct, we also need a slightly weaker version of the standard estimates, which we formulate in

Definition 1.39. We say that a kernel $K : \mathbf{R}^{2n}_{x\neq y} \to \mathscr{L}(X,Y)$ satisfies the *reduced standard R-estimates* if the following sets are R-bounded, with R-bounds uniformly bounded as u, v, u_0, v_0 range over $|u-v| > 2\,|u-u_0|\,, 2\,|v-v_0| > 0$:

$$\big\{2^{jn}\,|u-v|^n\,K(x+2^j u, x+2^j v) : x \in \mathbf{R}^n, j \in \mathbf{Z}\big\}\,,$$

$$\left\{2^{jn}\frac{|u-v|^{n+\gamma}}{|v-v_0|^{\gamma}}\big[K(x+2^j u, x+2^j v) - K(x+2^j u, x+2^j v_0)\big] : x \in \mathbf{R}^n, j \in \mathbf{Z}\right\},$$

and the set with the rôles of the first and second variable reversed.

Of course, $2^{jn}K(x+2^j u, x+2^j v)$ is simply the kernel of $T_x^{2^j}$ evaluated at (u,v). The proof of Theorem 1.18 will show that we could have replaced the standard R-estimates by their reduced versions in the statement of that result.

We can now state the following:

Theorem 1.40. *Let X, Y be UMD-spaces, and $g \in \mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))$. Then*

- *For every $p \in]1,\infty[$, the series* (1.30) *converges, for every $f \in X \otimes L^p(\mathbf{R}^n)$, weakly in $L^p(\mathbf{R}^n, Y) = (L^{p'}(\mathbf{R}^n, Y'))'$.*

- *The limit satisfies*

$$\|P(g,f(\cdot)\xi)\|_{L^p(\mathbf{R}^n,Y)} \leq C_p \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))} \|f\|_{L^p(\mathbf{R}^n)} |\xi|_X \tag{1.41}$$

for all $p \in]1,\infty[$. In particular, if we define

$$\langle \psi, P\phi\rangle\, x := \int_{\mathbf{R}^n} \psi(u) P(g,\phi(\cdot)x)(u)\,\mathrm{d}u$$

for $\phi,\psi \in \mathscr{S}(\mathbf{R}^n)$ and $x \in X$, we have $P \in \mathscr{L}(\mathscr{S}(\mathbf{R}^n),\mathscr{S}'(\mathbf{R}^n,\mathscr{L}(X,Y)))$, and P possesses the weak boundedness property.

- *P has an associated kernel which satisfies the reduced standard R-estimates.*
- *$P1 = g$, $P'1 = 0$.*

If, moreover, $g \in \mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y))$, then

- *P satisfies the weak R-boundedness property.*

As pointed out already, for Hilbert spaces X and Y we have

$$\mathrm{BMO}^2_R(\mathbf{R}^n,\mathscr{L}(X,Y)) = \mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y)),$$

with equivalence of norms. Thus Theorem 1.31 shows that the assumptions of Theorem 1.40 are insufficient, in general, to imply $P(g,\cdot) \in \mathscr{L}(L^p(\mathbf{R}^n,X),L^p(\mathbf{R}^n,Y))$.

A possible sufficient condition is obtained by requiring, in addition, the finiteness of the following BMO-type norm:

Definition 1.42 (The space $\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X,Y))$)**.** Let g be an $\mathscr{L}(X,Y)$-valued distribution on $\mathbf{R}^n$ which is strongly locally integrable, and let $q \in]1,\infty[$. We say that $g \in \mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X,Y))$, the "Littlewood–Paley-BMO" space, provided that the following quantity is finite:

$$\begin{aligned} &\|g\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X,Y))} \\ &\qquad := \sup_{\bar{B}} \left|\bar{B}\right|^{-1/q} \mathbf{E} \left\| \sum \varepsilon_j \Psi_j * (g - g_{\bar{B}}) 1_{\bar{B}} \right\|_{L^q(\mathbf{R}^n,\mathscr{L}(X,Y))}, \end{aligned} \tag{1.43}$$

where the supremum is taken over all balls $\bar{B} \subset \mathbf{R}^n$.

Observe that *if* the Littlewood–Paley decomposition were valid on the space $L^q(\mathbf{R}^n,\mathscr{L}(X,Y))$, then the randomized norm appearing in (1.43) would be equivalent to the L^q norm of $\left|\bar{B}\right|^{-1/q}(g - g_{\bar{B}})1_{\bar{B}}$, and then the supremum in (1.43) would simply be the BMO norm of g. This motivates the notation $\mathrm{BMO}_{\Psi,q}$.

Now we can formulate the following result for paraproducts:

Theorem 1.44. *Let X, Y be UMD-spaces, and*

$$g \in (\mathrm{BMO}_R \cap \mathrm{BMO}_{\Psi,q})(\mathbf{R}^n,\mathscr{L}(X,Y))$$

for some $q \in]1,\infty[$. Then $P(g,\cdot) \in \mathscr{L}(L^p(\mathbf{R}^n,X),L^p(\mathbf{R}^n,Y))$ for all $p \in]1,\infty[$.

By combining Theorems 1.18, 1.40 and 1.44, we obtain the following operator-valued "general $T1$ theorem" which, however, only provides sufficient conditions without the converse implication.

Corollary 1.45. *Let X, Y be UMD-spaces. Let*

$$T \in \mathscr{L}(\mathscr{S}(\mathbf{R}^n),\mathscr{S}'(\mathbf{R}^n,\mathscr{L}(X,Y)))$$

have an associated kernel K which satisfies the standard R-estimates. Suppose further that

- *T satisfies the weak R-boundedness property*; *and*

- *For some* $q_1, q_2 \in]1,\infty[$, *we have*

$$\begin{aligned} T1 &\in (\mathrm{BMO}^2_R \cap \mathrm{BMO}_{\Psi,q_1})(\mathbf{R}^n, \mathscr{L}(X,Y)), \\ T'1 &\in (\mathrm{BMO}^2_R \cap \mathrm{BMO}_{\Psi,q_2})(\mathbf{R}^n, \mathscr{L}(Y',X')). \end{aligned} \tag{1.46}$$

Then $T \in \mathscr{L}(L^p(\mathbf{R}^n,X), L^p(\mathbf{R}^n,Y))$ *for all* $p \in]1,\infty[$.

Proof. By Theorems 1.40 and 1.44, the paraproducts $P(T1,\cdot)$ and $P(T'1,\cdot)'$ define operators in $\mathscr{L}(L^p(\mathbf{R}^n,X), L^p(\mathbf{R}^n,Y))$. They also induce, in the canonical way described in Theorem 1.40, operators in $\mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y)))$ (which we denote by the same symbols) satisfying the weak R-boundedness property, and which have associated kernels verifying the reduced standard R-estimates. (As for $P(T'1,\cdot)'$, note that both the [reduced] standard R-estimates for the associated kernel and the weak R-boundedness are self-dual conditions, i.e., hold for T' if and only if for T; for details, see the discussion following Eq. (2.5) below.)

We define the new operator $T_0 := T - P(T1,\cdot) - P(T'1,\cdot)'$. It satisfies $T_0 1 = T_0' 1 = 0$. Its kernel verifies the reduced standard R-estimates, since the kernels of T, $P(T1,\cdot)$ and $P(T'1,\cdot)'$ verify them. The weak R-boundedness property holds for the same reason. Thus theorem 1.18 shows that T_0 is in the desired space $\mathscr{L}(L^p(\mathbf{R}^n,X), L^p(\mathbf{R}^n,Y))$. Since $P(T1,\cdot)$ and $P(T'1,\cdot)'$ are also, as already noted, the same conclusion holds for T. □

Remark 1.47. There are some prominent special cases in which the somewhat subtle $\mathrm{BMO}_{\Psi,q}$ norms can be estimated by a usual BMO norm:

- If g is finite-dimensionally valued, we can use an equivalent Hilbert space norm, and the Littlewood–Paley decomposition is valid, reducing the norm (1.43) to the usual norm topology BMO-norm $\|g\|_{\mathrm{BMO}(\mathbf{R}^n,\mathscr{L}(X,Y))}$. This of course contains the special case that $Y = X$ and g is scalar-valued.
- If $X = \mathbf{C}$ (resp. $Y = \mathbf{C}$), then $\mathscr{L}(X,Y) = Y$ (resp. X') is a UMD-space, and the Littlewood-Paley decomposition is valid. The same conclusion holds more generally if X or Y is finite-dimensional.
- If $Y = X$ is a Hilbert space, and g takes values not only in $\mathscr{L}(X)$, but more specifically in one of the *Schatten ideals* $S^p := \{A \in \mathscr{L}(X) : \|A\|_{S^p} := [\mathrm{trace}(A^*A)^{p/2}]^{1/p} < \infty\}$ for $p \in]1,\infty[$, we have

  $$\|g\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X))} \leq \|g\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,S^p)} \approx \|g\|_{\mathrm{BMO}(\mathbf{R}^n,S^p)},$$

  where the first estimate follows from the domination $\|A\|_{\mathscr{L}(X)} \leq \|A\|_{S^q}$, and the second from the fact that S^p is a UMD-space. Thus the condition $g \in \mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X))$ holds, for all $q \in]1,\infty[$, provided that $g \in \mathrm{BMO}(\mathbf{R}^n,S^p)$ for some $p \in]1,\infty[$.
- If $X = Y = L^q(M,\mu)$, $1 < q < \infty$, the space $HT^q(X)$ of *Hille–Tamarkin operators* consists of integral operators

  $$Tf(x) = \int_M k(x,y) f(y) \,\mathrm{d}\mu(y),$$

  where $k \in L^q(M, L^{q'}(M))$. The mixed-norm space $L^q(M, L^{q'}(M))$ is UMD, and the norm of the kernel in this space dominates the norm of the associated operator, so that $HT^q(X)$ is a UMD subspace embedded in $\mathscr{L}(X)$.
- For a separable Hilbert space H and a Banach space X, the space $\gamma(H,X)$ of γ*-radonifying operators* consists of those $T \in \mathscr{L}(H,X)$ for which, given an orthonormal basis $(e_n)_{n=1}^\infty$ of H and a sequence $(\gamma_n)_{n=1}^\infty$ of independent standard Gaussian random variables on some probability space $(\Omega, \mathbf{P})$, the

series $\sum_{n=1}^{\infty} \gamma_n T e_n$ converges in $L^2(\Omega, X)$. Equipped with the norm

$$\|T\|^2_{\gamma(H,X)} = \mathbf{E}\Big|\sum_{n=1}^{\infty} \varepsilon_n T e_n\Big|^2_X,$$

which dominates the usual operator norm, $\gamma(H, X)$ is identified with a closed subspace of $L^2(\Omega, X)$; thus it is UMD whenever X itself is. This space plays a decisive rôle in the theory of stochastic integration of vector-valued functions, as recently developed in [23, 24].

Remark 1.48. It is also possible to obtain an operator-valued version in UMD spaces of the Tb theorem of David, Journé and S. Semmes [8]. Such a program has been carried out by one of us in [15].

Organization of the paper. In the body of the paper, we establish the theorems stated in this Introduction. We first deal with the special case $T1 = T'1 = 0$, i.e., Theorem 1.18. In the treatment of the paraproduct operators, we depart from the order of presentation of the Introduction, and first prove the positive boundedness results, Theorems 1.40 and 1.44. The counterexample of Theorem 1.31 is constructed in the last section of the paper.

2. THE SPECIAL $T1$ THEOREM

The operators $T1$ and $T'1$. The action of T is not *a priori* defined on the constant function $1 \notin \mathscr{S}(\mathbf{R}^n)$, but we can make sense of the notion $T1$ by essentially the same procedure as in the scalar-valued situation: We define $T1$ as a linear operator acting on $\mathscr{D}^0(\mathbf{R}^n) := \{\phi \in \mathscr{D}(\mathbf{R}^n) : \int \phi(x)\,\mathrm{d}x = 0\}$, and with values in $\mathscr{L}(X, Y)$.

First, let us consider a (possibly vector-valued) distribution u which agrees with an integrable function in the complement of a compact set C. Then we define $\langle 1, u\rangle := \langle v_0, u\rangle + \langle v_1, u\rangle$, where $v_0 + v_1 \equiv 1$, $v_0 \in \mathscr{D}(\mathbf{R}^n)$, and v_1 vanishes in a neighbourhood of C. Then $\langle v_0, u\rangle$ has a sense as the usual duality pairing between test functions and distributions, and $\langle v_1, u\rangle$ can be evaluated as the (vector-valued) integral $\int v_1(x)u(x)\,\mathrm{d}x$. It is easily seen that the value of $\langle 1, u\rangle$ is independent of the decomposition $1 = v_0 + v_1$.

Now, if $\phi \in \mathscr{D}^0(\mathbf{R}^n)$ and $x \notin \operatorname{supp}\phi$, then

$$T\phi(x) = \int_{\mathbf{R}^n} [K(x, y) - K(x, y_0)]\,\phi(y)\,\mathrm{d}y, \qquad y_0 \in \operatorname{supp}\phi.$$

It follows from the standard estimates that this is an integrable function in the exterior of any neighbourhood of $\operatorname{supp}\phi$. Thus $\langle 1, T\phi\rangle$ is defined by the procedure above for $\phi \in \mathscr{D}^0(\mathbf{R}^n)$, and it remains to make the natural definition $\langle \phi, T'1\rangle := \langle 1, T\phi\rangle'$ for such ϕ. In an analogous way, we define $\langle T1, \phi\rangle \equiv \langle 1, T'\phi\rangle'$.

We obtain the representation

$$\langle T1, \phi\rangle = \langle Tv_0, \phi\rangle + \iint_{\mathbf{R}^n\times\mathbf{R}^n} [K(x, y) - K(x, y_0)]\,\phi(y)v_1(x)\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi \in \mathscr{D}^0(\mathbf{R}^n)$, $1 = v_0 + v_1$, $v_0 \in \mathscr{D}(\mathbf{R}^n)$, v_1 has disjoint support with ϕ, and $y_0 \in \operatorname{supp}\phi$.

A possibly more intuitively appealing definition of $T1$ is as follows: Given $\varphi \in \mathscr{D}(\mathbf{R}^n)$ with $\varphi(0) = 0$, we consider the distributions $T(\varphi(\epsilon\cdot))$. Then there exist constants $c_\epsilon \in \mathscr{L}(X, Y)$ such that the limit $\lim_{\epsilon\downarrow 0}(T(\varphi(\epsilon\cdot)) - c_\epsilon)$ exists in the distributional sense, and it can be shown to coincide with $T1$ as defined before. We refer the interested reader to [20], Lemma 2 of Ch. 8 for details (in the scalar case, but one can easily see that the argument goes through in our setting). We shall not make use of this alternative approach in this paper, however.

Decomposition of operators. The key ingredient in the original proof of the $T1$ theorem is the Cotlar–Stein lemma which gives sufficient conditions for the boundedness of a Hilbert space operator which is given as an "almost orthogonal" sum $T = \sum T_j$. While the Cotlar–Stein lemma is strictly a Hilbert space device, there are other methods of almost orthogonality, some of which carry over to a more general framework. We now turn to our decomposition of the operator.

Lemma 2.1. *There exist a radial real-valued* $\Phi \in \mathscr{D}(\mathbf{R}^n)$ *such that* $\operatorname{supp}\Phi \subset \bar{B}(0,1)$*,* $\int \Phi(x)\,\mathrm{d}x = 0$ *and* $\hat{\Phi}$ *is everywhere non-negative, and moreover* $\hat{\Phi}(x) \geq 1$ *for* $0 < a \leq |x| \leq 4a$*.*

Proof. Take any real, radial $\varphi \in \mathscr{D}(\mathbf{R}^n)$ satisfying the vanishing integral condition. Since φ is even, its Fourier transform is real. Then also $\phi := \varphi * \varphi$ is such a function, and moreover $\mathscr{F}[\varphi * \varphi] = \hat{\varphi}^2 \geq 0$. Since $\hat{\phi}$ is radial and not identically vanishing, it is strictly positive on some annulus $a_1 \leq |x| \leq a_2$. Taking as Φ a suitable linear combination, with positive coefficients, of dilates of ϕ, we ensure the condition $\hat{\Phi}(x) \geq 1$ for $0 < a \leq |x| \leq 4a$ for some $a > 0$; moreover, taking the dilations appropriately we also ensure the support condition. □

Corollary 2.2. *There exists a* Φ *as in Lemma* 2.1*, and a* $\Psi \in \mathscr{S}(\mathbf{R}^n)$ *with positive Fourier transform supported in the annulus* $\{a \leq |x| \leq 4a\}$*, and such that*

$$\sum_{-\infty}^{\infty} \hat{\Phi}(2^j x)\hat{\Psi}(2^j x) = 1 \qquad \forall\ x \in \mathbf{R}^n \setminus \{0\}. \tag{2.3}$$

Proof. Let $\hat{\psi} \in \mathscr{S}(\mathbf{R}^n)$ be constructed as in the usual dyadic resolution of the unity such that it is non-negative, supported in the annulus $\{a \leq |x| \leq 4a\}$, and satisfies $\sum \hat{\psi}(2^j x) = 1$ for $x \neq 0$. Then set $\hat{\Psi}(x) := \hat{\psi}(x)/\hat{\Phi}(x)$. □

Definition 2.4 (Resolution operators)**.** With Φ and Ψ as above, we denote $\Phi_j(x) := 2^{-nj}\Phi(2^{-j}x)$, $\Psi_j(x) := 2^{-nj}\Psi(2^{-j}x)$. Then let $P_j f = \Phi_j * f$, $Q_j f = \Psi_j * f$ for all tempered distributions f.

Observe that P_j and Q_j are commuting and self-adjoint operators. The equation (2.3) for the functions Φ and Ψ now transforms to the operator equality

$$\sum_{-\infty}^{\infty} P_j Q_j = I.$$

When applied to an f whose Fourier transform has a compact support not containing the origin, there is even no question of convergence, since only finitely many of the terms $P_j Q_j f$ are non-zero.

We can then decompose

$$\begin{aligned}\langle g, Tf\rangle &= \sum_j \sum_k \langle P_{j+k}Q_{j+k}g, TP_jQ_jf\rangle \\ &= \sum_{k\geq 0}\sum_j \langle Q_{j+k}g, P_{j+k}TP_jQ_jf\rangle + \sum_{k<0}\sum_j \langle P_jT'P_{j+k}Q_{j+k}g, Q_jf\rangle,\end{aligned} \tag{2.5}$$

and we are lead to investigate the partial operators $P_{j+k}TP_j$ and $P_jT'P_{j+k}$. In fact, it suffices to consider the first case, since T' satisfies exactly the same assumptions as T: This is an easy consequence of the fact that $(\operatorname{Rad} X)' \approx \operatorname{Rad}(X')$, with the natural duality pairing of $L^2(\Omega, X)$ and $L^2(\Omega, X')$, for a UMD-space X. Concerning the standard R-estimates on the kernel, we only need to note that

$$\|\mathfrak{T}'\eta'\|_{\operatorname{Rad} X'} \lesssim \sup_{\|\xi\|_{\operatorname{Rad} X}\leq 1} |\langle \mathfrak{T}'\eta', \xi\rangle| = \sup |\langle \eta', \mathfrak{T}\xi\rangle| \lesssim \|\eta'\|_{\operatorname{Rad} Y'} \|\mathfrak{T}\|_{\mathscr{L}(\operatorname{Rad} X, \operatorname{Rad} Y)},$$

from which it follows that $\mathscr{R}(\mathscr{T}') \leq C\mathscr{R}(\mathscr{T})$, where $\mathscr{T}' := \{T' : T \in \mathscr{T}\}$ and C is a geometric constant. Similarly, the weak R-boundedness property is seen to be self-dual by writing

$$\|\langle \phi, \mathfrak{T}\varphi\rangle\, \xi\|_{\mathrm{Rad}\, Y} \approx \sup_{\eta' \in \mathrm{Rad}\, Y'} |\langle \eta', \langle \phi, \mathfrak{T}\varphi\rangle\, \xi\rangle|$$

for $\mathfrak{T} = (T_x^{2^j})_{(j,x)\in F}$, where $F \subset \mathbf{Z} \times \mathbf{R}$.

To find a more explicit representation for the operators $P_{j+k}TP_j$, we recall that, in general, the combined action of T with two smooth convolution operators on both sides is given by

$$\begin{aligned}
\phi * T(\varphi * f)(x) &= \left\langle \phi(x - \cdot), T\left(\int \varphi(\cdot - y) f(y)\,\mathrm{d}y\right)\right\rangle \\
&= \int \langle \phi(x - \cdot), T(\varphi(\cdot - y))\rangle\, f(y)\,\mathrm{d}y = \int \left\langle \tilde{\phi}(\cdot - x), T(\varphi(\cdot - y))\right\rangle f(y)\,\mathrm{d}y.
\end{aligned}$$

In particular, we have the kernel representation

$$\Phi_{j+k} * T(\Phi_j * f)(x) \equiv \int K_{j+k,j}(x,y) f(y)\,\mathrm{d}y,$$

where the kernel is explicitly given by

$$\begin{aligned}
K_{j+k,j}(x,y) &:= \langle \Phi_{j+k}(\cdot - x), T(\Phi_j(\cdot - y))\rangle \\
&= \langle 2^{-jn}\Phi_k(2^{-j}(\cdot - x)), T[2^{-jn}\Phi(2^{-j}(\cdot - x) + 2^{-j}(x - y)))]\rangle \\
&= 2^{-jn} \left\langle \Phi_k, T_x^{2^j}[\Phi(\cdot + 2^{-j}(x - y))]\right\rangle.
\end{aligned}$$

The form

$$2^{jn} K_{j+k,j}(x, x + 2^j y) = \left\langle \Phi_k, T_x^{2^j}[\Phi(\cdot - y)]\right\rangle \tag{2.6}$$

will be particularly useful.

Notice that we have decomposed the operator T into a two-parameter family $T_{j+k,j}$ of partial operators, as opposed to the one-parameter decomposition in the original proof.

Formal implications of the kernel conditions. We now establish estimates for the partial operators $T_{j+k,j}$ that follow from the assumptions on T and K. We first deal with such properties which are rather formal consequences of these assumptions. In order to simplify the notation, we therefore omit all the reference to the vector-valued situation here, and simply write absolute value signs instead of the various norms.

For many purposes, slightly weaker forms of the standard estimates will be sufficient. We note in particular that (1.4) implies

$$\left(\frac{1}{r^n}\int_{r<|x-y|<2r} |K(x,y) - K(x,y_0)|^q\,\mathrm{d}x\right)^{1/q} \leq C\frac{|y - y_0|^\gamma}{r^{n+\gamma}} \qquad \forall x, x_0 \in \mathbf{R}^n,\ r > 2\,|y - y_0|. \tag{2.7}$$

for any $q \in [1, \infty[$, and this is the assumption we will be using for establishing the following estimates. It will turn out that we need to require the exponent q to be sufficiently large, but a finite value will always be sufficient.

Lemma 2.8. *Let T be an operator with an associated kernel K satisfying* (2.7). *Let φ be a normalized bump function associated with $\bar{B}(0,r)$ and having vanishing* *total integral. Then*

$$k_y(x) := T[\varphi(\cdot - y)](x) = \int_{\mathbf{R}^n} K(x,v)\varphi(v-y)\,\mathrm{d}v, \qquad x \notin \bar{B}(y,r),$$

satisfies, for $j = 1, 2, \ldots,$

$$\left(\frac{1}{(2^j r)^n}\int_{2^j r<|x-y|<2^{j+1}r} |k_y(x)|^q \,\mathrm{d}x\right)^{1/q} \leq C2^{-j(n+\gamma)}r^{-n}. \tag{2.9}$$

Proof. Let $2^j r < |x-y| < 2^{j+1}r$, where $j = 1, 2, \ldots,$ and write

$$k_y(x) = \int_{|v-y|<r} (K(x,v) - K(x,y))\,\varphi(v-y)\,\mathrm{d}v.$$

Then it follows that the quantity on the LHS of (2.9) is estimated by

$$\begin{aligned}&\int_{|v-y|<r}\left(\frac{1}{(2^j r)^n}\int_{2^j r<|x-y|<2^{j+1}r} |K(x,v)-K(x,y)|^q \,\mathrm{d}x\right)^{1/q} |\varphi(v-y)|\,\mathrm{d}v\\ &\leq \int_{|v-y|<r} C\frac{|v-y|^\gamma}{(2^j r)^{n+\gamma}}|\varphi(v-y)|\,\mathrm{d}v \leq C2^{-j(n+\gamma)}r^{-n}\|\varphi\|_{L^1} \leq C2^{-j(n+\gamma)}r^{-n};\end{aligned}$$

the first inequality follows from (2.7), and the last from the definition of a normalized bump function. ☐

Lemma 2.10. *Let* φ *and* ϕ *be normalized bump functions associated with* $\bar{B}(0,r)$ *and* $\bar{B}(0,R)$*, respectively, where* $R \geq r$ *and* $\int\varphi = 0$*. Let* T *verify the weak boundedness property and the condition* $T'1 = 0$*, and be associated with a kernel* K *which satisfies the estimate* (2.7)*. Then*

$$|\langle \phi(x-\cdot), T[\varphi(\cdot - y)]\rangle| \leq C\left(\frac{r}{R}\right)^\gamma (1+\log\frac{R}{r})^{\delta_{\gamma,1}} R^{-n}(1+R^{-1}|x-y|)^{-(\gamma+n/q')},$$

where the Kronecker symbol $\delta_{\gamma,1}$ *is* 1 *if* $\gamma = 1$ *and* 0 *otherwise.*

Proof. Since $T'1 = 0$ and $\varphi(\cdot - y) \in \mathscr{D}^0(\mathbf{R}^n)$, we have $\langle \phi(x-y), T[\varphi(\cdot - y)]\rangle = \phi(x-y)\langle T'1, \varphi(\cdot - y)\rangle = 0$. Let us fix a function υ_0 which equals 1 in $\bar{B}(0,2)$ and vanishes outside $\bar{B}(0,4)$, and let $\upsilon := \upsilon_0(\cdot) - \upsilon_0(2\cdot)$, so that $\operatorname{supp}\upsilon \subset \bar{B}(0,4)\setminus B(0,1)$. Then we write

$$\begin{aligned}\langle \phi(x-\cdot), T[\varphi(\cdot - y)]\rangle &= \langle (\phi(x-\cdot) - \phi(x-y))\upsilon_0((\cdot - y)/r), T[\varphi(\cdot - y)]\rangle\\ &+ \sum_{j=1}^\infty \big\langle (\phi(x-\cdot) - \phi(x-y))\upsilon((\cdot - y)/2^j r), T[\varphi(\cdot - y)]\big\rangle =: I_0 + \sum_{j=1}^\infty I_j.\end{aligned}$$

To estimate I_0, we are going to use the weak boundedness property. Obviously both $\varphi(\cdot - y)$ and $(\phi(x-\cdot) - \phi(x-y))\upsilon_0((\cdot - y)/r)$ are supported in $\bar{B}(y,4r)$, and $\varphi(\cdot - y)$ is, upto a numerical multiplicative constant, a normalized bump function associated to this ball. Concerning the other function, we note that $|(\phi(x-\cdot) - \phi(x-y))| \leq |\cdot - y|\,\|\nabla\phi\|_\infty \leq 4rR^{-(n+1)}$ on $\bar{B}(y,4r)$, and then from Leibniz' rule

$$\begin{aligned}&|D^\alpha[(\phi(x-\cdot) - \phi(x-y))\upsilon_0((\cdot - y)/r)]|\\ &\leq 4rR^{-(n+1)}r^{-|\alpha|} + \sum_{0\neq\theta\leq\alpha}\binom{\alpha}{\theta}R^{-n-|\theta|}r^{-|\alpha-\theta|} \leq C(r/R)^{n+1}r^{-n-|\alpha|},\end{aligned}$$

so that this other function is $C(r/R)^{n+1}$ times a normalized bump function associated with $\bar{B}(y,4r)$. The weak boundedness property then gives $|I_0| \leq CrR^{-n-1}$.

Concerning the I_j, $j = 1, 2, \ldots$, it is immediate in view of the support property of $\upsilon((\cdot - y)/2^j r)$ and Lemma 2.8 that

$$|I_j| \le \begin{cases} C2^j r \left\|\nabla\phi\right\|_\infty 2^{-j\gamma} & = C2^{j(1-\gamma)} r R^{-n-1} \\ C \left\|\phi\right\|_\infty 2^{-j\gamma} & = C2^{-j\gamma} R^{-n}, \end{cases}$$

where the two different estimates arise from the two obvious possibilities of estimating the difference $|\phi(x - \cdot) - \phi(x - y)|$.

Using the established estimates, we find that

$$\begin{aligned} \sum_{j=0}^{\infty} |I_j| &\le \sum_{j:2^j r \le R} CrR^{-n-1} 2^{j(1-\gamma)} + \sum_{j:2^j r > R} CR^{-n} 2^{-j\gamma} \\ &\le C\left(\frac{r}{R}\right)^{\gamma} (1 + \log \frac{R}{r})^{\delta_{\gamma,1}} R^{-n}. \end{aligned}$$

Estimate for $|x - y| > 9R$. To get the appropriate decay as a function of $|x - y|$, we need to take into account the support properties of our bump functions more carefully. We note that a necessary condition for the function $\phi(x - \cdot)\upsilon((\cdot - y)/2^j r)$ to be non-vanishing is $R + 2^{j+2} r \ge |x - y|$, i.e., $2^{j+2} r \ge |x - y| - R > 8/9 \cdot |x - y|$. Thus we now have

$$\langle \phi(x - \cdot), T[\varphi(\cdot - y)] \rangle = \sum_{j:2^j r > 2/9 \cdot |x-y|} \left\langle \phi(x - \cdot)\upsilon((\cdot - y)/2^j r), T[\varphi(\cdot - y)] \right\rangle,$$

where $2/9 \cdot |x - y| > 2R \ge 2r$, so that $j \ge 1$.

From Lemma 2.8, the L^q-norm of $\upsilon((\cdot - y)/2^j r) T[\varphi(\cdot - y)]$ has the upper bound $C2^{-j(n/q' + \gamma)} r^{-n/q'}$, and the $L^{q'}$-norm of $\phi(x - \cdot)$ is estimated by $CR^{-n} R^{n/q'}$. Thus

$$\begin{aligned} |\langle \phi(x - \cdot), T[\varphi(\cdot - y)] \rangle| &\le CR^{-n} \cdot R^{n/q'} r^{-n/q'} \sum_{j:2^j r > 8^{-1}|x-y|} 2^{-j(n/q' + \gamma)} \\ &\le CR^{-n} \left(\frac{r}{R}\right)^{\gamma} \left(\frac{|x - y|}{R}\right)^{-(n/q' + \gamma)}. \end{aligned}$$

These estimates prove the Lemma. $\square$

Corollary 2.11. *For the kernels* $K_{j+k,j}$ $(j \in \mathbf{Z},\ k = 0, 1, 2, \ldots)$ *defined in* (2.6), *the following estimate holds provided that* T *has the weak boundedness property,* $T'1 = 0$, *and* K *satisfies* (2.7)*:*

$$2^{jn} \left|K_{j+k,j}(x, x + 2^j y)\right| \le C(1 + k)^{\delta_{\gamma,1}} 2^{-k\gamma} \cdot 2^{-kn} (1 + 2^{-k} |y|)^{-(n/q' + \gamma)}.$$

Proof. This is immediate by applying Lemma 2.10 to $\varphi = \Phi$, which is associated to $\bar{B}(0, 1)$, and $\phi = \Phi_k$, which is associated to $\bar{B}(0, 2^k)$. $\square$

Vector-valued estimates. We now move from the formal estimates to ones where the analytic properties of the function spaces in question become crucial. Let $f \in X \otimes \hat{\mathscr{D}}_0(\mathbf{R}^n)$ and $g \in Y' \otimes \hat{\mathscr{D}}_0(\mathbf{R}^n)$, where

$$\hat{\mathscr{D}}_0(\mathbf{R}^n) := \left\{ \phi \in \mathscr{S}(\mathbf{R}^n) : \operatorname{supp} \hat{\phi} \text{ is a compact subset of } \mathbf{R}^n \setminus \{0\} \right\};$$

we are interested in obtaining an estimate $|\langle g, Tf \rangle| \le C \left\|g\right\|_{L^{p'}(Y')} \left\|f\right\|_{L^p(X)}$, which would imply the boundedness of T from $L^p(\mathbf{R}^n, X)$ to $L^p(\mathbf{R}^n, Y)$. To this end, we estimate the first part in the decomposition (2.5) as follows:

$$\begin{aligned}\Big|\sum_j \langle Q_{j+k}g, P_{j+k}TP_jQ_jf\rangle\Big| &= \Big|\mathbf{E}\Big\langle \sum_i \varepsilon_i Q_{i+k}g, \sum_j \varepsilon_j P_{j+k}TP_jQ_jf\Big\rangle\Big| \\ &\le \Big(\mathbf{E}\Big\|\sum \varepsilon_i Q_{i+k}g\Big\|^{p'}_{L^{p'}(Y')}\Big)^{1/p'}\Big(\mathbf{E}\Big\|\sum \varepsilon_j P_{j+k}TP_jQ_jf\Big\|^{p}_{L^{p}(Y)}\Big)^{1/p},\end{aligned}$$

and the first factor is bounded by $C\left\|g\right\|_{L^{p'}(Y')}$.

In an analogous fashion one shows that

$$\Big|\sum_j \langle P_jT'P_{j+k}Q_{j+k}g, Q_jf\rangle\Big| \le C\Big(\mathbf{E}\Big\|\sum \varepsilon_j P_jT'P_{j+k}Q_{j+k}g\Big\|^{p'}_{L^{p'}(X')}\Big)^{1/p'}\left\|f\right\|_{L^p(X)}.$$

Thus, showing the boundedness of T from $L^p(X)$ to $L^p(Y)$ amounts to proving that

$$\sum_{k=0}^{\infty}\Big[\mathbf{E}\Big\|\sum_{j\in\mathbf{Z}} \varepsilon_j P_{j+k}TP_jQ_jf\Big\|^p_{L^p(Y)}\Big]^{1/p} \le C\left\|f\right\|_{L^p(X)}, \tag{2.12}$$

as well as the dual inequality with T, f, p, X and Y replaced by T', g, p', Y' and X', respectively.

Lemma 2.13. *The k-th term in the series on the LHS of* (2.12) *is bounded by*

$$C\int_{\mathbf{R}^n}\sup_{x\in\mathbf{R}^n}\mathscr{R}(2^{nj}K_{j+k,j}(x,x+2^jy) : j\in\mathbf{Z})\log(2+|y|)\,\mathrm{d}y\cdot\left\|f\right\|_{L^p(X)}. \tag{2.14}$$

Proof. Changing the integration variable from y to $x+2^jy$

$$\begin{aligned}&\Big(\mathbf{E}\Big\|\sum\int \varepsilon_j K_{j+k,j}(x,x+2^jy)Q_jf(x+2^jy)2^{jn}\,\mathrm{d}y\Big\|^p_{L^p(\mathrm{d}x,Y)}\Big)^{1/p} \\ &\quad\le \int \mathrm{d}y\,\Big(\int\mathbf{E}\Big|\sum\varepsilon_j 2^{jn}K_{j+k,j}(x,x+2^jy)Q_jf(x+2^jy)\Big|^p_Y\,\mathrm{d}x\Big)^{1/p} \\ &\qquad\le \int\mathrm{d}y\sup_{x\in\mathbf{R}^n}\mathscr{R}(2^{nj}K_{j+k,j}(x,x+2^jy) : j\in\mathbf{Z}) \\ &\qquad\qquad\times\Big[\mathbf{E}\int\Big|\sum\varepsilon_jQ_jf(x+2^jy)\Big|^p_X\,\mathrm{d}x\Big]^{1/p}.\end{aligned}$$

The last factor above is bounded by

$$C\log(2+|y|)\Big[\mathbf{E}\int\Big|\sum\varepsilon_jQ_jf(x)\Big|^p_X\,\mathrm{d}x\Big]^{1/p} \le C\log(2+|y|)\left\|f\right\|_{L^p(X)}$$

by the lemma of Bourgain cited below, and the vector-valued Littlewood-Paley decomposition. □

We used above the following result, which is shown by Bourgain [3] in the case of $\mathbf{T}$ in place of $\mathbf{R}^n$. The version below can by obtained from that by standard transference techniques; see [12] for details.

Lemma 2.15 ([3]). *Let $p\in\,]1,\infty[$, and X be a UMD-space. Then there is a constant $C<\infty$ such that every $y\in\mathbf{R}^n$ and every finitely non-zero sequence $(f_j)_{-\infty}^{\infty}\subset L^p(\mathbf{R}^n,X)$ which has the property that* $\operatorname{supp}\hat{f}_j\subset\{x : |x|\le 2^{-j}\}$, *satisfy*

$$\mathbf{E}\Big\|\sum\varepsilon_jf_j(\cdot+2^jy)\Big\|_{L^p(\mathbf{R}^n,X)} \le C\log(2+|y|)\mathbf{E}\Big\|\sum\varepsilon_jf_j\Big\|_{L^p(\mathbf{R}^n,X)}.$$

Scalar-valued kernels. If the kernel K is scalar-valued, or else X has cotype 2 and Y has type 2 (in particular, if both X and Y are Hilbert spaces), then the R-bound appearing in (2.14) is simply

$$\sup_{x\in\mathbf{R}^n, j\in\mathbf{Z}} 2^{nj}\left|K_{j+k,j}(x,x+2^j y)\right| \leq C(1+k)^{\delta_{\gamma,1}}2^{-k\gamma}\cdot 2^{-kn}(1+2^{-k}|y|)^{-n/q+\gamma}$$

according to Cor. 2.11, provided that the assumptions of that Corollary are satisfied, of course. Then, after the change-of-variable $v=2^{-k}y$, the integral in (2.14) is readily seen to be bounded by $C(1+k)^{1+\delta_{\gamma,1}}2^{-k\gamma}\|f\|_{L^p(X)}$, provided that $\gamma>n/q$. Then it is clear that the inequality (2.12) is satisfied. Moreover, it is plain that the same assumptions for the dual operator T' yield the estimate "dual" to (2.12).

Thus the same assumptions as in the original scalar-valued case of the special $T1$ theorem suffice to give the boundedness of the operator T with a scalar-valued kernel K on $L^p(X)$, where X is an arbitrary UMD-space X; we also get the boundedness of T from $L^p(X)$ to $L^p(Y)$ for operator-valued kernels provided that X has cotype 2 and Y had type 2.

It is worth pointing out that the main analytic tools employed in obtaining this result were the Littlewood–Paley inequality and Bourgain's translation lemma. In the case when $p=2$ and X and Y are Hilbert spaces, both these reduce to trivialities about orthogonal expansions. Thus our general approach, specialized to the classical situation, provides a new proof of the scalar-valued special $T1$ theorem which is no harder than the original one, and, perhaps depending on personal taste, could even be considered simpler.

General kernels. In general, we need to cope with the presence of R-boundedness in (2.14). Let $\xi:=(\xi_j)_{-\infty}^{\infty}$ be a finitely non-zero sequence of elements of X. Then

$$\begin{aligned}\sum\varepsilon_j 2^{nj}K_{j+k,j}(x,x+2^j y)\xi_j &= \sum\varepsilon_j\left\langle\Phi_k, T_x^{2^j}[\Phi(\cdot-y)]\right\rangle\xi_j\\ &=\left\langle\Phi_k,\Big(\sum\varepsilon_j T_x^{2^j}\xi_j\Big)(\Phi(\cdot-y))\right\rangle =: \left\langle\Phi_k,\mathfrak{T}_{x\xi}(\Phi(\cdot-y))\right\rangle.\end{aligned}$$

Now the RHS is of the same form as (2.6); we have introduced the new operator $\mathfrak{T}_{x\xi}$ which has an associated kernel

$$\mathfrak{K}_{x\xi}(u,v)=\sum\varepsilon_j 2^{jn}K(x+2^j u, x+2^j v)\xi_j.$$

Thus it is clear that we can reach our desired conclusion provided only that we replace the original assumptions on T and K by the corresponding conditions for $\mathfrak{T}_{x\xi}$ and $\mathfrak{K}_{x\xi}$, the underlying space now being $\operatorname{Rad}X$. One readily finds that the analogues of the standard estimates (1.3) through (1.5) obtained by this procedure will be nothing else but the standard R-estimates as formulated in Remark 1.39, and so those conditions on K are exactly what we need to get the desired R-boundedness of the kernels $K_{j+k,j}$.

Similarly, substituting $\mathfrak{T}_{x\xi}$ in place of T in the weak boundedness condition and using appropriate norms, we end up with the requirement of the weak R-boundedness property (1.17).

Conclusion of the proof of Theorem 1.18. We have just seen above how the natural R-boundedness analogues of the classical conditions of the special $T1$ theorem give the estimates required to prove the boundedness of T in the operator-valued setting. We also observe that the weak R-boundedness property follows from the R-boundedness of the translates and dilates $T_x^{2^j}$, in just the same way as the weak boundedness follows from the boundedness of T in the scalar-valued case. What remains to be shown is the second assertion of Theorem 1.18 under the additional assumption that the spaces X and Y have the property (α). This follows easily by a trick recently found by M. Girardi and one of us [13]:

The asserted R-boundedness means the uniform boundedness of the operators $\mathfrak{T} = (T_x^{2^j})_{(x,j)\in F} : \sum \varepsilon_{x,j} f_{x,j} \mapsto \sum_{(x,j)\in F} \varepsilon_{x.j} T_x^{2^j} f_{x,j}$ (where $F \subset \mathbf{Z} \times \mathbf{R}^n$ is finite) from $\operatorname{Rad}(L^p(\mathbf{R}^n, X)) \approx L^p(\mathbf{R}^n, \operatorname{Rad} X)$ to $\operatorname{Rad}(L^p(\mathbf{R}^n, Y)) \approx L^p(\mathbf{R}^n, \operatorname{Rad} Y)$ (where the isomorphisms $\approx$ are consequences of the equivalence of the definitions of $\operatorname{Rad} X$ in terms of the various L^p-norms, and of Fubini's theorem). The kernel of $\mathfrak{T}$ [which is $\mathscr{L}(\operatorname{Rad} X, \operatorname{Rad} Y)$-valued] is $\mathfrak{K}(u,v) = (2^{jn} K(x+2^j u, x+2^j v))_{(x,j)}$, for which

$$(2^{in}\mathfrak{K}(y+2^i u, y+2^i v))_{(y,i)} = ((2^{(i+j)n} K((x+2^j y)+2^{i+j} u, (x+2^j y)+2^{i+j} v))_{(x,j)})_{(y,i)}.$$

Now the standard R-estimates for $\mathfrak{K}$ follow from those of K, and the fact that

$$\mathscr{R}(\{(T_i)_{i\in F} \in \mathscr{L}(\operatorname{Rad} X, \operatorname{Rad} Y) : F \text{ finite},\ T_i \in \mathscr{T}\}) \leq C\mathscr{R}(\mathscr{T}),$$

with $C < \infty$ geometric, whenever $\mathscr{T} \subset \mathscr{L}(X,Y)$ and X, Y have property (α). The weak R-boundedness property of $\mathfrak{T}$ follows from that of T by a similar consideration, and it is also not difficult to see that $T1 = 0$ implies $\mathfrak{T}1 = 0$, and the same for the adjoints. Thus the second assertion of Theorem 1.18 is actually a formal corollary of the first one, and the Theorem is now fully proved. □

We conclude this section with

Proof of Cor. 1.28. The proof is based on the same technique as the proof of the R-boundedness assertion just above. We now need to establish the uniform boundedness of the operators $\mathfrak{T} = (T_i)_{i\in F} : \sum \varepsilon_i f_i \mapsto \sum \varepsilon_i T_i f_i$, where $T_i = T_{K_i}$. The kernel of $\mathfrak{T}$ is $\mathfrak{K}(u,v) = (K_i(u,v))_{i\in F}$. Thus the standard R-estimates, the weak R-boundedness property, and the conditions $\mathfrak{T}1 = 0 = \mathfrak{T}'1$ are inherited by $\mathfrak{T}$ just like in the previous proof, and Cor. 1.28 is indeed seen to follow from Theorem 1.18.

The same reasoning can be repeated with H^1 in place of L^p. The indentification of $\operatorname{Rad}(H^1(\mathbf{R}^n, X))$ and $H^1(\mathbf{R}^n, \operatorname{Rad}(X))$ when X is a UMD-space follows easily from a "square function" characterization of $H^1(\mathbf{R}^n, X)$ from [14]. □

3. Boundedness of paraproducts: sufficient conditions

The general set-up. The operators we consider here are defined by the formal series (1.30), where $g \in \operatorname{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))$ is fixed, and our aim is to give a meaning to and prove the properties of $P(g,\cdot)$ asserted in Theorems 1.40 and 1.44. We will initially consider $P(g,f)$ for $f \in X \otimes \mathscr{S}(\mathbf{R}^n)$ and resort to density arguments once appropriate bounds are shown.

The meaning of Ψ_j and Φ_j is now slightly different compared to the previous section: the Ψ_j consist of a resolution of the identity in the frequency representation, whereas the Φ_j are the corresponding "partial sum functions". In this connection it is convenient to assume that the function Φ_0 gives rise to this resolution by $\Phi_j := 2^{nj}\Phi(2^j\cdot)$ (note that we have changed j and $-j$ compared to the previous section) and $\Psi_j := \Phi_j - \Phi_{j-1}$. Moreover, we assume that Φ_0 is a product of n one-dimensional functions, $\Phi_0(x) = \phi_0(x_1)\cdots\phi_0(x_n)$, where $\hat{\phi}_0(x) = 1$ for $|x| \leq 1$ and $\hat{\phi}_0(x) = 0$ for $|x| > 2$. We assume that $\hat{\phi}$ is even, non-negative, and non-increasing on $[0,\infty[$. Then $\hat{\Phi}_0(x) = 1$ for $|x|_\infty \leq 1$ and $\hat{\Phi}_0(x) = 0$ for $|x|_\infty > 2$. Now

$$\operatorname{supp}\hat{\Phi}_j \subset \{x : |x|_\infty \leq 2^{j+1}\}, \qquad \operatorname{supp}\hat{\Psi}_j \subset \{x : 2^{j-1} \leq |x|_\infty \leq 2^{j+1}\},$$

and thus, using $\operatorname{supp} h * \phi \subset \operatorname{supp} h + \operatorname{supp}\phi$, we have

$$\operatorname{supp}\mathscr{F}[(\Psi_{j+3} * g)(\Phi_j * f)] = \operatorname{supp}(\hat{\Psi}_{j+3}\hat{g}) * (\hat{\Phi}_j\hat{f}) \subset \{2^{j+1} \leq |x|_\infty \leq 2^{j+5}\}. \tag{3.1}$$

Hence the terms labelled j and j' in (1.30) overlap only if $|j - j'| \leq 4$, and each of them is compactly supported away from the origin.

We initially give a meaning to the formal series (1.30) as a functional acting on $\hat{\mathscr{D}}_0(\mathbf{R}^n) \otimes Y' \subset L^{p'}(\mathbf{R}^n, Y')$, which is a norming for $L^p(\mathbf{R}^n, Y)$. Since only finitely many non-zero terms appear in the series defining $\langle P(g,f), h\rangle$ for $h \in \hat{\mathscr{D}}_0(\mathbf{R}^n) \otimes Y'$, we find that it is sufficient to prove uniform bounds for the $L^p(\mathbf{R}^n, Y)$-norms of $\sum_{j\in F}(\Psi_{j+3} * g)(\Phi_j * f)$ for all finite $F \subset \mathbf{Z}$.

We then turn to the proofs of Theorems 1.40 and 1.44. They will be partially overlapping, and consist of several steps which are divided into the following subsections.

Preliminary estimates. By the support properties of the different terms appearing in (1.30), the Littlewood–Paley decomposition (valid on UMD-spaces) can be exploited, and we find that

$$\Big\| \sum (\Psi_{j+3} * g)(\Phi_j * f) \Big\|_{L^p(\mathbf{R}^n, Y)} \lesssim \mathbf{E} \Big\| \sum \varepsilon_j (\Psi_{j+3} * g)(\Phi_j * f) \Big\|_{L^p(\mathbf{R}^n, Y)}.$$

We then write

$$\begin{aligned} \sum \varepsilon_j (\Psi_{j+3} * g)(\Phi_j * f) &= \sum \varepsilon_j (\Psi_{j+3} * g)\left([\Phi_j - \Phi_{j+5}] * f\right) \\ &\quad + \sum \varepsilon_j \left[(\Psi_{j+3} * g)(\Phi_{j+5} * f) - \Phi_{j+5} * (\Psi_{j+3} * g) f\right] \\ &\quad + \sum \varepsilon_j \Phi_{j+5} * (\Psi_{j+3} * g) f. \end{aligned} \tag{3.2}$$

(Note that we are using the convention that the pointwise product has a higher precedence than the convolution product $*$.) For the first series appearing on the left-hand side above, we have

$$\begin{aligned} &\mathbf{E} \Big\| \sum \varepsilon_j (\Psi_{j+3} * g)\left([\Phi_j - \Phi_{j+5}] * f\right) \Big\|_{L^p(\mathbf{R}^n, Y)} \\ &\approx \Big(\int \mathbf{E} \Big| \sum \varepsilon_j (\Psi_{j+3} * g)(x)([\Phi_j - \Phi_{j+5}] * f)(x) \Big|_Y^p \,\mathrm{d}x \Big)^{1/p} \\ &\lesssim \Big(\int \mathscr{R}\left(\Psi_j * g(x) : j \in \mathbf{Z}\right) \mathbf{E} \Big| \sum \varepsilon_j ([\Phi_j - \Phi_{j+5}] * f)(x) \Big|_X^p \,\mathrm{d}x \Big)^{1/p} \\ &\lesssim \mathscr{R}\left(\Psi_j * g(x) : j \in \mathbf{Z}, x \in \mathbf{R}^n\right) \|f\|_{L^p(\mathbf{R}^n, X)}, \end{aligned} \tag{3.3}$$

where the last estimate follows again from the Littlewood–Paley decomposition, since the functions $\Phi_j - \Phi_{j+5}$ make up a resolution of the unity (or more precisely, a "resolution of -5").

Observe that

$$\sum_{x,j} \varepsilon_{xj} \Psi_j * g(x) \xi_{xj} = \int \Psi_0(-y) \Big(\sum_{x,j} \varepsilon_{x,-j} g(2^j y + x) \xi_{x,-j} \Big) \,\mathrm{d}y$$

by simple manipulation. By the assumption that $g \in \mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))$, the expression in $(\cdots)$ belongs to $\mathrm{BMO}(\mathbf{R}^n, \mathrm{Rad}(Y))$, with the norm estimated by $\|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} \|\xi\|_{\mathrm{Rad}\, X}$. Thus one can merely copy the classical argument (see e.g. [26]) which shows that $\|\psi * g\|_{L^\infty(\mathbf{R}^n)} \leq C(\psi) \|g\|_{\mathrm{BMO}(\mathbf{R}^n)}$, where $C(\psi) < \infty$ for $\psi \in \mathscr{S}(\mathbf{R}^n)$ with vanishing integral, to prove that

$$\mathscr{R}\left(\Psi_j * g(x) : j \in \mathbf{Z}, x \in \mathbf{R}^n\right) \lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))}.$$

Thus the first series on the right of (3.2) has been estimated from above by

$$\|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} \|f\|_{L^p(\mathbf{R}^n, X)}. \tag{3.4}$$

We then come to the second series on the right of (3.2). The Fourier transform of the jth term is

$$\begin{aligned}(\hat{\Psi}_{j+3}\hat{g})*(\hat{\Phi}_{j+5}\hat{f})(x)-\hat{\Phi}_{j+5}(x)(\hat{\Psi}_{j+3}\hat{g}*\hat{f})(x)&\\ =\Big\langle\hat{\Psi}_{j+3}\hat{g},\Big(\hat{\Phi}_{j+5}(x-\cdot)-\hat{\Phi}_{j+5}(x)\Big)\,\hat{f}(x-\cdot)\Big\rangle&.\end{aligned}\tag{3.5}$$

Recall that $\hat{\Psi}_{j+3}$ is supported in the set $2^{j+2}\leq|y|\leq 2^{j+4}$ (where we, for the moment, denote the ℓ^∞ norm of $\mathbf{R}^n$ by $|\cdot|$). For y in this range, we have

- if $|x-y|\leq 2^{j+4}$, then $|x|\leq 2^{j+5}$, and so $\hat{\Phi}_{j+5}(x-y)=\hat{\Phi}_{j+5}(x)=1$;
- if $|x-y|\geq 2^{j+7}$, then $|x|\geq 2^{j+6}$, and so $\hat{\Phi}_{j+5}(x-y)=\hat{\Phi}_{j+5}(x)=0$.

Hence, for y in the given range, $\hat{\Phi}_{j+5}(x-y)-\hat{\Phi}_{j+5}(x)$ can only be non-vanishing when $2^{j+4}<|x-y|<2^{j+7}$. Let $\hat{\eta}_0\in\mathscr{D}(\mathbf{R}^n)$ be 1 for $2^4\leq|y|\leq 2^7$ and vanish outside $2^3\leq|y|\leq 2^8$. Let $\hat{\eta}_j:=\hat{\eta}_0(2^{-j}\cdot)$. Then $\hat{\eta}_j(x-y)=1$ for $2^{j+4}\leq|x-y|\leq 2^{j+7}$, and we can add it to the right-hand side of (3.5) without affecting anything:

$$(3.5)=\Big\langle\hat{\Psi}_{j+3}\hat{g},\Big(\hat{\Phi}_{j+5}(x-\cdot)-\hat{\Phi}_{j+5}(x)\Big)\,\hat{\eta}_j(x-\cdot)\hat{f}(x-\cdot)\Big\rangle.$$

Now the second series in (3.2) can be estimated by

$$\begin{aligned}&\leq\mathbf{E}\Big\|\sum\varepsilon_j(\Psi_{j+3}*g)(\Phi_{j+5}*\eta_j*f)\Big\|_{L^p(\mathbf{R}^n,X)}\\ &\qquad\qquad+\mathbf{E}\Big\|\sum\varepsilon_j\Phi_{j+5}*(\Psi_{j+3}*g)(\eta_j*f)\Big\|_{L^p(\mathbf{R}^n,X)}.\end{aligned}$$

For the first term above we immediately get the same bound (3.4), even by exactly the same reasoning as for the first series in (3.2), since the functions $\Phi_{j+5}*\eta_j$ now serve as our resolution of unity. To estimate the second term, we need the following:

Lemma 3.6. *Let* $\Phi(x)=\phi(x_1)\ldots\phi(x_n)$ *where* $\hat{\phi}$ *is a bounded, non-negative, even function which is non-increasing on* $[0,\infty[$. *Denote* $\Phi^r:=r^n\Phi(r\cdot)$. *Then the set of convolution operators* $\{\Phi^r*:r>0\}$ *is R-bounded on* $L^p(\mathbf{R}^n,X)$ *whenever* X *is a UMD-space and* $p\in\,]1,\infty[$.

Proof. Assume first that $\hat{\phi}=\sum_{i=1}^m a_i 1_{I_i}$, where the I_i are intervals centred at the origin and $a_i>0$. Then

$$\hat{\Phi}=\sum_{1\leq i_1,\ldots,i_n\leq m}a_{i_1}\cdots a_{i_n}1_{I_{i_1}\times\cdots\times I_{i_n}}=:\sum_\kappa\alpha_\kappa 1_{R_\kappa},$$

where the R_κ are rectangles with sides parallel to the coordinate axes, and $\alpha_\kappa>0$. Observe that $\sum_\kappa\alpha_\kappa=\hat{\Phi}(0)$.

We further have

$$\Phi*f=\sum\alpha_\kappa P_{R_\kappa}f,\qquad\Phi^r*f=\sum\alpha_\kappa P_{rR_\kappa}f,$$

where P_R is the spectral projection defined by $P_Rf:=\mathscr{F}^{-1}(1_R\hat{f})$. F. Zimmermann [29] has shown that the set $\mathcal{P}:=\{P_R:R\in\mathcal{R}\}$ is R-bounded on $L^p(\mathbf{R}^n,X)$ (for X UMD and $p\in\,]1,\infty[$), where $\mathcal{R}$ denotes the collection of all rectangles in $\mathbf{R}^n$ with sides parallel to the axes. Thus we have

$$\begin{aligned}\mathbf{E}\Big\|\sum_r\varepsilon_r\Phi^r*f_r\Big\|_{L^p(\mathbf{R}^n,X)}&=\mathbf{E}\Big\|\sum_r\varepsilon_r\sum_\kappa\alpha_\kappa P_{rQ_\kappa}f_r\Big\|_{L^p(\mathbf{R}^n,X)}\\ \leq\sum_\kappa\alpha_\kappa\mathbf{E}\Big\|\sum_r\varepsilon_r P_{rQ_\kappa}f_r\Big\|_{L^p(\mathbf{R}^n,X)}&\leq\hat{\Phi}(0)\,\mathscr{R}(\mathcal{P})\,\mathbf{E}\Big\|\sum_r\varepsilon_r f_r\Big\|_{L^p(\mathbf{R}^n,X)}.\end{aligned}$$

This completes the proof for $\hat{\phi}$ of the special form. For general $\hat{\phi}$, we simply approximate by functions of the special form, and make a standard limiting argument. □

Remark 3.7. If X is a UMD-space with property (α), the conclusion of the above lemma holds for other kinds of Φ's, too. In fact, if the multipliers m_λ, $\lambda \in \Lambda$, satisfy the conditions of the n-dimensional Mihlin theorem uniformly, then the corresponding family of Fourier multiplier transformations is R-bounded on $L^p(\mathbf{R}^n, X)$ for such spaces [13], and this clearly applies to the dilations of a single multiplier (provided, of course, that this single multiplier satisfies Mihlin's conditions). However, it is important to us that the statement of Lemma 3.6 is valid for arbitrary UMD-spaces. In fact, this Lemma can be thought of as a "harmonic analysis analogue" of a probabilistic lemma due to Bourgain [3], which is used by Figiel in the proof of Theorem 1.10 (see [11]).

Using Lemma 3.6, we obtain the bound

$$\begin{aligned}\mathbf{E}\left\|\sum \varepsilon_j \Phi_{j+5} * (\Psi_{j+3} * g)(\eta_j * f)\right\|_{L^p(\mathbf{R}^n,X)}& \\ \lesssim \mathbf{E}\left\|\sum \varepsilon_j (\Psi_{j+3} * g)(\eta_j * f)\right\|_{L^p(\mathbf{R}^n,X)}&,\end{aligned}$$

and this can be estimated exactly as before by (3.4).

Having estimated the first two sums on the right of (3.2), all together we have shown that

$$\begin{aligned}&\left\|\sum (\Psi_{j+3} * g)(\Phi_j * f)\right\|_{L^p(\mathbf{R}^n,Y)} \\ &\lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))}\|f\|_{L^p(\mathbf{R}^n,X)} + \mathbf{E}\left\|\sum \varepsilon_j \Phi_{j+2} * (\Psi_j * g) f\right\|_{L^p(\mathbf{R}^n,\mathrm{Rad}\, Y)},\end{aligned} \tag{3.8}$$

where a change of variable $j+3 \to j$ was made.

The key estimate for paraproducts. We still need to estimate the second term on the right of (3.8), and finding appropriate estimates for this constitutes the key step in the boundedness proof of operator-valued paraproducts. The proof follows the ideas of Figiel [10] for scalar kernels in the dyadic case: We proceed via establishing (L^∞, BMO) and (H^1, L^1) boundedness and then interpolating. More precisely, we are going to show the following:

Lemma 3.9. *Suppose that* $g \in (\mathrm{BMO}_s \cap \mathrm{BMO}_{\Psi,q})(\mathbf{R}^n, \mathscr{L}(X,Y))$ *for some* $q \in \,]1,\infty[$. *Then the mapping* $f \mapsto \sum \varepsilon_j \Phi_{j+2} * (\Psi_j * g) f$ *is bounded*

- *from* $L^\infty(\mathbf{R}^n, X)$ *to* $\mathrm{BMO}(\mathbf{R}^n, \mathrm{Rad}\, Y)$, *and*
- *from* $H^1(\mathbf{R}^n, X)$ *to* $L^1(\mathbf{R}^n, \mathrm{Rad}\, Y)$; *thus, by interpolation,*
- *from* $L^p(\mathbf{R}^n, X)$ *to* $L^p(\mathbf{R}^n, \mathrm{Rad}\, Y)$ *for all* $p \in \,]1,\infty[$,

and in fact its norm is estimated by

$$\|g\|_{\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))} + \|g\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n,\mathscr{L}(X,Y))}.$$

Proof of the (L^∞, BMO)*-boundedness.* To show the desired boundedness, we need to estimate the mean oscillations on an arbitrary ball $\bar{B} = \bar{B}(x_0, r)$ as in the definition of the space BMO. For this purpose, we first write

$$\begin{aligned}\sum_j \varepsilon_j \Phi_{j+2} * (\Psi_j * g) f &= \sum_{j:2^j r \geq 1} \varepsilon_j \Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c}) f \\ + \sum_{j:2^j r<1} \varepsilon_j &\left[\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c}) f - (\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c}) f)(x_0)\right] \\ &+ c_{\bar{B}} + \sum_j \varepsilon_j \Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{2\bar{B}}) f,\end{aligned} \tag{3.10}$$

where $2\bar{B} := \bar{B}(x_0, 2r)$, and $c_{\bar{B}}$ is just the same constant (the sum of point evaluations at x_0) that was subtracted one line earlier. (When saying that $c_{\bar{B}}$ is a constant, we mean that it is a fixed element of the space $\operatorname{Rad} Y$; it is still a function of the ε_j's.) Note also that there is no harm of the "extra" terms $-\Psi_j * g_{2\bar{B}} 1_{(2\bar{B})^c}$ and $-\Psi_j * g_{2\bar{B}} 1_{2\bar{B}}$, since their sum is just $-\Psi_j * g_{2\bar{B}} = 0$ as $\int \Psi_j = 0$ and $g_{2\bar{B}}$ is a constant.

Now we need to estimate the $L^q(\bar{B}, \operatorname{Rad} Y)$ norms of the various terms in (3.10):

Case $2^j r < 1$. For $x \in \bar{B}(x_0, r)$, we have

$$\begin{aligned}
&\left|\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c} f(x) - \Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c} f(x_0)\right|_Y \\
&\leq |x - x_0| \left\|\nabla \Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c}) f\right\|_{L^\infty(\mathbf{R}^n, Y)^n} \\
&\leq r \left\|\nabla \Phi_{j+2}\right\|_{L^1(\mathbf{R}^n)^n} \left\|(\Psi_j * (g - g_{2\bar{B}})) f - (\Psi_j * (g - g_{2\bar{B}}) 1_{2\bar{B}}) f\right\|_{L^\infty(\mathbf{R}^n, Y)} \\
&\lesssim r 2^j \left[\left\|\Psi_j * g\right\|_{L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))} + \left\|\Psi_j * (g - g_{2\bar{B}}) 1_{2\bar{B}}\right\|_{L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))}\right] \|f\|_{L^\infty(\mathbf{R}^n, X)} \\
&\lesssim r 2^j \left[\|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} + \left\|\Psi_j\right\|_{L^\infty(\mathbf{R}^n)} \left\|(g - g_{2\bar{B}}) 1_{2\bar{B}}\right\|_{L^1_s(\mathbf{R}^n, \mathscr{L}(X,Y))}\right] \|f\|_{L^\infty} \\
&\lesssim r 2^j \left[1 + 2^{jn} r^n\right] \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} \|f\|_{L^\infty(\mathbf{R}^n, X)} \cdot
\end{aligned}$$

By summing the geometric series, we find that the second term on the right of (3.10) is dominated by

$$\|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} \|f\|_{L^\infty(\mathbf{R}^n, X)} \cdot \tag{3.11}$$

Since this is uniform in $x \in \bar{B}$ and independent of the ε_j's, a bound for the $L^q(\bar{B}, \operatorname{Rad} Y)$ norm is simply the quantity in (3.11) times $\left|\bar{B}\right|^{1/q}$.

Case $2^j r \geq 1$. We start by estimating the quantity $\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c}(x)$. Let $\xi \in X$, and let first $x \in \bar{B}(x_0, r)$. Then

$$\begin{aligned}
&\left|(\Psi_j * (g - g_{2\bar{B}}) 1_{(2\bar{B})^c})(x) \xi\right|_Y \leq \int_{|y - x_0| \geq 2r} \left|\Psi_j(x - y)\right| \left|(g(y) - g_{2\bar{B}}) \xi\right|_Y \,\mathrm{d}y \\
&\lesssim \sum_{i=1}^\infty \int_{2^i r \leq |y - x_0| \leq 2^{i+1} r} 2^{jn} (2^j 2^i r)^{-N} \left|(g(y) - g_{2\bar{B}}) \xi\right|_Y \,\mathrm{d}y \\
&\lesssim (2^j r)^{(n-N)} \sum_{i=1}^\infty 2^{i(n-N)} (2^i r)^{-n} \int_{\bar{B}(x_0, 2^{i+1} r)} \left|(g(y) - g_{\bar{B}(x_0, 2r)}) \xi\right|_Y \,\mathrm{d}y \\
&\lesssim (2^j r)^{(n-N)} \sum_{i=1}^\infty 2^{i(n-N)} i \cdot \|g(\cdot)\xi\|_{\mathrm{BMO}(\mathbf{R}^n, Y)} \lesssim (2^j r)^{-\delta} \|g(\cdot)\xi\|_{\mathrm{BMO}(\mathbf{R}^n, Y)},
\end{aligned}$$

provided we take $\delta := N - n > 0$.

We next consider $x \notin \bar{B}(x_0, r)$, i.e., $\Delta := |x - x_0| > r$. Then

$$\begin{aligned}
&\left|(\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})(x)\xi\right|_Y \\
&\leq \int_{2r \leq |y - x_0| \leq 2\Delta} 2^{jn}(2^j r)^{-N} \left|(g(y) - g_{2\bar{B}})\xi\right|_Y \,\mathrm{d}y \\
&\qquad + \sum_{i=1}^{\infty} \int_{2^i\Delta \leq |y - x_0| \leq 2^{i+1}\Delta} 2^{jn}(2^j 2^i \Delta)^{-N} \left|(g(y) - g_{2\bar{B}})\xi\right|_Y \,\mathrm{d}y \\
&\lesssim 2^{j(n-N)} r^{-N} \Delta^n (1 + \log\frac{\Delta}{r}) \left\|g(\cdot)\xi\right\|_{\mathrm{BMO}(\mathbf{R}^n, Y)} \\
&\qquad + \sum_{i=1}^{\infty} (2^j 2^i \Delta)^{n-N} (i + \log\frac{\Delta}{r}) \left\|g(\cdot)\xi\right\|_{\mathrm{BMO}(\mathbf{R}^n, Y)} \\
&= (2^j \Delta)^{n-N} \left[\frac{\Delta^N}{r^N}(1 + \log\frac{\Delta}{r}) + \sum_{i=1}^{\infty} 2^{i(n-N)} (i + \log\frac{\Delta}{r})\right] \left\|g(\cdot)\xi\right\|_{\mathrm{BMO}} \\
&\lesssim (2^j |x - x_0|)^{-\delta} \left[1 + \left(\frac{|x - x_0|}{r}\right)^{n+\delta} \log\frac{|x - x_0|}{r}\right] \left\|g(\cdot)\xi\right\|_{\mathrm{BMO}(\mathbf{R}^n, Y)}.
\end{aligned}$$

Finally, we estimate $(\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})f)(x)$ for $x \in \bar{B}$:

$$\begin{aligned}
&\left|(\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})f)(x)\right|_Y \\
&\leq \int \left|\Phi_{j+2}(y)\right| \left|\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c}(x - y)\right|_{\mathscr{L}(X,Y)} \left\|f\right\|_{L^\infty(\mathbf{R}^n, X)} \,\mathrm{d}y \\
&\lesssim \int_{|y| \leq 2r} 2^{jn} \cdot (2^j r)^{-\delta} \left\|g\right\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} \left\|f\right\|_{L^\infty(\mathbf{R}^n, X)} \,\mathrm{d}y \\
&\qquad + \int_{|y| > 2r} 2^{jn} (2^j |y|)^{-M} \cdot (2^j |y|)^{-\delta} \left[1 + \left(\frac{|y|}{r}\right)^{n+\delta} \log\frac{|y|}{r}\right] \left\|g\right\|_{\mathrm{BMO}_s} \left\|f\right\|_{L^\infty} \mathrm{d}y \\
&\lesssim \left[(2^j r)^{n-\delta} + 2^{j(n-M-\delta)} \int_{|y| > 2r} \left(1 + \left(\frac{|y|}{r}\right)^{n+\delta} \log\frac{|y|}{r}\right) \frac{\mathrm{d}y}{|y|^{M+\delta}}\right] \left\|g\right\|_{\mathrm{BMO}_s} \left\|f\right\|_{L^\infty} \\
&\lesssim \left[(2^j r)^{n-\delta} + (2^j r)^{n-M-\delta}\right] \left\|g\right\|_{\mathrm{BMO}_s} \left\|f\right\|_{L^\infty}
\end{aligned}$$

provided we take $M > 2n$. We also choose $\delta > n$. Then, again, we can sum the geometric series, now over $j : 2^j r \geq 1$, to get the bound (3.11) also for the first term on the right of (3.10).

The last term in (3.10). For this one deduces immediately from Lemma 3.6 the estimates

$$\begin{aligned}
&\left\|\sum \varepsilon_j \Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}})1_{2\bar{B}})f\right\|_{L^q(\mathbf{R}^n, \operatorname{Rad} Y)} \\
&\lesssim \mathbf{E} \left\|\sum \varepsilon_j (\Psi_j * (g - g_{2\bar{B}})1_{2\bar{B}})f\right\|_{L^q(\mathbf{R}^n, Y)} \\
&\leq \mathbf{E} \left\|\sum \varepsilon_j \Psi_j * (g - g_{2\bar{B}})1_{2\bar{B}}\right\|_{L^q(\mathbf{R}^n, \mathscr{L}(X,Y))} \left\|f\right\|_{L^\infty(\mathbf{R}^n, X)} \\
&\leq \left|\bar{B}\right|^{1/q} \left\|g\right\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \mathscr{L}(X,Y))} \left\|f\right\|_{L^\infty(\mathbf{R}^n, X)}.
\end{aligned}$$

Now we have estimated all the terms in (3.10), and shown that

$$\begin{aligned}
&\left\|\left(\sum \varepsilon_j \Phi_{j+2} * (\Psi_j * g)f - c_{\bar{B}}\right)1_{\bar{B}}\right\|_{L^q(\mathbf{R}^n, \operatorname{Rad} Y)} \\
&\lesssim \left|\bar{B}\right|^{1/q} \left(\left\|g\right\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} + \left\|g\right\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \mathscr{L}(X,Y))}\right) \left\|f\right\|_{L^\infty(\mathbf{R}^n, X)},
\end{aligned} \tag{3.12}$$

and this is exactly the boundedness that we wanted to prove. □

Proof of the (H^1, L^1)-boundedness. Let now f be an atom of $H^1(\mathbf{R}^n, X)$, i.e., $f \in L^\infty(\mathbf{R}^n, X)$ with $\operatorname{supp} f \subset \frac{1}{2}\bar{B} := \bar{B}(x_0, r/2)$, $\int f(x)\,\mathrm{d}x = 0$, and $\|f\|_{L^\infty(\mathbf{R}^n,X)} \le \left|\bar{B}\right|^{-1}$. We need to estimate the norm of $\sum \varepsilon_j \Phi_{j+2} * (\Psi_j * g)f$ in $L^1(\mathbf{R}^n, \operatorname{Rad} Y)$. Inside the ball $\bar{B}$, we can exploit the estimate (3.12) which we already have, but we still need an estimate for the renormalization constant (denoted by $c_{\bar{B}}$ above), and for the integral of our function outside the ball $\bar{B}$.

The renormalization constant. Recall that the constant to be estimated is

$$c_{\bar{B}} = \sum_{j:2^j r<1} \varepsilon_j (\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})f)(x_0).$$

Using the support and moment properties of f, we have

$$\begin{aligned}
&\left|(\Phi_{j+2} * (\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})f)(x_0)\right|_Y \\
&\le \int_{\bar{B}} \big|\big[\Phi_{j+2}(x_0 - y)(\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})(y) \\
&\qquad\qquad - \Phi_{j+2}(0)(\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})(x_0)\big]\, f(y)\big|_Y \,\mathrm{d}y \\
&\le \int_{\bar{B}} |\Phi_{j+2}(x_0 - y) - \Phi_{j+2}(0)| \left|\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c})(y)f(y)\right|_Y \mathrm{d}y \\
&\quad + \int_{\bar{B}} |\Phi_{j+2}(0)| \left|\left[\Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c}(y) - \Psi_j * (g - g_{2\bar{B}})1_{(2\bar{B})^c}(x_0)\right] f(y)\right|_Y \mathrm{d}y \\
&\lesssim \int_{\bar{B}} 2^{j(n+1)} r \Big(\|\Psi_j * g\|_{L^\infty} + \|\Psi_j\|_{L^\infty} \|(g - g_{2\bar{B}})1_{2\bar{B}}\|_{L^1_s(\mathbf{R}^n,\mathscr{L}(X,Y))}\Big) |f(y)|_X \,\mathrm{d}y \\
&\quad + \int_{\bar{B}} 2^{jn} \|\nabla\Psi_j * (g - g_{2\bar{B}})1_{2\bar{B}}\|_{L^\infty(\mathbf{R}^n,\mathscr{L}(X,Y)^n)}\, r\, |f(y)|_X \,\mathrm{d}y \\
&\lesssim 2^{j(n+1)} r \left(1 + 2^{jn} r^n\right) \|g\|_{\mathrm{BMO}_s} \|f\|_{L^1(\mathbf{R}^n,X)} + 2^{jn} 2^{j(n+1)} r^n r \|f\|_{L^1(\mathbf{R}^n,X)} \\
&\lesssim 2^{jn}(2^j r)\left(1 + (2^j r)^n\right) \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))},
\end{aligned}$$

where finally the size condition on f was taken into account.

Summing over $j : 2^j r < 1$, it is plain that $c_{\bar{B}}$ is estimated by $r^{-n}\|g\|_{\mathrm{BMO}_s}$, and hence its integral over $\bar{B}$ by $\|g\|_{\mathrm{BMO}_s}$.

The integral over $\bar{B}^c$. Let $x \in \bar{B}^c$, i.e., $|x - x_0| > r$. Then we can estimate

$$\begin{aligned}
&|\Phi_{j+2} * (\Psi_j * g)f(x)|_Y \le \int_{\bar{B}/2} |\Phi_{j+2}(x - y)|\, |(\Psi_j * g)(y)f(y)|_Y \,\mathrm{d}y \\
&\lesssim \int_{\bar{B}/2} 2^{jn}(2^j|x - y|)^{-N} \|g\|_{\mathrm{BMO}_s} |f(y)|_X \,\mathrm{d}y \lesssim 2^{jn}(2^j|x - x_0|)^{-N} \|g\|_{\mathrm{BMO}_s},
\end{aligned}$$

and thus

$$\int_{\bar{B}^c} |\Phi_{j+2} * (\Psi_j * g)f(x)|_Y \,\mathrm{d}x \lesssim (2^j r)^{-\delta} \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))} \tag{3.13}$$

provided we take $\delta := N - n > 0$.

On the other hand, we also have the bounds

$$
\begin{aligned}
&|\Phi_{j+2} * (\Psi_j * g) f(x)|_Y \\
&\leq \int_{\bar{B}/2} |[\Phi_{j+2}(x-y)(\Psi_j * g)(y) - \Phi_{j+2}(x-x_0)(\Psi_j * g)(x_0)]\, f(y)|_Y \,\mathrm{d}y \\
&\leq \int_{\bar{B}/2} |\Phi_{j+2}(x-y) - \Phi_{j+2}(x-x_0)|\, \|\Psi_j * g\|_{L^\infty} |f(y)|_X \,\mathrm{d}y \\
&\qquad + \int_{\bar{B}/2} |\Phi_{j+2}(x-x_0)|\, \|\Psi_j * g(y) - \Psi_j * g(x_0)\|_{\mathscr{L}(X,Y)} |f(y)|_X \,\mathrm{d}y \\
&\lesssim \int_{\bar{B}/2} 2^{j(n+1)} r \left\|\nabla \Phi_0(2^j \cdot)\right\|_{L^\infty([x-y,x-x_0])} \|g\|_{\mathrm{BMO}_s} |f(y)|_X \,\mathrm{d}y \\
&\qquad + \int_{\bar{B}/2} 2^{jn} (2^j |x-x_0|)^{-N} 2^j r \|g\|_{\mathrm{BMO}_s} |f(y)|_X \,\mathrm{d}y \\
&\lesssim 2^{j(n+1)} r (2^j |x-x_0|)^{-N},
\end{aligned}
$$

and integrating we get

$$
\int_{\bar{B}^c} |\Phi_{j+2} * (\Psi_j * g) f(x)|_Y \,\mathrm{d}x \lesssim (2^j r)^{1-\delta} \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))}, \tag{3.14}
$$

provided, again, that $\delta := N - n > 0$.

Choosing $\delta \in]0,1[$, and using (3.13) for $2^j r \geq 1$ and (3.14) for $2^j r < 1$, we can sum the geometric series to the conclusion that

$$
\sum_j \|[\Phi_{j+2} * (\Psi_j * g) f]\, 1_{\bar{B}^c}\|_{L^1(\mathbf{R}^n, Y)} \lesssim \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))}.
$$

This completes the proof of the asserted (H^1, L^1)-boundedness, and then interpolation completes the proof of Lemma 3.9. $\square$

A summary of results obtained so far. Now we have shown that

$$
\begin{aligned}
&\|P(g,f)\|_{L^p(\mathbf{R}^n, Y)} \\
&\qquad \leq C \left(\|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} + \|g\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \mathscr{L}(X,Y))} \right) \|f\|_{L^p(\mathbf{R}^n, X)},
\end{aligned} \tag{3.15}
$$

which is the assertion of Theorem 1.44.

Note that the estimate (1.41) in Theorem 1.40 also follows from this: In fact,

$$
P(g, f(\cdot)\xi) = P(g(\cdot)\xi, f),
$$

and to bound this, we just need to estimate

$$
\|g(\cdot)\xi\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \mathscr{L}(\mathbf{C},Y))} \lesssim \|g(\cdot)\xi\|_{\mathrm{BMO}(\mathbf{R}^n, Y)} \leq \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n, \mathscr{L}(X,Y))} |\xi|_X
$$

(where the crucial first estimate follows from the fact that $\mathscr{L}(\mathbf{C}, Y) = Y$ is a UMD-space), and

$$
\|g(\cdot)\xi\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(\mathbf{C},Y))} \leq \|g(\cdot)\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} |\xi|_X
$$

(which follows at once from the definitions of the various norms).

From the norm estimate (1.41) now shown, the first assertion of Theorem 1.40 follows by routine arguments, using the density of $\hat{\mathscr{D}}_0(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$ for $p \in]1, \infty[$.

The kernel of the paraproduct operator. Let $f \in \mathscr{D}(\mathbf{R}^n)\otimes X$, and $x \notin \operatorname{supp} f$. Then

$$(\Psi_{j+3} * g)(\Phi_j * f)(x) = \int (\Psi_{j+3} * g)(x)\Phi_j(x-y)f(y)\,\mathrm{d}y,$$

and

$$\sum_j (\Psi_{j+3}*g)(\Phi_j*f)(x) = \int \sum_j (\Psi_{j+3}*g)(x)\Phi_j(x-y)f(y)\,\mathrm{d}y =: \int K(x,y)f(y)\,\mathrm{d}y,$$

where the series defining $K(x,y)$ is absolutely convergent for $x \neq y$. In fact, $\|\Psi_{j+3} * g(x)\|_{\mathscr{L}(X,Y)} \lesssim \|g\|_{\mathrm{BMO}_s(\mathbf{R}^n,\mathscr{L}(X,Y))}$, and for $|\Phi_j(x-y)|$ we have the estimates 2^{jn} and $2^{jn}(2^j|x-y|)^{-N}$, so that using one or the other depending on whether $2^j|x-y| \geq 1$ or $2^j|x-y| < 1$, we deduce $\sum_j |\Phi_j(x-y)| \lesssim |x-y|^{-n}$, and thus $\|K(x,y)\|_{\mathscr{L}(X,Y)} \lesssim \|g\|_{\mathrm{BMO}_s} |x-y|^{-n}$.

Using similar estimates, one also readily shows that

$$\|K(u,v)-K(u,w)\|_{\mathscr{L}(X,Y)} + \|K(v,u)-K(w,u)\|_{\mathscr{L}(X,Y)} \lesssim \|g\|_{\mathrm{BMO}_s} \frac{|v-w|}{|u-w|^{n+1}},$$

so that the standard estimated are satisfied by K.

However, we need the R-versions of these bounds, and to this end, we consider the randomized kernel

$$\begin{aligned}
\mathfrak{K}_\xi(u,v) &= \sum_{x,i} \varepsilon_{x,i} 2^{in} K(x+2^i u, x+2^i v)\xi_{x,i} \\
&= \sum_{x,i} \varepsilon_{x,i} 2^{in} \sum_j (\Psi_{j+3} * g)(x+2^i u)\Phi_j(2^i(u-v))\xi_{x,i} \\
&= \sum_{x,i} \varepsilon_{x,i} \sum_j (\Psi_{j+i+3} * g(2^i\cdot+x))(u)\Phi_{j+i}(u-v)\xi_{x,i} \\
&= \sum_j \Psi_{j+3} * \left[\sum_{x,i} \varepsilon_{x,i} g(2^i\cdot+x)\xi_{x,i}\right](u)\Phi_j(u-v) \\
&=: \sum_j (\Psi_{j+3} * \mathfrak{g}_\xi)(u)\Phi_j(u-v).
\end{aligned}$$

This is of the same form as K, but with $\mathfrak{g}_\xi := \sum_{x,i} \varepsilon_{x,i} g(2^i\cdot+x)\xi_{x,i}$ in place of g. Thus, by repeating the same steps that lead to the standard estimates for K, we deduce

$$\begin{aligned}
\|\mathfrak{K}_\xi(u,v)\|_{\operatorname{Rad} Y} &\lesssim \|\mathfrak{g}_\xi\|_{\mathrm{BMO}(\mathbf{R}^n,\operatorname{Rad} Y)} |u-v|^{-n} \\
&\leq \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} |u-v|^{-n},
\end{aligned}$$

and

$$\begin{aligned}
\|\mathfrak{K}_\xi(u,v) - \mathfrak{K}_\xi(u,v_0)\|_{\operatorname{Rad} Y} &\lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} \frac{|v-v_0|}{|u-v|^{n+1}}, \\
\|\mathfrak{K}_\xi(u,v) - \mathfrak{K}_\xi(u_0,v)\|_{\operatorname{Rad} Y} &\lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n,\mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} \frac{|u-u_0|}{|u-v|^{n+1}},
\end{aligned}$$

which are the standard R-estimates for K.

The operators $P(g,1)$ and $P(g,\cdot)'1$. Let $\varphi \in Y' \otimes \mathscr{D}^0(\mathbf{R}^n)$, and υ_0, υ_1 be as in the definition of $\langle T1, \varphi\rangle$, where $\xi \in X$.

$$\begin{aligned}
\langle P(g,1),\varphi\rangle\,\xi &= \langle P(g(\cdot)\xi,\upsilon_0),\varphi\rangle + \iint_{\mathbf{R}^n\times\mathbf{R}^n} [K(x,y)-K(x_0,y)]\xi\upsilon_1(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{j=-\infty}^{\infty} \langle(\Psi_{j+3} * g(\cdot)\xi)(\Phi_j * \upsilon_0),\varphi\rangle \\
&\qquad + \iint_{\mathbf{R}^n\times\mathbf{R}^n} \sum_{j=-\infty}^{\infty} [(\Psi_{j+3} * g)(x)\Phi_j(x-y) \\
&\qquad\qquad -(\Psi_{j+3} * g)(x_0)\Phi_j(x_0-y)]\,\xi\upsilon_1(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\overset{(\star)}{=} \lim_{N\to\infty}\sum_{-N}^{N}\Big\{\iint(\Psi_{j+3} * g(\cdot)\xi)(x)\Phi_j(x-y)(\upsilon_0(y)+\upsilon_1(y))\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad - \iint(\Psi_{j+3} * g(\cdot)\xi)(x_0)\Phi_j(x_0-y)\upsilon_1(y)\varphi(x)\,\mathrm{d}y\,\mathrm{d}x\Big\} \\
&\overset{(*)}{=} \lim_{N\to\infty}\sum_{-N}^{N}\Big\{\int(\Psi_{j+3} * g(\cdot)\xi)(x)\varphi(x)\,\mathrm{d}x + 0\Big\} \\
&= \lim_{N\to\infty}\Big\langle g(\cdot)\xi, \sum_{-N+3}^{N+3}\Psi_j * \varphi\Big\rangle = \langle g(\cdot)\xi,\varphi\rangle\,.
\end{aligned}$$

In $(\star)$, the absolute convergence of $\iint\sum$ permitted the change of the order of summation and integration, and in $(*)$ we used the facts that $\upsilon_0+\upsilon_1 \equiv 1$, $\Phi_j * 1 = 1$, and $\int\varphi(x)\,\mathrm{d}x = 0$. The last equality follows from the continuity of $g(\cdot)\xi \in \mathrm{BMO}(\mathbf{R}^n, Y)$ as a functional on $H^1(\mathbf{R}^n, Y')$, and from the convergence of $\sum\Psi_j * \varphi$ to φ in $H^1(\mathbf{R}^n, Y')$. This shows that $P(g,1) = g$.

As for the assertion $P(g,\cdot)'1 = 0$, we need to show that $\langle 1, P(g,\varphi)\rangle = 0$ for all $\varphi \in \mathscr{D}^0(\mathbf{R}^n)$. Note that $P(g(\cdot)\xi,\cdot)$, for $\xi \in X$, is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n, Y)$ for $p \in \,]1,\infty[$, and its kernel verifies the standard estimates. Hence it is also bounded from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n, Y)$ by the classical Calderón–Zygmund theory. Since $\mathscr{D}^0(\mathbf{R}^n) \subset H^1(\mathbf{R}^n)$, we know that $P(g(\cdot)\xi,\varphi)$ is an integrable function, and so

$$\langle 1, P(g,\varphi)\rangle\,\xi = \langle 1, P(g(\cdot)\xi,\varphi)\rangle = \int P(g(\cdot)\xi,\varphi)(x)\,\mathrm{d}x = \mathscr{F}[P(g(\cdot)\xi,\varphi)](0).$$

Let first $\varphi \in \hat{\mathscr{D}}_0(\mathbf{R}^n)$. Then $\Phi_j * \varphi = 0$ for all $j < j_0$, say, and from (3.1) we conclude that $\operatorname{supp}\mathscr{F}[(\Psi_j * g(\cdot)\xi)(\Phi_j * \varphi)] \subset \{x : |x|_\infty \geq 2^{j_0+1}\}$ for all j. In particular, $\mathscr{F}[P(g(\cdot)\xi,\varphi)]$ vanishes even in a neighbourhood of the origin in this case. For a general $\varphi \in \mathscr{D}^0(\mathbf{R}^n)$ [or even $\varphi \in H^1(\mathbf{R}^n)$], we use the continuity of

$$P(g(\cdot)\xi,\cdot) : H^1(\mathbf{R}^n) \to L^1(\mathbf{R}^n, Y) \qquad \text{and of} \qquad f \mapsto \hat{f}(0) : L^1(\mathbf{R}^n, Y) \to Y$$

to get the full assertion.

Weak R-boundedness of the paraproduct operators. Note that the condition (1.17) for $T = P(g,\cdot)$ requires that $\|\langle\phi, \mathfrak{T}_\xi\varphi\rangle\|_{\operatorname{Rad} Y} \leq C\,\|\xi\|_{\operatorname{Rad} X}$, for ϕ,φ normalized bump functions associated with the unit ball, where the operator $\mathfrak{T}_\xi$

(which acts on **C**-valued functions) looks as follows:

$$\begin{aligned}
\mathfrak{T}_\xi f(u) &= \sum_{x,i} \varepsilon_{x,i} T[f(2^{-i}(\cdot - x))\xi_{x,i}](2^i u + x) \\
&= \sum_{x,i} \varepsilon_{x,i} \sum_j (\Psi_{j+3} * g)(2^i u + x) \left[\Phi_j * f(2^{-i}(\cdot - x))\right] (2^i u + x)\xi_{x,i} \\
&= \sum_{x,i} \varepsilon_{x,i} \sum_j (\Psi_{j+i+3} * g(2^i \cdot + x))(u)(\Phi_{j+i} * f)(u)\xi_{x,i} \\
&= \sum_j \Big(\Psi_{j+3} * \Big[\sum_{x,i} \varepsilon_{x,i} g(2^i \cdot + x)\xi_{x,i}\Big]\Big)(u)(\Phi_j * f)(u) \\
&= P(\mathfrak{g}_\xi, f)(u);
\end{aligned}$$

$\mathfrak{g}_\xi$ has the same meaning as in the previous subsection.

Now we can apply the boundedness result (3.15) which we have already proved. Note that in place of X we now have $\mathbf{C}$ and in place of Y we have $\operatorname{Rad} Y$. We also have the obvious identification $\mathscr{L}(\mathbf{C}, \operatorname{Rad} Y) = \operatorname{Rad} Y$. Thus we get

$$\|P(\mathfrak{g}_\xi, f)\|_{L^p(\mathbf{R}^n, \operatorname{Rad} Y)} \lesssim \Big[\|\mathfrak{g}_\xi\|_{\mathrm{BMO}_R(\mathbf{R}^n, \operatorname{Rad} Y)} + \|\mathfrak{g}_\xi\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \operatorname{Rad} Y)}\Big] \|f\|_{L^p(\mathbf{R}^n)}$$

for $p, q \in \,]1, \infty[$.

For the first term, the estimate

$$\|\mathfrak{g}_\xi\|_{\mathrm{BMO}_R(\mathbf{R}^n, \operatorname{Rad} Y)} \le \|g\|_{\mathrm{BMO}^2_R(\mathbf{R}^n, \operatorname{Rad} Y)} \|\xi\|_{\operatorname{Rad} X} \tag{3.16}$$

follows more or less directly from the definitions of the various norms.

The other BMO-norm is also readily estimated under the standing assumption of the UMD property:

Lemma 3.17. *If Y has UMD and $q \in \,]1, \infty[$, then*

$$\|g_\xi\|_{\mathrm{BMO}_{\Psi,q}(\mathbf{R}^n, \operatorname{Rad} Y)} \lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} .$$

Proof. Since $\operatorname{Rad} Y$ has UMD when Y has, we get

$$\begin{aligned}
\mathbf{E} \Big\| \sum \varepsilon_j \Psi_j * (\mathfrak{g}_\xi - (\mathfrak{g}_\xi)_{\bar B}) 1_{\bar B} \Big\|_{L^q(\mathbf{R}^n, \operatorname{Rad} Y)} &\approx \|(\mathfrak{g}_\xi - (\mathfrak{g}_\xi)_{\bar B}) 1_{\bar B}\|_{L^q(\mathbf{R}^n, \operatorname{Rad} Y)} \\
&\le |\bar B|^{1/q} \|\mathfrak{g}_\xi\|_{\mathrm{BMO}(\mathbf{R}^n, \operatorname{Rad} Y)} ,
\end{aligned}$$

and moreover the estimate

$$\|\mathfrak{g}_\xi\|_{\mathrm{BMO}(\mathbf{R}^n, \operatorname{Rad} Y)} \le \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} .$$

is a matter of definition. ☐

Conclusion of the proof of Theorem 1.40. Combining (3.16) and Lemma 3.17 with our estimate for $P(\mathfrak{g}_\xi, f)$ above, we have shown that

$$\|P(\mathfrak{g}_\xi, f)\|_{L^p(\mathbf{R}^n, \operatorname{Rad} Y)} \lesssim \|g\|_{\mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))} \|\xi\|_{\operatorname{Rad} X} \|f\|_{L^p(\mathbf{R}^n)}$$

provided that $p \in \,]1, \infty[$, the spaces X and Y have property (α), and Y also has UMD. Since the L^p norms of all normalized bump functions associated with the unit ball are uniformly bounded, this implies the desired weak R-boundedness property of $P(g, \cdot)$, and finishes the proof of Theorem 1.40. ☐

Lemmata on BMO-spaces. We conclude this section by giving the proofs of Lemmata 1.37 and 1.38 from the Introduction.

Proof of Lemma 1.37. We need only show "$\supset$". Since Y has (α), the double random series in the definition of BMO^2_R can be estimated by a random series with a single independent family ε_{xyij} in place of $\varepsilon_{xj}\varepsilon'_{yi}$, and then we can use the estimate coming from $g \in \mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))$. Thus all we need to show is (after an application of Khinchin–Kahane)

$$\Big(\mathbf{E}\Big|\sum_{x,j}\sum_{y,i}\varepsilon_{xyij}\lambda_{yi}\xi_{xj}\Big|_X^2\Big)^{1/2} \leq C\left\|\lambda\right\|_{\ell^2}\left\|\xi\right\|_{\mathrm{Rad}\,X}.$$

We consider the sequence of random variables $\gamma_{xj} := \sum_{y,i}\varepsilon_{xyij}\lambda_{yi}$. This sequence is sign-invariant and the L^s norm of γ_{xj} is bounded by $C_s\left\|\lambda\right\|_{\ell^2}$ with $C_s < \infty$ for every finite s by Khinchin's inequality. Thus Theorem 12.27 of [9] applies to give

$$\Big(\mathbf{E}\Big|\sum_{x,j}\gamma_{x,j}\xi_{x,j}\Big|_X^2\Big)^{1/2} \leq C_{s,r}(X)\sup_{x,j}\left\|\gamma_{x,j}\right\|_{L^s}\left\|\xi\right\|_{\mathrm{Rad}\,X} \leq C_{s,r}(X)\left\|\lambda\right\|_{\ell^2}\left\|\xi\right\|_{\mathrm{Rad}\,X}$$

whenever $r < s < \infty$ and X has cotype r. (The cited theorem in [9] is stated for Gaussian random variables γ_{xj}, but an inspection of the proof reveals that only the sign invariance of the sequence and the finiteness of the sth moment [for some $s > r$, where X has cotype r] is really required.) □

Proof of Lemma 1.38. We again consider some sign invariant random variables, this time the following:

$$\gamma_{yi}(\omega,\omega',u) := \varepsilon'_{yi}(\omega')\sum_{x,j}\varepsilon_{xj}(\omega)(g(2^i(2^j u + x) + y) - g_{2^i(2^j\bar{B}+x)+y})\lambda_{xj}$$

on $(\Omega\times\Omega'\times\bar{B},\ \mathrm{d}\mathbf{P}\times\mathrm{d}\mathbf{P}'\times\mathrm{d}u/\left|\bar{B}\right|)$. Let $r<\infty$ be a cotype for X, let $r < s < \infty$, and denote by m_s the supremum of the L^s norms of the γ_{yi}. Then we have, by the same variant of Theorem 12.27 of [9] as in the proof of Lemma 1.37,

$$\Big(\frac{1}{\left|\bar{B}\right|}\int_{\bar{B}}\mathbf{E}\mathbf{E}'\Big|\sum_{y,i}\gamma_{yi}\xi_{yi}\Big|_X^2\,\mathrm{d}u\Big)^{1/2} \leq C_{s,r}(X)m_s\left\|\xi\right\|_{\mathrm{Rad}(X)},$$

where, by the John–Nirenberg and the Khinchin-Kahane inequalities combined,

$$\begin{aligned}
m_s &\leq \sup_{y,i,\bar{B}}\Big(\frac{1}{\left|\bar{B}\right|}\int_{\bar{B}}\mathbf{E}\Big|\sum_{x,j}\varepsilon_{x,j}(g(2^{j+i}u+2^i x+y) - g_{2^{j+i}\bar{B}+2^i x+y})\lambda_{xj}\Big|^s\mathrm{d}u\Big)^{1/s} \\
&\lesssim \sup_{y,i,\bar{B}}\Big(\frac{1}{\left|\bar{B}\right|}\int_{\bar{B}}\mathbf{E}\Big|\sum_{x,j}\varepsilon_{x,j}(g(2^{j+i}u+2^i x+y) - g_{2^{j+i}\bar{B}+2^i x+y})\lambda_{xj}\Big|^2\mathrm{d}u\Big)^{1/2} \\
&= \sup_{y,i,\bar{B}}\Big(\frac{1}{\left|\bar{B}\right|}\int_{\bar{B}}\sum_{x,j}\lambda_{xj}^2\left|g(2^{j+i}u+2^i x+y) - g_{2^{j+i}\bar{B}+2^i x+y}\right|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{x,j}\lambda_{xj}^2\left\|g\right\|_{BMO(\mathbf{R}^n)}^2\Big)^{1/2} = \left\|g\right\|_{BMO(\mathbf{R}^n)}\left\|\lambda\right\|_{\mathrm{Rad}(\mathbf{C})}.
\end{aligned}$$

□

4. Boundedness of paraproducts: a counterexample

In this section we construct a counterexample to show that the assumption $g \in \mathrm{BMO}_R(\mathbf{R}^n, \mathscr{L}(X,Y))$ fails, in general, to imply the boundedness of the paraproduct operator $P(g,\cdot)$ from $L^p(\mathbf{R}^n, X)$ to $L^p(\mathbf{R}^n, Y)$. We do this in the case where $n = 1$, X is a(n infinite dimensional) Hilbert space and Y is the field of scalars (which of course provides a counterexample for the case where X is Hilbert and Y is any

Banach space). Recall that in this situation $\mathscr{L}(X,Y) = X' \approx X$, and the BMO-norm of interest is the one in (1.32).

Our counterexample is a modification of one due to F. Nazarov, S. Treil and A. Volberg [21] for the vector-valued dyadic Carleson embedding theorem. (In the dyadic setting, there is an isomorphic correspondence between [Hilbert space] operator-valued Carleson measures and paraproducts, as explained in [22], 0.5.) A similar result is also proved in [22], with the sharp dimensional estimate as in the dyadic case, for the harmonic Carleson embedding theorem; however, this is no longer in a direct correspondence to the boundedness of paraproducts as it was in the dyadic case. Here, we consider the smooth paraproduct, and neglect the search for a sharp estimate but content ourselves by giving a constructive proof of Theorem 1.31.

Let us briefly describe the heuristic idea behind the construction in the dyadic setting of Nazarov *et al.*: It is based on a delicate interplay between two orthogonal systems of functions, the Haar functions $h_I(x) := (1_{[0,1/2[}-1_{[1/2,1[})(|I|^{-1}(x-\inf I))$ related to the dyadic intervals $I \subset [0,1[$, and the Rademacher functions $r_i := \sum_{|I|=2^{-i}} h_I$, which are even independent as random variables on $[0,1[$ with Lebesgue measure. While the h_I and the r_i have equal positive and negative mass, the products $h_I r_i = |h_I|$ for $|I| = 2^{-i}$ are positive functions, and this "cancellation of cancellation" is exploited in [21] to build "small" $f \in L^2$ and $g \in \mathrm{BMO}$ with a "large" paraproduct $P(g,f)$.

In order to imitate this counterexample in our smooth situation, we want to redefine the systems h_I and r_i so as to preserve the essence of the algebraic structure in the dyadic setting (in particular: the orthogonality of the h_I, the independence of the r_i, and a restricted version of the positivity of the appropriate products $h_I r_i$), but also to have the h_I appropriately localized in frequency to ensure neat behaviour of the smooth paraproduct $P(\cdot,\cdot)$ on these functions. We start from the construction of these auxiliary function systems.

A smooth Haar-like system of functions. We want to construct a function h_0 of the following kind, which will serve the rôle of the basic Haar function $1_{[0,2^{-1}[} - 1_{[2^{-1},1[}$:

Lemma 4.1. *For every sufficiently large $j_0 \in \mathbf{Z}_+$, there exists a real-valued $h_0 \in \mathscr{S}(\mathbf{R})$ with the following properties:*

- $h_0(x) \geq c2^{j_0/2}$ *for* $x \in \hat{I}_0$, *where* $|\hat{I}_0| = c2^{-j_0}$;
- $h_0(x) \leq 0$ *for* $x \in \tilde{I}_0$, *where* $|\tilde{I}_0| = |\hat{I}_0|$;
- $\|h_0\|_{L^\infty(\mathbf{R})} \leq C2^{j_0/2}$;
- $|h_0(x+1/2)| \leq C_m 2^{j_0(1/2-m)}|x|^{-m}$ *for* $|x| \geq 1/2$ *and* $m = 0,1,\ldots$;
- $\operatorname{supp}\hat{h}_0 = \cup \pm [1/3,4/3]\cdot 2^{j_0}$;
- *the functions* $2^{j/2}h_0(2^j x - k)$ $(j = 0,1,\ldots;\ k \in \mathbf{Z})$ *are orthonormal*

Here $\hat{I}_0$ and $\tilde{I}_0$ are subintervals of $I_0 := [0,1[$ with dyadic endpoints, and c, C and C_m are constants independent of j_0.

Proof. Our starting point is Y. Meyer's Littlewood–Paley-type wavelet (see [19])

$$\psi(x+\frac{1}{2}) = \int_{-\infty}^{\infty}\theta(\xi)e^{\mathbf{i}2\pi x\xi}\,\mathrm{d}\xi = 2\int_0^{\infty}\theta(\xi)\cos(2\pi x\xi)\,\mathrm{d}\xi,$$

where $\theta \in \mathscr{D}(\mathbf{R})$ is even, non-negative, and supported on $\cup \pm [1/3,4/3]$.

Since $\cos t \geq 2^{-1/2}$ for $|t| \leq \pi/4$, and $|2\pi x\xi| \leq \pi/4$ for $|x| \leq 3/32$ and $|\xi| \leq 4/3$, we deduce at once that $\psi(x+1/2) \geq 2^{-1/2}\|\theta\|_{L^1}$ for $x \in [-3,3]\cdot 2^{-5}$, i.e., $\psi(x) \geq$

$2^{-1/2}\|\theta\|_{L^1}$ for $x \in [13,19]\cdot 2^{-5}$. Also,

$$|x^m\psi(x+1/2)| = \left|(-2\pi\mathbf{i})^{-m}\int D^m\theta(\xi)e^{\mathbf{i}2\pi x\xi}\,\mathrm{d}\xi\right| \le (2\pi)^{-m}\|D^m\theta\|_{L^1}.$$

Now we consider the function $h_0(x) := \psi_{j_0,2^{j_0-1}}(x) \equiv 2^{j_0/2}\psi(2^{j_0}x - 2^{j_0-1}) = 2^{j_0/2}\psi(2^{j_0}(x-1/2))$, where $j_0 \in \mathbf{Z}_+$.

Clearly $h_0(x+1/2) = 2^{j_0/2}\psi(2^{j_0}x) \ge 2^{(j_0-1)/2}\|\theta\|_{L^1}$ for $x \in 2^{-j_0-5}\cdot[13,19]$, and $\|h_0\|_{L^\infty} \le 2^{j_0/2}\|\theta\|_{L^1}$. Since $\int\psi(x)\,\mathrm{d}x = \theta(0) = 0$, the wavelet ψ, and so also h_0, must attains some negative values, and we can now fix the intervals $\hat{I}_0$ and $\tilde{I}_0$ with the required properties, provided that j_0 is large enough for some negative values of ψ to fall inside $2^{j_0-1}\cdot[-1,1]$.

For $|x| \ge 2^{-1}$, we have $|h_0(x+1/2)| = 2^{j_0/2}\left|\psi(2^{j_0}x)\right| \le 2^{j_0/2}C_m\left|2^{j_0}x - 1/2\right|^{-m}$, and $\left|2^{j_0}x - 1/2\right| \ge 2^{j_0}|x| - 1/2 \ge (2^{j_0}-1)|x| \ge 2^{j_0-1}|x|$, so that $|h_0(x+1/2)| \le C_m 2^{j_0(1/2-m)}|x|^{-m}$ as required.

The support property of $\hat{h}_0$ follows at once by scaling from that of $\hat{\psi}_0$. The asserted orthonormality is a consequence of the orthonormality of the wavelets $\psi_{j,k} = 2^{j/2}\psi(2^j\cdot{}-k)$ $(j,k \in \mathbf{Z})$, since $2^{j/2}h_0(2^jx-k) = 2^{(j+j_0)/2}\psi(2^{j+j_0}x - (2^{j_0}k + 2^{j_0-1}))$. □

Next, we want to construct a suitable set of translates and dilates of h_0, indexed by a subset of the dyadic intervals. The following Haar-like properties should hold: If I and J are dyadic intervals and $I \subsetneq J$, then h_J should not change sign on the set where h_I differs substantially from zero; if I and J are disjoint, then h_I and h_J should be essentially non-overlapping.

Take some $m_0 \in \mathbf{Z}_+$, to be fixed later. For $I = 2^{-m_0}(I_0+k)$ $(k \in \mathbf{Z})$, we denote $h_I(x) \equiv h_{1,k}(x) := h_0(2^{m_0}x - k)$. If I is of the above form for some $k = 0,1,\ldots,2^{m_0}-1$, we say that I is a *rank* one interval, and denote this by $\mathrm{rk}(I) = 1$. We also define $\mathrm{rk}(I_0) := 0$. Let $\mathcal{Z} = \mathcal{Z}(j_0)$ be the set of zeros of h_0 on I_0, and $N(j_0) := \#\mathcal{Z}$. This number is finite, by the finiteness of the zero set of a holomorphic function on a compact set, since ψ can be extended to an entire function by the substitution $x \leftarrow z$ in the defining equation. Actually, applying Jensen's formula (e.g. [25], 15.18), one can readily estimate the number of zeros of $\psi(z+1/2)$ in $D(0,R)$ by CR, from which $N(j_0) \le C2^{j_0}$; however, this precision is not needed here.

Let $\varepsilon_0 > 0$ – again, to be fixed later. We then choose $m_0 = m_0(N(j_0),\varepsilon_0)$ large enough so that, on the one hand $N(j_0) \le \varepsilon_0(2^{m_0}-1)$, and on the other hand the endpoints of the intervals $\hat{I}_0$ and $\tilde{I}_0$ have the form $p2^{-m_0}$ for some integers p.

Now let $\mathcal{B}_1$ (for *bad*) be the set of those rank one intervals I which intersect with $\mathcal{Z}$; thus $\#\mathcal{B}_1 \le N(j_0)$. Then the set $\mathcal{G}_1$ of *good* rank one intervals (defined in an obvious way) satisfies $\#\mathcal{G}_1 \ge 2^{m_0} - N(j_0) > (1-\varepsilon_0)2^{m_0}$.

We proceed inductively:

For $I = 2^{-im_0}(I_0+k)$ $(k = 0,1,\ldots,2^{im_0}-1)$, let $h_I(x) \equiv h_{i,k}(x) := h_0(2^{im_0}x - k)$, and call these I the rank i intervals. Let $\mathcal{B}_i$ be the set of those I, $\mathrm{rk}(I) = i$, which contain a zero of *some* h_J with $J \supsetneq I$. Since h_J has $N(j_0)$ zeros on J, the total number of zeros appearing in the definition of $\mathcal{B}_i$ is at most

$$\sum_{j=0}^{i-1}2^{jm_0}N(j_0) = \frac{2^{im_0}-1}{2^{m_0}-1}N(j_0) \le \varepsilon_0(2^{im_0}-1)$$

thus $\#\mathcal{B}_i < \varepsilon_0 2^{im_0}$, and $\#\mathcal{G}_i > (1-\varepsilon_0)2^{im_0}$, where the definition of $\mathcal{G}_i$ is clear.

When $I = 2^{-jm_0}(I_0+k)$, we let $\hat{I} := 2^{-jm_0}(\hat{I}_0+k)$, and $\tilde{I}$ is defined similarly. We finally denote $\mathcal{B} := \cup_{i=0}^n\mathcal{B}_i$ and $\mathcal{G} := \cup_{i=0}^n\mathcal{G}_i$, and $\mathcal{I} := \mathcal{B}\cup\mathcal{G}$, where $\mathcal{G}_0 := \{I_0\}$,

$\mathcal{B}_0 := \emptyset$. The important properties of the function system we constructed are summarized in the following:

Lemma 4.2. *For every sufficiently large* $j_0 \in \mathbf{Z}_+$ *and every* $\varepsilon_0 > 0$, *there exists an orthogonal system of functions* $h_I(x) = h_0(|I|^{-1}(x - \inf I))$, $I \in \mathcal{I}$, *with the following additional properties:*

- $h_I(x) \geq c2^{j_0/2}$ *for* $x \in \hat{I}$, *where* $|\hat{I}| = c2^{-j_0}|I|$;
- $h_I(x) \leq 0$ *for* $x \in \tilde{I}$, *where* $|\tilde{I}| = |\hat{I}|$.
- $\|h_I\|_{L^\infty(\mathbf{R})} \leq C2^{j_0/2}$, $\|h_I\|^2_{L^2(\mathbf{R})} = |I|$;
- $|h_I(x + c_I)| \leq C_m 2^{j_0(1/2-m)}|I|^m |x|^{-m}$ *for* $|x| \geq |I|/2$ *and* $m = 0, 1, \ldots$, *where* $c_I := 2^{-1}(\inf I + \sup I)$;
- $\operatorname{supp} \hat{h}_I = \cup \pm [1/3, 4/3] \cdot 2^{j_0} \cdot |I|^{-1}$;
- *the functions* $|I|^{-1/2} h_I$ *are Meyer's Littlewood–Paley-type wavelets.*

The set $\mathcal{I}$ *consists of the* 2^{m_0}*-adic subintervals of* $[0, 1[$ *of rank* $\leq n$. $\hat{I}$ *and* $\tilde{I}$ *are subintervals of* I, *and they are exact unions of certain* $J \in \mathcal{I}$ *such that* $\operatorname{rk}(J) = \operatorname{rk}(I) + 1$ *(if* $\operatorname{rk}(I) < n$*).*

Moreover, we have a disjoint decomposition $\mathcal{I} = \mathcal{G} \cup \mathcal{B}$, *so that if* $I \in \mathcal{G}$ *and* $I \subsetneq J \in \mathcal{I}$, *then* $\operatorname{sgn} h_J$ *is constant on* I. *Denoting by* $\mathcal{G}_i$ *the set of those* $I \in \mathcal{G}$ *with* $\operatorname{rk}(I) = i$, *we have* $\#\mathcal{G}_i > (1 - \varepsilon_0)2^{im_0}$ *or every* $i = 0, 1, \ldots, n$.

The functions f and g of the counterexample. Our Rademacher-like system of functions is defined as follows:

$$r_i := \sum_{I \in \mathcal{I}:\operatorname{rk}(I) = i} (1_{\hat{I}} - 1_{\tilde{I}}), \qquad i = 0, 1, \ldots, n.$$

It follows easily that

$$\int_K r_i(x) r_j(x)\,\mathrm{d}x = 2|\hat{I}_0| \cdot |K|\,\delta_{i,j}, \qquad \int_K r_i(x)\,\mathrm{d}x = 0, \qquad K \in \mathcal{I},\ i, j \geq \operatorname{rk}(K),$$

and the r_i are independent random variables on $[0, 1[$. Moreover, we have

$$r_i h_I 1_I = |h_I|\, 1_{\hat{I} \cup \tilde{I}} \qquad \text{for} \qquad \operatorname{rk}(I) = i, \tag{4.3}$$

which is the "restricted positivity" of the products.

Let us denote by $(e_j)_{j=0}^n$ an orthonormal basis of $\ell^2(n + 1)$, and by $(a_j)_{j=1}^n$ a sequence of positive numbers to be fixed later.

Then we are ready to define the BMO-function of our counterexample:

$$g(\cdot) := \sum_{I \in \mathcal{G}} \varphi_I h_I(\cdot), \quad \text{where} \quad \varphi_I := \sum_{J \supsetneq I} a_{\operatorname{rk}(I:J)} r_{\operatorname{rk}(J)}(I) e_{\operatorname{rk}(J)} = \sum_{j=0}^{i-1} a_{i-j} r_j(I) e_j,$$

where $i = \operatorname{rk}(I)$, $\operatorname{rk}(I : J) := |\operatorname{rk}(I) - \operatorname{rk}(J)|$, and we denote by $r_j(I)$ the constant value of r_j on the interval I with $\operatorname{rk}(I) > j$.

When estimating the operator norm of $P(g, \cdot)$ from below, we are going to evaluate the paraproduct of g with the L^2-function

$$f = \sum_{I \in \mathcal{I}} h_I e_{\operatorname{rk}(I)} = \sum_{i=0}^n e_i \sum_{I:\operatorname{rk}(I) = i} h_I.$$

By the properties of the functions h_I from Lemma 4.2, we see that $\|f\|^2_{L^2(\mathbf{R}, \ell^2(n+1))} = n + 1$.

The BMO**-norm of** $(g(\cdot),e)_{\ell^2}$**.** Let $e=\sum_{j=0}^{n}b_je_j\in\ell^2$. Now

$$(g(\cdot),e)_{\ell^2}=\sum_{I\in\mathcal{G}}(\varphi_I,e)_{\ell^2}\,|I|^{1/2}\cdot|I|^{-1/2}\,h_I(\cdot)$$

is a scalar-valued function, and the above series is its expansion in the wavelet basis (ψ_J), since $|I|^{-1/2}\,h_I$ are Littlewood–Paley-type wavelets. Then Meyer's wavelet characterization of $\mathrm{BMO}(\mathbf{R})$ ([19], Theorem 5.4) shows that

$$\|(g(\cdot),e)_{\ell^2}\|^2_{\mathrm{BMO}(\mathbf{R})}\approx\sup_K|K|^{-1}\sum_{I\subset K}|(\varphi_I,e)_{\ell^2}|^2\cdot|I|\,,\tag{4.4}$$

where the supremum is taken over all intervals $K\subset\mathbf{R}$.

A further equivalent norm is obtained by restricting the supremum to all 2^{m_0}-adic intervals: If $K\subset\mathbf{R}$ is an arbitrary interval, let $k\in\mathbf{Z}$ be chosen so that $2^{m_0(k-1)}<|K|\leq 2^{m_0k}$. Then $K\subset J_1\cup J_2$, where the J_i are two consecutive 2^{m_0}-adic intervals of length 2^{m_0k}, and

$$\frac{1}{|K|}\sum_{I\subset K}(\cdot)\leq\frac{1}{|K|}\sum_{I\subset J_1\cup J_2}(\cdot)=\sum_{i=1}^{2}\frac{|J_i|}{|K|}\frac{1}{|J_i|}\sum_{I\subset J_i}(\cdot)\leq 2^{m_0}\sum_{i=1}^{2}\frac{1}{|J_i|}\sum_{I\subset J_i}(\cdot).$$

For our particular g which has non-zero wavelet coefficient only for $I\subset\mathcal{G}\subset\mathcal{I}$, one easily sees that we can still further restrict just to $K\in\mathcal{I}$.

We also note that

$$(\varphi_I,e)_{\ell^2}=\sum_{j=0}^{i-1}a_{i-j}r_j(I)b_j,\qquad|(\varphi_I,e)_{\ell^2}|^2\leq\sum_{j=1}^{n}a_j^2\cdot|e|^2_{\ell^2}\,.$$

We now fix a $K\in\mathcal{I}$ and search a bound for $\sum_{I\subsetneq K}|(\varphi_I,e)_{\ell^2}|\,|I|$. Let us denote $k:=\mathrm{rk}(K)$.

We first note that, for $I\subset K\subsetneq J$, we have $r_{\mathrm{rk}(J)}(I)=r_{\mathrm{rk}(J)}(K)$, and so, denoting $i:=\mathrm{rk}(I)$,

$$\sum_{J\supsetneq K}a_{\mathrm{rk}(I:J)}r_{\mathrm{rk}(J)}(I)e_{\mathrm{rk}(J)}=\sum_{j=0}^{k-1}a_{i-j}r_j(K)e_j=:f_i,$$

i.e., this expression depends on I only through $i=\mathrm{rk}(I)$.

We then investigate the following function, whose L^2 norm is related to the expression needed on the right of (4.4):

$$\begin{aligned}\phi_i(x):=&\sum_{I\subset K,\,\mathrm{rk}(I)=i}(\varphi_I,e)_{\ell^2}\,1_I(x)\\=&\sum_{I\subset K,\,\mathrm{rk}(I)=i}\sum_{j=0}^{i-1}a_{i-j}r_j(I)b_j1_I(x)\\=&\Big(\sum_{j=k}^{i-1}+\sum_{j=0}^{k-1}\Big)a_{i-j}b_jr_j(x)\sum_{I\subset K,\,\mathrm{rk}(I)=i}1_I(x)\\=&\sum_{j=k}^{i-1}a_{i-j}b_jr_j(x)1_K(x)+(f_i,e)_{\ell^2}\,1_K(x).\end{aligned}$$

Evaluating the L^2-norm of ϕ_i by the first and last expressions above (using the orthogonality properties of the r_j's for the latter), we get

$$\sum_{I\subset K,\,\mathrm{rk}(I)=i}|(\varphi_I,e)_{\ell^2}|^2\,|I|=\Big(2|\hat{I}_0|\sum_{j=k}^{i-1}a_{i-j}^2\,|b_j|^2+|(f_i,e)_{\ell^2}|^2\Big)\,|K|\,.$$

Thus

$$\sum_{I\subsetneq K} |(\varphi_I, e)_{\ell^2}|^2 \, |I| \le |K| \Big(\sum_{i=k+1}^{n} \sum_{j=k}^{i-1} a_{i-j}^2 \, |b_j|^2 + |e|_{\ell^2}^2 \sum_{i=k+1}^{n} |f_i|_{\ell^2}^2 \Big).$$

The double sum equals $\sum_{j=k}^{n} |b_j|^2 \sum_{i=j+1}^{n} a_{i-j}^2 \le \sum_{j=k}^{n} |b_j|^2 \sum_1^n a_i^2 \le |e|_{\ell^2}^2 \sum_1^n a_i^2$.

Using the definition of the f_i's, we get

$$\sum_{i=k+1}^{n} |f_i|_{\ell^2}^2 = \sum_{j=0}^{k-1} \sum_{i=k-j+1}^{n-j} a_i^2 = \sum_{i=2}^{n} a_i^2 \sum_{j=(i+k-n)\vee 1}^{(i-1)\wedge k} 1 \le \sum_{i=2}^{n} a_i^2 (i-1).$$

We conclude the following:

Lemma 4.5. *For the g defined above, we have the norm estimate*

$$\|g\|_{\mathrm{BMO}_w(\mathbf{R},\ell^2(n+1))} \le C \Big(\sum_{i=1}^{n} i a_i^2 \Big)^{1/2}.$$

Lower estimate for the paraproduct $P(g,f)$. Recall from Lemma 4.2 that $\hat{h}_I$ is supported on an annulus whose outer and inner radii have ratio 4. Also note that in our standard dyadic partition of unity, $\sum_{j=0}^{2} \hat{\Psi}_j$ is constantly $= 1$ on such an annulus, namely, the support of $\hat{\Psi}_1$. Of course we can choose the scale in such a way that these annuli coincide. Then we may readily evaluate the modified paraproduct-like expression

$$\begin{aligned} \sum_j \sum_{\ell=3}^{5} (\Psi_{3j+\ell} * g)(\Phi_{3j} * f) &= \sum_{i=1}^{n} \sum_{I\in\mathcal{G}_i} h_I \Big(\varphi_I, \sum_{J:\mathrm{rk}(J)<i} h_I e_{\mathrm{rk}(I)} \Big)_{\ell^2} \\ &= \sum_{i=1}^{n} \sum_{I\in\mathcal{G}_i} h_I \sum_{J:j=\mathrm{rk}(J)<i} a_{i-j} r_j(I) h_J, \end{aligned} \tag{4.6}$$

and the summands with different i index have disjoint frequencies, and are therefore orthogonal in $L^2(\mathbf{R})$.

On the other hand, we should note that (4.6) is not too far away from the actual paraproduct, as

$$\begin{aligned} \sum_j \sum_{\ell=3}^{5} (\Psi_{3j+\ell} * g)(\Phi_{3j} * f) &= \sum_j (\Psi_j * g)(\Phi_{j-\ell(j)} * f) \\ &= P(g,f) + \sum_j (\Psi_j * g)((\Phi_{j-\ell(j)} - \Phi_{j-3}) * f), \end{aligned} \tag{4.7}$$

where $\ell(j) := 3, 4, 5$, respectively, depending on whether $j \equiv 0, 1, 2 \mod 3$. For the last sum, we can essentially imitate the estimate (3.3) to bound its L^2 norm by $C \|g\|_{\mathrm{BMO}_w(\mathbf{R},\ell^2)} \|f\|_{L^2(\mathbf{R},\ell^2)}$ (cf. (3.4)). Thus proving our desired lower estimate for $P(g,f)$ is equivalent to finding one for the expression in (4.6).

To estimate this quantity, we start by decomposing the summation as follows:

$$\begin{aligned} \sum_{I\in\mathcal{G}_i} h_I \sum_{J:j=\mathrm{rk}(J)<i} r_j(I) h_J a_{i-j} &= \sum_{I\in\mathcal{G}_i} h_I 1_I \sum_{J\supsetneq I} r_j(I) h_J a_{i-j} \\ &+ \sum_{I\in\mathcal{G}_i} h_I 1_I \sum_{\substack{J:j=\mathrm{rk}(J)<i \\ J\not\supseteq I}} r_j(I) h_J a_{i-j} + \sum_{I\in\mathcal{G}_i} h_I 1_{I^c} \sum_{J:j=\mathrm{rk}(J)<i} r_j(I) h_J a_{i-j}. \end{aligned} \tag{4.8}$$

Consider the first of the three terms above. Note that there $1_I r_j(I) = 1_I 1_J r_j$, and then $1_J r_j h_J = |h_J|\,1_{\hat{J}\cup\tilde{J}}$ by (4.3). Thus the first sum can further be written as

$$\sum_{I:\mathrm{rk}(I)=i} h_I 1_I \sum_{J\supsetneq I} |h_J|\,1_{\hat{J}\cup\tilde{J}} a_{i-j} - \sum_{I\in\mathcal{B}_i} h_I 1_I \sum_{J\supsetneq I} |h_J|\,1_{\hat{J}\cup\tilde{J}} a_{i-j}. \tag{4.9}$$

Now we start estimating the various terms in the decomposition:

The first term in (4.9). If $\mathrm{rk}(J) < \mathrm{rk}(I)$ and $J \not\supset I$, then $1_I 1_{\hat{J}\cup\tilde{J}} = 0$, and thus the double sum considered is equal to

$$A := \sum_{I:\mathrm{rk}(I)=i} h_I 1_I \sum_{J:j=\mathrm{rk}(J)<i} |h_J|\,1_{\hat{J}\cup\tilde{J}} a_{i-j}$$

We observe the lower bounds

$$\Big|\sum_{I:\mathrm{rk}(I)=i} h_I 1_I\Big| \geq \sum_{I:\mathrm{rk}(I)=i} c2^{j_0/2} 1_{\hat{I}},$$

$$\sum_{J:j=\mathrm{rk}(J)<i} |h_J|\,1_{\hat{J}\cup\tilde{J}} a_{i-j} \geq \sum_{J:j=\mathrm{rk}(J)<i} c2^{j_0/2} 1_{\hat{J}} a_{i-j}.$$

Thus

$$\|A\|_{L^2(\mathbf{R})} \geq c2^{j_0}\Big(\int\Big[\sum_{I:\mathrm{rk}(I)=i} 1_{\hat{I}}(x) \sum_{j=0}^{i-1} a_{i-j} \sum_{J:\mathrm{rk}(J)=j} 1_{\hat{J}}(x)\Big]^2 \,\mathrm{d}x\Big)^{1/2}.$$

We denote $X_j := \sum_{J:\mathrm{rk}(J)=j} 1_{\hat{J}}$. Then the X_j's are independent, $\{0,1\}$-valued random variables on $[0,1]$, and we continue the estimate with

$$\begin{aligned}
&= c2^{j_0}\Big(\mathbf{E}\Big[X_i^2\Big\{\sum_{j=0}^{i-1} a_{i-j} X_j\Big\}^2\Big]\Big)^{1/2} \\
&= c2^{j_0}\Big(\mathbf{E}X_i^2\Big)^{1/2}\Big(\mathbf{E}\Big\{\sum_{j=0}^{i-1} a_{i-j}^2 X_j^2 + 2\sum_{j<j'} a_{i-j}a_{i-j'}X_j X_{j'}\Big\}\Big)^{1/2} \\
&= c2^{j_0}(c2^{-j_0})^{1/2}\Big(\sum_{j=0}^{i-1} a_{i-j}^2 c2^{-j_0} + 2\sum_{j<j'} a_{i-j}a_{i-j'}(c2^{-j_0})^2\Big)^{1/2} \\
&\geq c2^{j_0}(c2^{-j_0})^{1/2}(c2^{-j_0})\Big(\sum_{j=0}^{i-1} a_{i-j}\Big) = c2^{-j_0/2}\sum_{j=1}^{i} a_j.
\end{aligned}$$

This is the lower bound we wanted. For the rest of the terms to be estimated, we are looking for upper norm bounds.

The second term in (4.9). We know that $|h_J| \leq C2^{j_0/2}$, and hence

$$\sum_{J\supsetneq I} |h_J|\,1_{\hat{J}\cup\tilde{J}} a_{i-j} \leq C2^{j_0/2}\sum_{j=0}^{i-1} a_{i-j}.$$

Thus

$$\begin{aligned}
\Big\| \sum_{I\in\mathcal{B}_i} h_I 1_I \sum_{J\supsetneq I} |h_J|\, 1_{\hat{J}\cup\tilde{J}} a_{i-j} \Big\|_{L^2(\mathbf{R})} &\leq C2^{j_0/2} \sum_{j=1}^{i} a_j \Big\| \sum_{I\in\mathcal{B}_i} h_I 1_I \Big\|_{L^2(\mathbf{R})} \\
= C2^{j_0/2} \sum_{j=1}^{i} a_j \Big(\sum_{I\in\mathcal{B}_i} \|h_I 1_I\|_{L^2(\mathbf{R})}^2 \Big)^{1/2} &\leq C2^{j_0/2} \sum_{j=1}^{i} a_j \Big(\sum_{I\in\mathcal{B}_i} |I| \Big)^{1/2} \\
= C2^{j_0/2} \sum_{j=1}^{i} a_j (\#\mathcal{B}_i/2^{im_0})^{1/2} &\leq C2^{j_0/2} \varepsilon_0^{1/2} \sum_{j=1}^{i} a_j.
\end{aligned}$$

The second term in (4.8). To bound this sum, we first want to estimate the quantity $\sum_{J:\mathrm{rk}(J)=j, J\not\supset I} |h_J(x)|$ for $x \in I$. For any J appearing in the sum, we have $x \notin J$, and hence $|x - c_J| \geq |J|/2$, and moreover this distance increases by increments of $|J|$ as we move to intervals further away from I. Using Lemma 4.2, we then get the upper bound

$$\sum_{p=0}^{\infty} C_m 2^{j_0(1/2-m)} (1/2+p)^{-m} \leq \tilde{C}_m 2^{j_0(1/2-m)} \tag{4.10}$$

provided that $m \geq 2$. Now we get

$$\begin{aligned}
&\Big\| \sum_{I\in\mathcal{G}_i} h_I 1_I \sum_{\substack{J:j=\mathrm{rk}(J)<i \\ J\not\supset I}} r_j(I) h_J a_{i-j} \Big\|_{L^2(\mathbf{R})} \\
&\qquad \leq C_m 2^{j_0(1/2-m)} \sum_{j=0}^{i-1} a_j \Big\| \sum_{I\in\mathcal{G}_i} h_I 1_I \Big\|_{L^2(\mathbf{R})} \leq C_m 2^{j_0(1/2-m)} \sum_{j=0}^{i-1} a_j.
\end{aligned}$$

The third term in (4.8). By applying the bound $|h_J(x)| \leq C2^{j_0/2}$, to the J for which $J \ni x$, and the the estimate (4.10) to the rest of the J with $\mathrm{rk}(J) = j$, we find that

$$\sum_{J:\mathrm{rk}(J)=j} |h_J(x)| \leq C2^{j_0/2} + C_m 2^{j_0(1/2-m)} \leq \tilde{C} 2^{j_0/2}.$$

Thus

$$\Big\| \sum_{J:j=\mathrm{rk}(J)<i} r_j(I) h_J a_{i-j} \Big\|_{L^\infty(\mathbf{R})} \leq C2^{j_0/2} \sum_{j=1}^{i} a_j. \tag{4.11}$$

We then concentrate on $\sum_{I\in\mathcal{G}_i} \left| h_I(x) 1_{I^c(x)} \right|$. By a similar reasoning as that leading to (4.10), we find the same upper bound $C_m 2^{j_0(1/2-m)}$, valid for all $x \in \mathbf{R}$. For x well outside $[0,1[$, we need a sharper estimate, which is obtained by observing that the distance of x to the centre of the nearest $I \in \mathcal{G}_i$ is at least $d := d(x,[0,1]) + |I|/2$ for $x \notin [0,1]$. Thus

$$\sum_{I\in\mathcal{G}_i} |h_I(x) 1_{I^c}(x)| \leq \sum_{p=0}^{\infty} C_m 2^{j_0(1/2-m)} (d/|I| + p)^{-m} \leq C_m 2^{j_0(1/2-m)} |I|^{m-1} d^{1-m}.$$

Using one or the other of the above estimates for $x \in [-1/2, 3/2]$ and $x \notin [-1/2, 3/2]$, say, we get $\left\| \sum_{I\in\mathcal{G}_i} h_I 1_{I^c} \right\|_{L^2(\mathbf{R}^n)} \leq C2^{j_0(1/2-m)}$. Combining this with (4.11), we have obtained the upper bound $C_m 2^{j_0(1-m)} \sum_{j=1}^{i} a_j$ for the L^2-norm of the third term in (4.8).

Conclusion of the proof of Theorem 1.31. Combining the estimates obtained above for the various terms appearing in (4.8) and (4.9), we have shown that

$$\Big\| \sum_{I\in\mathcal{G}_i} h_I \sum_{J:j=\mathrm{rk}(J)<i} r_j(I) h_J a_{i-j} \Big\|_{L^2(\mathbf{R})} \geq \Big(c2^{-j_0/2} - C2^{j_0/2}\varepsilon_0^{1/2} - C_m 2^{j_0(1/2-m)} - C_m 2^{j_0(1-m)} \Big) \sum_{j=1}^{i} a_j,$$

where some $m \geq 2$ is fixed. We now finally exploit our freedom to choose the constants j_0 and ε_0 as desired. First, we take j_0 large enough so that $c2^{-j_0/2} - C_m 2^{j_0(1/2-m)} - C_m 2^{j_0(1-m)} > 0$. Once j_0 is fixed like this, we choose ε_0 small enough so that the whole $(\cdots)$ above is strictly positive. Now that j_0 and ε_0 are fixed once and for all, we can drop the references to them from the constants, and simply denote the positive constant in $(\cdots)$ above by c.

From orthogonality we conclude that the L^2-norm of the quantity in (4.6) is bounded from below by $c\left(\sum_{i=1}^n \left[\sum_{j=1}^i a_j\right]^2\right)^{1/2}$. From (4.7) and the observations following it, and from the estimate in Lemma 4.5, we conclude that

$$\frac{\|P(g,f)\|_{L^2(\mathbf{R})}}{\|g\|_{\mathrm{BMO}_w(\mathbf{R},\ell^2(n+1))} \|f\|_{L^2(\mathbf{R},\ell^2(n+1))}} \geq c \left[\frac{\sum_{i=1}^n \left(\sum_{j=1}^i a_j\right)^2}{(n+1)\sum_{i=1}^n i a_i^2} \right]^{1/2} - C$$

Now the final step is to choose $a_j := j^{-1}$, so that $\sum_{i=1}^n \left(\sum_{j=1}^i a_j\right)^2 \approx n \log^2 n$, whereas $(n+1)\sum_{i=1}^n i a_i^2 \approx n \log n$. This completes the proof of Theorem 1.31. $\square$

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