# Convolutions, multipliers and maximal regularity on vector-valued Hardy spaces

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**Abstract**

We extend the recently developed $L^p$-theory for the maximal regularity of the abstract Cauchy problem $[\dot{u} + Au = f,\ u(0) = 0]$ and the related Fourier multiplier techniques to the real-variable Hardy space $H^1$. Some results for $H^p$, $0 < p < 1$, are also proved.

**AMS subject classification** Primary 34G10; Secondary 42B30

**Keywords** Hardy spaces of vector-valued functions, operator-valued Fourier multipliers, abstract Cauchy problem, maximal regularity.

## 1 Introduction

The aim of this paper is to consider the limiting case $p = 1$ of some recent results on maximal $L^p$-regularity of abstract evolution equations, as well as the related continuity problems of convolution and Fourier multiplier transformations with operator-valued kernel (see [1, 6, 7, 8, 13, 15, 21, 23]). Since these are essentially questions of vector-valued harmonic analysis, the real-variable Hardy space $H^1$ will provide a natural substitute for the space $L^1$.

Besides theoretical interest, such an extension is also motivated by the rôle of the Hardy spaces in applications like non-linear equations of elasticity (cf. [4], Sect. 1.2). While our framework is not precisely the one of the cited paper, the results given here appear to be the first steps towards bringing the recent successful $R$-boundedness techniques to the $H^1$ setting.

Let us consider the following abstract Cauchy problem (ACP):

$$\dot{u}(t) + Au(t) = f(t) \quad \text{for } t \geq 0, \qquad u(0) = 0, \tag{1.1}$$

with $A$ a closed linear operator with dense domain in the underlying Banach space $X$ and $f \in L^1_{\mathrm{loc}}(\bar{\mathbf{R}}_+; X)$.

By *maximal regularity* of the ACP one means that, with any given data $f$ in a certain function class, there exists a unique solution $u$ (in an appropriate sense) such that both the terms on the left-hand side of (1.1) possess the same regularity (e.g., in the $L^p$-setting, are integrable to the same power on the positive half-line) as $f$.

If $-A$ is the generator of a strongly continuous semigroup $(T^t)_{t\geq 0}$, it is wel-known that an $L^p$-solution [$u \in W^{1,p}(\bar{\mathbf{R}}_+;X)$ with $u(t) \in \mathcal{D}(A)$ for a.e. $t \geq 0$, and which satisfies (1.1) a.e.], when it exists, is necessarily given by the *variation-of-constants formula* $u(t) = \int_0^t T^{t-s} f(s)\, \mathrm{d}s$, and this formula can always be used to *define* what is called the *mild solution*.

Moreover, if $(T^t)$ is bounded and analytic and $f$ is appropriate, then

$$\hat{u}(\xi) = (\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi), \qquad \mathscr{F}[Au](\xi) = A(\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi), \tag{1.2}$$

where $\hat{\ } \equiv \mathscr{F}$ is the Fourier transform.

Proving first an operator-valued Mihlin multiplier theorem, and applying it to the particular multiplier in (1.2), L. Weis [23] obtained the following characterization of maximal regularity in the setting of UMD Banach spaces (cf. [18]):

**Theorem 1.3 (Weis 2001).** *Let $X$ be a UMD-space and $-A$ the generator of a bounded analytic semigroup. Then the following are equivalent:*

($W_1$) *ACP has maximal $L^p$-regularity for all $p \in ]1, \infty[$.*

($W_2$) *The collection $\{A(\mathbf{i}2\pi\xi + A)^{-1} |\ \xi \in \mathbf{R} \setminus \{0\}\}$ is R-bounded.*

An important companion to this result is the theorem of N. Kalton and G. Lancien [16], which shows the non-triviality of the conditions ($W_1$) and ($W_2$), in the sense that every $X \neq \ell^2$ with an unconditional basis has generators $-A$ for which ($W_1$), and hence ($W_2$), fails.

In the present paper, we extend these equivalences to the maximal regularity in the case $p = 1$ as follows:

**Theorem 1.4.** *Let $X$ be a UMD-space and $A$ a closed, linear, densely defined operator on $X$. Then the following are equivalent:*

($C_1$) *ACP has maximal $L^p$-regularity for all $p \in ]1, \infty[$.*

($C_2$) *ACP has maximal $H^1$-regularity.*

($C_3$) *ACP has $(H^1, L^1)$-regularity.*

($C_4$) *$-A$ generates a bounded analytic semigroup and $\{A(\mathbf{i}2\pi\xi + A)^{-1} |\ \xi \in \mathbf{R} \setminus \{0\}\}$ is R-bounded.*

*Moreover, any of these is sufficient to*

($C_5$) *ACP has maximal $H^p$-regularity for all $p \in ]0, 1[$.*

*The implications $C_1 \Rightarrow C_2 \Rightarrow C_3 \Rightarrow C_4$ and $C_1 \Rightarrow C_5$ hold in fact for any Banach space $X$.*

By $(H^1, L^1)$-regularity of the ACP we mean that for every $f \in H^1(\bar{\mathbf{R}}_+;X)$ there exists a unique $u \in W^{1,1}_{\mathrm{loc}}(\bar{\mathbf{R}}_+;X)$ such that $u(t) \in \mathcal{D}(A)$ and $\dot{u}(t) + Au(t) = f(t)$ for a.e. $t > 0$, and moreover

$$\|\dot{u}\|_{L^1(\bar{\mathbf{R}}_+;X)} + \|Au\|_{L^1(\bar{\mathbf{R}}_+;X)} \leq C \|f\|_{H^1(\bar{\mathbf{R}}_+;X)},$$

where $C < \infty$ is independent of $f$. For the definition of maximal $H^1$-regularity, replace $L^1$ by $H^1$.

Concerning maximal $H^p$-regularity when $0 < p < 1$, we simply require that the map $f \mapsto Au$ is well-defined and bounded on a dense subspace of $H^p(\bar{\mathbf{R}}_+; X)$ consisting of proper functions.

As tools for proving Theorem 1.4, we also establish extensions of the multiplier theorems used in the proof of Theorem 1.3. Moreover, we show that these general results also apply to other equations than just the ACP.

The paper is organized as follows: Sect. 2 is preliminary, collecting some general notation and facts to be used. A density lemma concerning Hardy spaces is postponed to Sect. 8. In Sect. 3 we prove the necessity of $-A$ generating an analytic semigroup (Theorem 3.1) for the $(H^1, L^1)$-regularity of (1.1). We go on with necessary conditions for regularity in Sect. 4, where we prove the necessity of $R$-boundedness for $(H^1, L^1)$-multipliers (Theorem 4.1). Sufficient conditions for the boundedness of our operators are then taken up in Sect. 5. We return to the problem of maximal regularity in Sect. 6, where we complete the proof of Theorem 1.4. Analogous results for two other model problems are stated in Sect. 7.

**Acknowledgement** The results of this paper are part of the authors doctoral dissertation [14], written under the supervision of Prof. Stig-Olof Londen. This particular topic for my research was suggested by Prof. Philippe Clément, who also gave many fruitful suggestions during its progress. Helpful discussions with Professors Stefan Geiss and Lutz Weis are also gratefully acknowledged.

## 2 General preliminaries

Let us fix some notation. The set of natural numbers is $\mathbf{N} := \{0, 1, 2, \ldots\}$ and that of positive integers is $\mathbf{Z}_+ := \{1, 2, \ldots\}$. Moreover, $\mathbf{R}_+ := ]0, \infty[$ and $\bar{\mathbf{R}}_+ := [0, \infty[$. For $\ell > 0$, we denote by $\lfloor \ell \rfloor$ the greatest integer at most $\ell$, and by $[\![ \ell ]\!]$ the greatest integer strictly less than $\ell$. Thus both functions give the integer part of a non-integer $\ell$, but $[\![ m ]\!] = m - 1$, $\lfloor m \rfloor = m$ for $m \in \mathbf{Z}_+$.

$X$ and $Y$ are complex Banach spaces. The *Lebesgue–Bôchner spaces* of $X$-valued functions on $\Omega$ [usually $\mathbf{R}^n$ or $\bar{\mathbf{R}}_+$, always equipped with the Lebesgue measure] are denoted by $L^p(\Omega; X)$, and the *Hardy spaces* [whose definition is given later in this section] by $H^p(\Omega; X)$. If $X = \mathbf{C}$, we omit it from the notation and simply write $L^p(\mathbf{R}^n)$ etc.

**Test function spaces** $\mathscr{S}(\mathbf{R}^n; X)$ denotes the *Schwartz class* of infinitely differentiable, rapidly decreasing $X$-valued functions. The $X$-valued tempered distributions are defined by $\mathscr{S}'(\mathbf{R}^n; X) := \mathscr{L}(\mathscr{S}(\mathbf{R}^n), X)$, where $\mathscr{L}(A, B)$ denotes continuous linear operators between the topological vector spaces $A$ and $B$.

Other useful function classes are $\mathscr{D}(\mathbf{R}^n;X) := \mathscr{C}_c^\infty(\mathbf{R}^n;X) \subset \mathscr{S}(\mathbf{R}^n;X)$, where the subscript $c$ indicates compact support, and

$$\hat{\mathscr{D}}_0(\mathbf{R}^n;X) := \left\{ \psi \in \mathscr{S}(\mathbf{R}^n;X) \,\middle|\, \hat{\psi} \in \mathscr{D}(\mathbf{R}^n;X),\ 0 \notin \operatorname{supp} \hat{\psi} \right\},$$

where $\hat{\psi}$ stands for the Fourier transform of $\psi$. It is well-known that all the test-function classes mentioned so far are dense in $L^p(\mathbf{R}^n;X)$ for $p \in \,]1,\infty[$; in fact, this is true even for the algebraic tensor products $X \otimes \hat{\mathscr{D}}_0(\mathbf{R}^n)$ etc.

**Fourier transform and multipliers** The Fourier transform of $f$ is denoted by $\mathscr{F}f$ or $\hat{f}$ and the inverse Fourier transform by $\mathscr{F}^{-1}f$ or $\check{f}$.

Given $m \in L^1_{\mathrm{loc}}(\mathbf{R}^n;\mathscr{L}(X,Y))$, we can consider the *Fourier multiplier operator* $T = T_m$, initially defined on $\hat{\mathscr{D}}_0(\mathbf{R}^n;X)$, say, by $Tf := \mathscr{F}^{-1}[m\hat{f}]$; or more explicitly,

$$Tf(t) := \int_{\mathbf{R}^n} m(t)\hat{f}(t)e^{\mathbf{i}2\pi\xi\cdot t}\,\mathrm{d}\xi. \tag{2.1}$$

A Banach space $X$ is called a UMD-space if $m(\xi) := -\mathbf{i}\operatorname{sgn}(\xi)$ defines by means of (2.1) a bounded operator $\mathscr{H}$, the Hilbert transform, on $L^p(\mathbf{R};X)$ for one (and then all) $p \in \,]1,\infty[$ (cf. [18]).

As another notion from the geometry of Banach spaces, we recall that a Banach space $X$ is said to have *Fourier-type* $p$ if the Hausdorff–Young inequality

$$\|\hat{f}\|_{L^{p'}(\mathbf{R}^n;X)} \leq C\,\|f\|_{L^p(\mathbf{R}^n;X)}\,, \tag{2.2}$$

is true for every $f \in (L^1 \cap L^p)(\mathbf{R}^n;X)$ with some finite $C$. Every Banach space satisfies this inequality with $p = 1$, and no Banach space (except $\{0\}$) with any $p > 2$. By interpolation the inequality holds for $q \in [1,p]$ if it holds for $p$. $X$ is said to have a non-trivial Fourier-type, if it has a Fourier-type $p > 1$. Every UMD-space is known to possess this property.

***R*-boundedness** We denote by $\varepsilon_j$, $j = 1,2,\ldots$, the Rademacher system of independent random variables on some probability space $(\Omega,\Sigma,\mathbf{P})$ which satisfy $\mathbf{P}(\varepsilon_j = 1) = \mathbf{P}(\varepsilon_j = -1) = 1/2$. $\mathbf{E}$ denotes the expectation related to the probability measure $\mathbf{P}$.

Then $\mathscr{T} \subset \mathscr{L}(X;Y)$ is called *R-bounded*, if for some $p \in \,]0,\infty[$ and $C < \infty$ and for all $N \in \mathbf{Z}_+$, $x_j \in X$, $T_j \in \mathscr{T}$ the inequality

$$\Big(\mathbf{E}\Big|\sum_{j=1}^N \varepsilon_j T_j x_j\Big|_Y^p\Big)^{\frac{1}{p}} \leq C\Big(\mathbf{E}\Big|\sum_{j=1}^N \varepsilon_j x_j\Big|_X^p\Big)^{\frac{1}{p}}$$

holds. By J.-P. Kahane's inequality this condition actually holds true either for all $p \in \,]0,\infty[$ (with $C$ possibly depending on $p$) or for none. We shall be mostly concerned with the case $p = 1$, and we refer to the smallest $C$ in this inequality as the $R$-bound of $\mathscr{T}$ and denote it by $\mathscr{R}(\mathscr{T})$.

A basic tool related to $R$-boundedness is the *contraction principle* (also of Kahane) stating that $\mathscr{R}(\Lambda \cdot I) \leq 2\sup_{\lambda\in\Lambda}|\lambda|$, for $\Lambda \subset \mathbf{C}$ and $I$ the identity operator on any Banach space. More on $R$-boundedness can be found in [6, 23].

**Atomic Hardy spaces** We define the Hardy spaces in terms of the atomic decomposition as follows: (For a discussion of differently defined Hardy spaces and their relations in the vector-valued setting, we refer to O. Blasco [3].)

$$H^p(\mathbf{R}^n;X) := \Big\{ \mathscr{S}'\text{-}\sum_{k=0}^{\infty} \lambda_k a_k \ : a_k \text{ an } H^p\text{-atom}, \ \lambda_k \in \mathbf{C}, \ \sum_{k=0}^{\infty} |\lambda_k|^p < \infty \Big\}$$

equipped with the $p$-norm $\|f\|_{H^p(\mathbf{R}^n;B)}^p := \inf \sum_{k=0}^{\infty} |\lambda_k|^p$, where the infimum is taken over all atomic decompositions of $f \in H^p$ as above.

The definition of the atoms appearing here is the same as in the scalar-valued context: We say that $a \in L^q(\mathbf{R}^n;B)$ is a $(p,q,N)$-atom, where $0 < p \leq 1 < q \leq \infty$ and $N \in \mathbf{N}$, provided that

- $a$ is supported in a ball $\bar{B}$,
- $\|a\|_{L^q} \leq \left|\bar{B}\right|^{q^{-1}-p^{-1}}$, and
- $\int x^\alpha a(x)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq N$.

The three requirements above are referred to as the *support* condition, the *size* condition and the *moment* condition, respectively.

We say that $a$ is a $(p,q)$-atom if it is a $(p,q,N)$-atom for some $N \in \mathbf{N}$, and that $a$ is an $H^p$-atom of $L^q$-type if it is a $(p,q,N)$-atom for some $N \geq n(p^{-1}-1)$. Finally, $a$ is an $H^p$-atom, if it is an $H^p$-atom of some $L^q$-type, $q > 1$. In the definition of $H^p$ above, we require that the $a_k$ are $(p,q)$-atoms for some fixed $q > 1$. The spaces obtained with different values of $q$ coincide and the norms are equivalent. In the sequel, we assume that a certain $q \in ]1,\infty[$ has been fixed, and the $H^p$-norms are always defined in terms of this $q$. [We exclude $q = \infty$ in order to have access to density arguments in the $L^q$ norm.]

**Hardy spaces on a half-line** For the purposes of studying the Cauchy problem, we need a notion of Hardy spaces on the half-line $\bar{\mathbf{R}}_+$. We define $H^p(\bar{\mathbf{R}}_+;X)$ simply as the subspace of $H^p(\mathbf{R},X)$ consisting of all $f$ whose distributional support lies on $\bar{\mathbf{R}}_+$. It is useful to know that the atomic decomposition of such an $f$ can also be chosen to be supported on $\bar{\mathbf{R}}_+$; more precisely:

**Lemma 2.3.** *There exists $C < \infty$ such that every $f \in H^p(\bar{\mathbf{R}}_+;X)$ has an atomic decomposition $f = \mathscr{S}'\text{-}\sum_{k=1}^{\infty} \lambda_k a_k$ with $\operatorname{supp} a_k \subset \bar{\mathbf{R}}_+$ and $\sum_{k=0}^{\infty} |\lambda_k|^p \leq C \|f\|_{H^p(\mathbf{R};X)}^p$.*

*Proof.* This is based on a standard reflection argument, which we only sketch. Given any atomic decomposition $\sum_{k=0}^{\infty} \lambda_k a_k$ of $f \in H^p(\bar{\mathbf{R}}_+;X)$, the idea is to construct new atoms $A_k$ as follows: Write $a_k = a_k^+ + a_k^-$, where $a_k^{\pm} := a_k \chi_{\mathbf{R}_\pm}$, and define $\tilde{a}_k^- := \sum_{j=0}^{N} b_j a_k^-(-c_j\cdot)$, where $b_j \in \mathbf{R}$ and $c_j > 0$ are appropriately chosen so that $A_k := a_k^+ + \tilde{a}_k^-$ has the same required vanishing moments as $a_k$. Then one can show that $\sum_{k=0}^{\infty} \lambda_k A_k = f$, and this decomposition has the desired properties. The only non-trivial part of the proof is to notice that an element of $H^p(\mathbf{R};X)$ cannot have a one-point support (in this case, the origin), for which fact, see e.g. [20], Ch. III, §5.5(c). □

**Dense subsets** Since our line of attack to the maximal regularity and the related problems is via density extensions of estimates for sufficiently regular functions, we need to have a sufficiently ample collection of dense subspaces at our disposal. It turns out that the problem of finding dense subspaces of the Hardy spaces of vector-valued functions reduces to the corresponding task in the scalar-valued context:

**Lemma 2.4.** *Let $Z$ be a dense subspace of $X$, and $G$ a dense subspace of $H^p(\mathbf{R}^n)$ resp. $H^p(\bar{\mathbf{R}}_+)$. Then the algebraic tensor product $Z \otimes G$ is a dense subspace of $H^p(\mathbf{R}^n; X)$ resp. $H^p(\bar{\mathbf{R}}_+; X)$. In particular,*

- *$Z \otimes (\mathscr{D}(\mathbf{R}^n) \cap H^p(\mathbf{R}^n))$ and $Z \otimes \hat{\mathscr{D}}_0(\mathbf{R}^n)$ are dense in $H^p(\mathbf{R}^n; X)$,*
- *$Z \otimes (\mathscr{C}_c^\infty(\mathbf{R}_+) \cap H^p(\bar{\mathbf{R}}_+))$ is dense in $H^p(\bar{\mathbf{R}}_+; X)$.*

The proof of this lemma is postponed to Sect. 8.

Recall that $\mathscr{D}(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$ coincides with the set of those $f \in \mathscr{D}(\mathbf{R}^n)$ which have the same vanishing moments as required for an $H^p$-atom, and the same is true with $\mathscr{D}(\mathbf{R}^n)$ replaced by $\mathscr{S}(\mathbf{R}^n)$.

# 3 Necessity of analyticity for $H^1$-regularity

It is a classical result due to P. E. Sobolevskiĭ [19] that if $A$ is a closed, linear, densely defined operator on any Banach space $X$ and the corresponding ACP has maximal $L^p$-regularity, then necessarily $-A$ is the generator of a bounded analytic semigroup. We now show that the same necessary condition remains to hold for $(H^1, L^1)$-regularity:

**Theorem 3.1.** *Let $X$ be a complex Banach space and $A$ a densely defined, closed, linear operator in $X$. If the ACP has $(H^1, L^1)$-regularity, then $-A$ generates a bounded analytic semigroup.*

*If the solution $u$ of the ACP even satisfies $\|u\|_{L^1(\bar{\mathbf{R}}_+;X)} \leq C \|f\|_{H^1(\bar{\mathbf{R}}_+;X)}$, then $A$ is boundedly invertible.*

The proof is a modification of that by Sobolevskiĭ for the $L^p$ case, which is also found in [9]. The only difference is that instead of the auxiliary functions $f_\lambda(t) = e^{\lambda t}\chi_{[0,1/\operatorname{Re}\lambda]}(t)$ used there, we will need [in order to ensure membership in $H^1$] the slightly more complicated expression

$$f_\lambda(t) := (A_\lambda e^{\lambda t} + B_\lambda)\chi_{[0,1/\operatorname{Re}\lambda]}(t) \qquad \text{for } \operatorname{Re}\lambda > 0, \tag{3.2}$$

where the constant $A_\lambda$ and $B_\lambda$ are chosen to satisfy the following conditions:

$$0 \equiv \int_0^\infty f_\lambda(t)\,\mathrm{d}t = A\frac{e^{\lambda/\operatorname{Re}\lambda} - 1}{\lambda} + B\frac{1}{\operatorname{Re}\lambda} = \frac{1}{\operatorname{Re}\lambda}\left(A\frac{e^{1+\mathbf{i}\theta} - 1}{1+\mathbf{i}\theta} + B\right), \tag{3.3}$$

where $\theta := \operatorname{Im}\lambda/\operatorname{Re}\lambda$, which is the requirement $f_\lambda \in H^1$, and

$$\frac{1}{\operatorname{Re}\lambda} \equiv \int_0^\infty e^{-\lambda t} f_\lambda(t)\,\mathrm{d}t = \frac{1}{\operatorname{Re}\lambda}\left(A + B\frac{1 - e^{-1-\mathbf{i}\theta}}{1+\mathbf{i}\theta}\right), \tag{3.4}$$

which is simply a normalization.

It is standard to show the following lemma which ensures that we can choose $A_\lambda$ and $B_\lambda$ with the desired properties in a uniform manner:

**Lemma 3.5.** *For every $\theta \in \mathbf{R}$, the pair of equations* $\begin{cases} A\frac{e^{1+\mathrm{i}\theta}-1}{1+\mathrm{i}\theta} + B & = 0 \\ A + B\frac{1-e^{-1-\mathrm{i}\theta}}{1+\mathrm{i}\theta} & = 1 \end{cases}$ *has a unique solution $(A(\theta), B(\theta))$, and $|A(\theta)| + |B(\theta)| \leq C$ for some constant $C$ independent of $\theta$.*

Now we are ready to prove the semigroup generation. Recall (cf. e.g. [11], Sect. II.4.a, where the terminology is slightly different though) that the condition that $-A$ generates a bounded analytic semigroup is equivalent to saying that $A$ is sectorial of angle $\omega < \pi/2$. We recall the definition of sectoriality:

**Definition 3.6.** We say that the linear operator $A$, with dense domain $\mathcal{D}(A)$ in $X$, is *sectorial* of angle $\omega \in ]0, \pi[$ if

- the spectrum of $A$ satisfies $\sigma(A) \subset \bar{\Sigma}_\omega$, where $\Sigma_\omega := \{\zeta \in \mathbf{C} \setminus \{0\} : |\arg(\zeta)| < \omega\}$, and
- for all $\theta \in ]\omega, \pi[$ there exists a $C_\theta < \infty$ such that $\left\|\zeta(\zeta - A)^{-1}\right\|_{\mathscr{L}(X)} \leq C_\theta$ for all $\zeta \notin \bar{\Sigma}_\theta$.

For later use, note that the above estimate holds with $\zeta \in \mathbf{iR} \setminus \{0\}$ when $\omega < \pi/2$, and then also the similar estimate with $A(\zeta - A)^{-1}$ in place of $\zeta(\zeta - A)^{-1}$, since their difference is just the identity. Then finally to the proof:

*Proof of Theorem 3.1.* Let $\operatorname{Re}\lambda > 0$. As we saw above, we can define $f_\lambda$ by (3.2) so that $|A_\lambda| + |B_\lambda| \leq C$ (independent of $\lambda$) and (3.3), (3.4) hold. Then $f_\lambda \in H^1(\bar{\mathbf{R}}_+)$, and more precisely,

$$\|f_\lambda\|_{H^1} \leq \|f_\lambda\|_{L^\infty} |\operatorname{supp} f_\lambda| \leq (|A_\lambda| \cdot e + |B_\lambda|) \frac{1}{\operatorname{Re}\lambda} \leq \frac{C}{\operatorname{Re}\lambda}.$$

Then for every $x \in X$, we have $f_\lambda(\cdot)x \in H^1(\bar{\mathbf{R}}_+; X)$, and there is a unique $u =: \mathcal{U}(f_\lambda x)$ with the properties listed in the assumptions.

Let us then define

$$R_\lambda x := \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t} \mathcal{U}(f_\lambda x)(t)\,\mathrm{d}t = \frac{\operatorname{Re}\lambda}{\lambda} \int_0^\infty e^{-\lambda t} \dot{\mathcal{U}}(f_\lambda x)(t)\,\mathrm{d}t. \tag{3.7}$$

With the help of $(\lambda + A)u = \lambda u + f - \dot{u}$ and (3.4), we find that

$$(\lambda + A)R_\lambda x = \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t} f_\lambda(t)\,\mathrm{d}t\, x = x,$$

so $R_\lambda$ is a right inverse of $\lambda + A$. That it is also a left inverse depends on the equality $A\mathcal{U}(f_\lambda x) = \mathcal{U}(f_\lambda A x)$ for $x \in \mathcal{D}(A)$, which can be proved in a similar way as in [10] in the $L^p$ case. (Cf. [14] for the details of the $H^1$ situation.)

Thus $\mathbf{C}_+ \subset \rho(-A)$ [the resolvent set of $-A$], and from (3.7) we have

$$\left|(\lambda + A)^{-1}x\right|_X \leq \frac{\operatorname{Re}\lambda}{|\lambda|} \left\|\dot{\mathcal{U}}(f_\lambda x)\right\|_{L^1(\bar{\mathbf{R}}_+;X)} \leq \frac{\operatorname{Re}\lambda}{|\lambda|} C \left\|f_\lambda x\right\|_{H^1(\bar{\mathbf{R}}_+;X)} \leq \frac{\tilde{C}}{|\lambda|} |x|_X .$$

The estimate $\left\|\lambda(\lambda + A)^{-1}\right\|_{\mathscr{L}(X)} \leq C$, which is uniform in the sector of angle $\pi/2$, continues to hold in a slightly larger sector by a standard power series argument, possibly adjusting the constant. Thus $-A$ is the generator of a bounded analytic semigroup.

Under the extra assumption on the regularity of $u$, we also have from the first form of $R_\lambda x$ in (3.7) that

$$\left|(\lambda + A)^{-1}x\right|_X \leq \operatorname{Re}\lambda \left\|\mathcal{U}(f_\lambda x)\right\|_{L^1(\bar{\mathbf{R}}_+;X)} \leq \operatorname{Re}\lambda \cdot C \left\|f_\lambda x\right\|_{H^1(\bar{\mathbf{R}}_+;X)} \leq \tilde{C} |x|_X ,$$

and the bounded invertibility of $A$ follows from the uniformity of this inequality as $\lambda \downarrow 0$. □

# 4 Necessity of $R$-boundedness for $H^1$-multipliers

In this section we show the necessity of $R$-boundedness for an operator-valued Fourier-multiplier from $H^1(\mathbf{R}^n; X)$ to $H^1(\mathbf{R}^n; Y)$, and in fact, even for a multiplier from $H^1(\mathbf{R}^n; X)$ to $L^1(\mathbf{R}^n; Y)$. When applied to the ACP, this result immediately yields the necessity of $R$-boundedness for the maximal $H^1$-regularity of the ACP. We recall that the corresponding $L^p$ result, namely that $m \in L^1_{\text{loc}}(\mathbf{R}^n; \mathscr{L}(X,Y))$ induces a bounded multiplier operator $T_m f := \mathscr{F}^{-1}[m\hat{f}]$ from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$ for some $p \in [1, \infty[$ only if $\{m(y) |\ y$ strong Lebesgue point of $m\}$ is $R$-bounded, is due to Ph. Clément and J. Prüss [7].

**Theorem 4.1.** *Suppose $m \in L^1_{\text{loc}}(\mathbf{R}^n; \mathscr{L}(X,Y))$ is such that the multiplier operator $T_m f := \mathscr{F}^{-1}[m\hat{f}]$ acts boundedly from $H^1(\mathbf{R}^n; X)$ to $L^1(\mathbf{R}^n; Y)$.*

*Then $m$ is strongly continuous away from the origin and moreover*

$$\mathscr{R}\left(\{m(y) |\ y \neq 0\}\right) \leq C_n \left\|T_m\right\|_{\mathscr{L}(H^1(\mathbf{R}^n;X);L^1(\mathbf{R}^n;Y))} ,$$

*where the constant $C_n$ depends only on the dimension $n$. In particular, $m \in L^\infty(\mathbf{R}^n; \mathscr{L}(X,Y))$.*

We need two lemmata. First, we give a tool for estimating the $H^1$-norms we will encounter. Let $B_r$ be the ball in $\mathbf{R}^n$ of radius $r$ centered at the origin and $A_{r,R} := B_R \setminus B_r$ the annulus with inner and outer radii $r$ and $R$, respectively.

**Lemma 4.2.** *Let $\varphi \in \mathscr{S}(\mathbf{R}^n; X)$ satisfy $\int \varphi(x)\,\mathrm{d}x = 0$. Then $\varphi \in H^1(\mathbf{R}^n; X)$, and the norm is estimated by*

$$\left\|\varphi\right\|_{H^1(\mathbf{R}^n;X)} \leq \sum_{k=1}^{\infty} |B_k|^{\frac{1}{2}} \left\|\varphi\chi_{A_{k-1,k}}\right\|_{L^2(\mathbf{R}^n;X)} + (1 + 2^{\frac{n}{2}}) \sum_{k=1}^{\infty} \left\|\varphi\chi_{B_k^c}\right\|_{L^1(\mathbf{R}^n;X)}$$

It is easy to see that the sum is indeed finite for a rapidly decreasing $\varphi$.

*Proof.* Let us denote

$$\varphi_k := \left(\varphi - \frac{1}{|B_k|}\int_{B_k}\varphi(y)\,\mathrm{d}y\right)\chi_{B_k} = \left(\varphi + \frac{1}{|B_k|}\int_{B_k^c}\varphi(y)\,\mathrm{d}y\right)\chi_{B_k},$$

where it is clear from the first form that $\int \varphi_k(x)\,\mathrm{d}x = 0$, and the latter equality follows from the assumption that the total integral of $\varphi$ vanishes. Then

$$\begin{aligned}|\varphi(x) - \varphi_k(x)|_X &\leq |\varphi(x)|_X\,\chi_{B_k^c}(x) + \frac{1}{|B_k|}\int_{B_k^c}|\varphi(y)|_X\,\mathrm{d}y \\ &\leq \max_{|y|\geq k}|\varphi(y)|_X + \frac{1}{|B_k|}\int_{B_k^c}|\varphi(y)|_X\,\mathrm{d}y \underset{k\to\infty}{\longrightarrow} 0;\end{aligned}$$

thus $\varphi_k \to \varphi$ uniformly as $k \to \infty$.

We then define $\phi_1 := \varphi_1$ and $\phi_k := \varphi_k - \varphi_{k-1}$ for $k > 1$ so that $\sum_{k=1}^N \phi_k = \varphi_N \to \varphi$ uniformly as $N \to \infty$. Thus we have $\varphi = \sum_{k=1}^\infty \phi_k$, where $\operatorname{supp}\phi_k \subset B_k$ and $\int \phi_k(x)\,\mathrm{d}x = 0$. This is hence an atomic decomposition of $\varphi$, and we have

$$\|\varphi\|_{H^1(\mathbf{R}^n;X)} \leq \sum_{k=1}^{\infty} |B_k|^{\frac{1}{2}}\,\|\phi_k\|_{L^2(\mathbf{R}^n;X)}.$$

Hence it remains to estimate the $L^2$-norm of

$$\phi_k = \varphi\chi_{A_{k-1,k}} + \frac{\chi_{B_k}}{|B_k|}\int_{B_k^c}\varphi(y)\,\mathrm{d}y - \frac{\chi_{B_{k-1}}}{|B_{k-1}|}\int_{B_{k-1}^c}\varphi(y)\,\mathrm{d}y,$$

where the last term is interpreted as 0 for $k = 1$, and this yields

$$\begin{aligned}\|\phi_k\|_{L^2(\mathbf{R}^n;X)} \leq \left\|\varphi\chi_{A_{k-1,k}}\right\|_{L^2(\mathbf{R}^n;X)} &+ \frac{1}{|B_k|^{\frac{1}{2}}}\left\|\varphi\chi_{B_k^c}\right\|_{L^1(\mathbf{R}^n;X)} \\ &+ \frac{1}{|B_{k-1}|^{\frac{1}{2}}}\left\|\varphi\chi_{B_{k-1}^c}\right\|_{L^1(\mathbf{R}^n;X)}.\end{aligned}$$

Multiplying by $|B_k|^{\frac{1}{2}}$, observing that $|B_k|^{\frac{1}{2}} / |B_{k-1}|^{\frac{1}{2}} = (k/(k-1))^{\frac{n}{2}} \leq 2^{\frac{n}{2}}$ and summing over $k$ we obtain the asserted estimate. □

The following simple result handles the easy part of the main theorem.

**Lemma 4.3.** *If $m \in L^1_{\mathrm{loc}}(\mathbf{R}^n;\mathscr{L}(X,Y))$ defines a bounded multiplier operator $T_m f := \mathscr{F}^{-1}[m\hat{f}]$, which maps $H^1(\mathbf{R}^n;X)$ boundedly into $L^1(\mathbf{R}^n;Y)$, then $m$ is strongly continuous at every $y \neq 0$. In particular, every $y \neq 0$ is a strong Lebesgue point of $m$.*

*Proof.* Let $y_0 \neq 0$. Then there exists a test function $\hat{\varphi} \in \mathscr{D}(\mathbf{R})$, which is supported away from the origin and equals unity in a neighbourhood of $y_0$. Then for $x \in X$ we have $\varphi(\cdot)x \in \mathscr{S}(\mathbf{R}^n; X)$ and $\int \varphi(y)x \, \mathrm{d}y = \hat{\varphi}(0)x = 0$. Hence $\varphi(\cdot)x \in H^1(\mathbf{R}; X)$, and thus $T_m[\varphi(\cdot)x] \in L^1(\mathbf{R}; Y)$. The Fourier transform of this latter function is $m(y)\hat{\varphi}(y)x$, and in a neighbourhood of $y_0$, this is just $m(y)x$. But the Fourier transform of an $L^1$-function is continuous, thus $y \mapsto m(y)x$ is continuous in a neighbourhood of $y_0$, and this being true for every $x \in X$ the assertion is established. □

*Proof of Theorem 4.1.* Let $N \in \mathbf{Z}_+$ and $x_1, \ldots, x_N \in X$, and let first

$$y_1, \ldots, y_N \in \{y = (y^1, \ldots, y^n) \in \mathbf{R}^n |\ y^n \geq 0,\ y \neq 0\},$$

i.e., the points are taken from the closed upper half-space, excluding the origin. Let us choose a (real-valued) test-function $\psi \in \mathscr{D}(\mathbf{R}^n)$ with support strictly contained in the lower half-space $\{y \in \mathbf{R}^n |\ y^n < 0\}$ and such that

$$\int_{\mathbf{R}^n} \psi^2(y) \, \mathrm{d}y = 1.$$

This function will be exploited in building an appropriate approximation of the identity; the reason for the support condition will become clear later. Since $y_j$ is a Lebesgue point of $y \mapsto m(y)x_j$ by Lemma 4.3, we have

$$m(y_j)x_j = \lim_{k\to\infty} \int_{\mathbf{R}^n} m(y)x_j \psi^2(k(y_j - y))k^n \, \mathrm{d}y,$$

the convergence being in the norm of $Y$. Thus

$$\mathbf{E}\Big| \sum_{j=1}^N \varepsilon_j m(y_j)x_j \Big|_Y = \lim_{k\to\infty} k^n \mathbf{E}\Big| \int_{\mathbf{R}^n} \sum_{j=1}^N \varepsilon_j m(y)\psi(k(y_j - y))x_j \psi(k(y_j - y)) \, \mathrm{d}y \Big|_Y.$$

We then write

$$\begin{aligned} m(y)\psi(k(y_j - y))x_j &= m(y)\mathscr{F}\mathscr{F}^{-1}[\psi(k(y_j - \cdot))x_j](y) \\ &= m(y)\mathscr{F}[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n = \mathscr{F}T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n, \end{aligned}$$

and using the duality equality $\int \hat{g} f \, \mathrm{d}y = \int g \hat{f} \, \mathrm{d}y$ of the Fourier transform we can further equate $\mathbf{E}\left| \sum_{j=1}^N \varepsilon_j m(y_j)x_j \right|_Y$ with

$$\begin{aligned} &\lim_{k\to\infty} k^{-n} \mathbf{E}\Big| \int_{\mathbf{R}^n} \sum_{j=1}^N \varepsilon_j T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y) e^{-\mathbf{i}2\pi y_j\cdot y}\hat{\psi}(-y/k) \, \mathrm{d}y \Big|_Y \\ &\quad \leq \liminf_{k\to\infty} k^{-n} \|\hat{\psi}\|_{L^\infty} \mathbf{E} \int_{\mathbf{R}^n} \Big| \sum_{j=1}^N \varepsilon_j e^{-\mathbf{i}2\pi y_j\cdot y} T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y) \Big|_Y \, \mathrm{d}y. \end{aligned}$$

We now invoke the contraction principle to get rid of the exponential factors $e^{-\mathbf{i}2\pi y_j\cdot y}$ and then the assumed boundedness of the operator $T_m$ to yield

$$\begin{aligned}\le 2\|\hat{\psi}\|_{L^\infty}\,\|T_m\|_{\mathscr{L}(H^1,L^1)}\liminf_{k\to\infty}k^{-n}\mathbf{E}\Big\|\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j\Big\|_{H^1(\mathbf{R}^n;X)}\\ =2\|\hat{\psi}\|_{L^\infty}\,\|T_m\|_{\mathscr{L}(H^1,L^1)}\liminf_{k\to\infty}\mathbf{E}\Big\|\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\Big\|_{H^1(\mathbf{R}^n;X)},\end{aligned}\tag{4.4}$$

where the last equality follows from the dilation property of the $H^1$-norm.

So far the proof has been completely parallel to that in [7] concerning the $L^p$ situation, except for the choice of our auxiliary function $\psi$, but now we are faced with the $H^1$-norm, with which the contraction principle can no longer be applied. Instead, we invoke Lemma 4.2 for the evaluation of this norm. Let us first check that the assumptions of the lemma are satisfied by

$$\varphi(y):=\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)x_j:$$

Certainly $\hat{\psi}\in\mathscr{S}(\mathbf{R}^n)$ since $\psi\in\mathscr{D}(\mathbf{R}^n)$, and since the exponential factors are $\mathscr{C}^\infty$ with bounded derivatives of all orders, the entire function $\varphi$ belongs to $\mathscr{S}(\mathbf{R}^n;X)$. Moreover, recognizing the formula of the inverse Fourier transform, we have

$$\int_{\mathbf{R}^n}e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)\,\mathrm{d}y=\psi(ky_j)=0,$$

since $k>0$ and $y_j$ is in the upper half-space, whereas $\psi$ is supported in the lower half-space.

Hence we get, for the $H^1$-norm appearing in (4.4), the estimate

$$\begin{aligned}&\mathbf{E}\Big\|\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\Big\|_{H^1(\mathbf{R}^n;X)}\\&\qquad\le\sum_{\ell=1}^\infty|B_\ell|^{\frac12}\,\mathbf{E}\Big\|\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\chi_{A_{\ell-1,\ell}}(\cdot)x_j\Big\|_{L^2(\mathbf{R}^n;X)}\\&\qquad\qquad+(1+2^{\frac{n}{2}})\sum_{\ell=1}^\infty\mathbf{E}\Big\|\sum_{j=1}^N\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\chi_{B_\ell^c}(\cdot)x_j\Big\|_{L^1(\mathbf{R}^n;X)}.\end{aligned}$$

Now we are back to $L^p$-norms, and the contraction principle applies again:

$$\leq 2\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\Big(\mathbf{E}\Big\|\sum_{j=1}^{N}\varepsilon_j\hat{\psi}\chi_{A_{\ell-1,\ell}}(\cdot)x_j\Big\|^2_{L^2(\mathbf{R}^n;X)}\Big)^{\frac{1}{2}} + (1+2^{\frac{n}{2}})2\sum_{\ell=1}^{\infty}\mathbf{E}\Big\|\sum_{j=1}^{N}\varepsilon_j\hat{\psi}\chi_{B_\ell^c}(\cdot)x_j\Big\|_{L^1(\mathbf{R}^n;X)}$$

$$= 2\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\left\|\hat{\psi}\chi_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbf{R}^n)}\Big(\mathbf{E}\Big|\sum_{j=1}^{N}\varepsilon_j x_j\Big|^2_X\Big)^{\frac{1}{2}} + (1+2^{\frac{n}{2}})2\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\chi_{B_\ell^c}\right\|_{L^1(\mathbf{R}^n)}\mathbf{E}\Big|\sum_{j=1}^{N}\varepsilon_j x_j\Big|_X.$$

Finally, combining (4.4) with the estimate above and applying Kahane's inequality $\sqrt{\mathbf{E}\left|\sum\varepsilon_j x_j\right|_X^2} \leq \sqrt{2}\mathbf{E}\left|\sum\varepsilon_j x_j\right|_X$, we get

$$\mathbf{E}\Big|\sum_{j=1}^{N}\varepsilon_j m(y_j)x_j\Big|_Y \leq C_n\left\|T_m\right\|_{\mathscr{L}(H^1(\mathbf{R}^n;X);L^1(\mathbf{R}^n;Y))}\mathbf{E}\Big|\sum_{j=1}^{N}\varepsilon_j x_j\Big|_X,$$

where the constant

$$C_n = 4\|\hat{\psi}\|_{L^\infty(\mathbf{R}^n)}\left(\sqrt{2}\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\left\|\hat{\psi}\chi_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbf{R}^n)} + (1+2^{\frac{n}{2}})\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\chi_{B_\ell^c}\right\|_{L^1(\mathbf{R}^n)}\right) < \infty$$

depends only on the dimension $n$ and the choice of the auxiliary function $\psi$, thus fixing one $\psi$ once and for all, only on the dimension $n$.

It is clear that we can repeat the same argument for points $y_1,\ldots,y_N$ in the lower half-space, exploiting another auxiliary function $\tilde{\psi}\in\mathscr{D}(\mathbf{R}^n)$ supported in the upper half-space (e.g., the reflection of $\psi$ about the hyperplane $\{y\in\mathbf{R}^n|\ y^n=0\}$). Thus we get the $R$-boundedness of $\{m(y)|\ y\neq 0\}$ with an $R$-bound of the asserted form. □

# 5 Sufficient conditions for $H^p$-multipliers

Turning to sufficient conditions for the continuity of our operators, we first investigate the case where we have the *a priori* boundedness from $L^{\tilde{p}}(\mathbf{R}^n;X)$ to $L^{\tilde{p}}(\mathbf{R}^n;Y)$ for some $\tilde{p}\in\,]1,\infty[$. In this situation, it turns out that conditions introduced by Strömberg and Torchinsky ([22], p. 151) for checking the $H^p$-boundedness in the scalar case have immediate generalizations to the vector-valued function spaces.

For $H^1(\mathbf{R}^n;X)$-to-$L^1(\mathbf{R}^n;Y)$ boundedness, nothing more than the vector-valued Hörmander integral condition is needed (cf. [12], V.3.4), but $H^p(\mathbf{R}^n;X)$-to-$H^p(\mathbf{R}^n;Y)$ type boundedness results require somewhat stronger (and more technical) assumptions:

**Definition 5.1.** We say that a function $k$, with values in $\mathscr{L}(X,Y)$, belongs to the class $K(q,\ell;X,Y)$ [or just $K(q,\ell;X)$ if $Y=X$], where $1\le q<\infty$ and $\ell>0$, provided that $k\in\mathscr{C}^{\llbracket\ell\rrbracket}(\mathbf{R}^n\setminus\{0\};\mathscr{L}(X,Y))$ and satisfies

$$\left(\frac{1}{r^n}\int_{r<|t|<2r}|D^\alpha k(t)x|_Y^q\,\mathrm{d}t\right)^{\frac{1}{q}}\le A\,r^{-n-|\alpha|}\,|x|_X \tag{5.2}$$

for all $r>0$, $x\in X$, and $\alpha\in\mathbf{N}^n$ with $|\alpha|\le\llbracket\ell\rrbracket$, and moreover, for all $r>0$, $s\in\mathbf{R}^n$ with $|s|<r/2$, $x\in X$, and $\alpha\in\mathbf{N}^n$ with $|\alpha|=\llbracket\ell\rrbracket$,

$$\left(\frac{1}{r^n}\int_{r<|t|<2r}|(D^\alpha k(t)-D^\alpha k(t-s))x|_Y^q\,\mathrm{d}t\right)^{\frac{1}{q}}\le A\varrho(\frac{|s|}{r};\ell-\llbracket\ell\rrbracket)r^{-n-\llbracket\ell\rrbracket}\,|x|_X\,,$$

where $\varrho(t;u):=t^u$ for $u\in]0,1[$ and $\varrho(t;1):=t\log(t^{-1})$.

The conditions $K(\infty,\ell;X,Y)$ is defined by the usual modification.

*Remark* 5.3. The estimate (5.2) is verified if $\|D^\alpha k(t)\|_{\mathscr{L}(X,Y)}\le A\,|t|^{-n-|\alpha|}$.

The conditions $K(q,\ell;X,Y)$ provide very satisfactory control over the action of the convolution $k*\cdot$ on atoms of Hardy spaces, and thus on general elements of $H^p(\mathbf{R}^n;X)$. In fact, the scalar result in [22] is "generalized" to the vector-valued case by a repetition of the same argument:

**Theorem 5.4 (Strömberg, Torchinsky 1989).** *Let $k\in K(q,\ell;X,Y)$, where $q\in]1,\infty[$, and suppose that the operator of convolution by $k$ maps*

$$f\in L^q(\mathbf{R}^n;X)\mapsto k*f\in L^q(\mathbf{R}^n;Y)\qquad\textit{boundedly.}$$

*Then also*

$$f\in H^p(\mathbf{R}^n;X)\mapsto k*f\in H^p(\mathbf{R}^n;Y)\qquad\textit{boundedly for all } p\in\left]\frac{1}{1+\ell/n},1\right].$$

We next state related conditions for the multiplier $m$; these are due to Kurtz and Wheeden [17] for integral $\ell$, whereas the general definition again comes from [22]:

**Definition 5.5.** We say that a function $m\in L^\infty(\mathbf{R}^n;\mathscr{L}(X,Y))$ belongs to the class $M(q,\ell;X,Y)$ [or just $M(q,\ell;X)$ if $Y=X$] provided that $m\in\mathscr{C}^{\lfloor\ell\rfloor}(\mathbf{R}^n\setminus\{0\};\mathscr{L}(X,Y))$ and satisfies

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r}|D^\alpha m(\xi)x|_Y^q\,\mathrm{d}\xi\right)^{\frac{1}{q}}\le A\,r^{-|\alpha|}\,|x|_X \tag{5.6}$$

for all $r > 0$, $\alpha \in \mathbf{N}^n$ with $|\alpha| \le \lfloor \ell \rfloor$ and $x \in X$, and moreover, if $\ell \notin \mathbf{Z}_+$,

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} |(D^\alpha m(\xi) - D^\alpha m(\xi-\zeta))x|_Y^q \,\mathrm{d}\xi\right)^{\frac{1}{q}} \le A\left(\frac{|\zeta|}{r}\right)^{\ell-\lfloor\ell\rfloor} r^{-|\alpha|}\,|x|_X$$

for all $r > 0$, $\zeta \in \mathbf{R}^n$ with $|\zeta| < r/2$, $\alpha \in \mathbf{N}^n$ with $|\alpha| = \lfloor \ell \rfloor$ and $x \in X$.

*Remark* 5.7. The estimate (5.6) holds if $\|D^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} \le A\,|\xi|^{-|\alpha|}$.

The condition $m \in M(q, \ell; X, Y)$ implies (in a sense elaborated in Lemma 5.8) that $k := \check{m} \in K(\tilde{q}, \tilde{\ell}; X, Y)$ for certain $\tilde{q}$ and $\tilde{\ell}$. The proof again copies the scalar argument, but we need to assume the appropriate Fourier-type in order to apply the Hausdorff–Young inequality.

The statement of the lemma involves the following partition of unity: Let $\eta \in \mathscr{D}(\mathbf{R}^n)$ be non-negative, equal to unity in $\bar{B}(0,1)$ and supported in $\bar{B}(0,2)$. Let $\phi(\xi) := \eta(\xi) - \eta(2\xi)$. Then $\phi(2^{-i}\cdot)$ is supported in the annulus $2^{i-1} \le |x| \le 2^{i+1}$, and $\eta(\xi) + \sum_{i=1}^{\infty} \phi(2^{-i}\xi) = 1$ for all $\xi \in \mathbf{R}^n$.

**Lemma 5.8 (Strömberg, Torchinsky 1989).** *Let $m \in L^\infty(\mathbf{R}^n; \mathscr{L}(X,Y))$ satisfy $M(q, \ell; X, Y)$, where $Y$ has Fourier-type $q \in [1,2]$. If we define*

$$m_0(\xi) := \eta(\xi)m(\xi), \quad m_i(\xi) := \phi(2^{-i}\xi)m(\xi), \text{ for } i \in \mathbf{Z}_+, \quad \text{and} \quad k_i := \check{m}_i,$$

*then the kernels $k^N := \sum_{i=0}^{N} k_i$ satisfy the condition $K(q', \ell - n/q; X, Y)$ uniformly in $N$.*

Now the following multiplier theorem is a corollary of Theorem 5.4.

**Theorem 5.9 (Strömberg, Torchinsky 1989).** *Let $m \in L^\infty(\mathbf{R}^n; \mathscr{L}(X,Y))$, and let the corresponding multiplier operator $T$ be bounded from $L^{\tilde{p}}(\mathbf{R}^n; X)$ to $L^{\tilde{p}}(\mathbf{R}^n; Y)$ for some $\tilde{p} \in \,]1,\infty[$. Suppose further that $m \in M(q, \ell; X, Y)$ for some $q$ such that*

$$Y \text{ has Fourier-type } q, \qquad 1 \le q \le \tilde{p}' \qquad \text{and} \qquad \ell > n/q.$$

*Then $T$ extends boundedly to*

$$f \in H^p(\mathbf{R}^n; X) \mapsto Tf \in H^p(\mathbf{R}^n; Y) \qquad \text{for all } p \in \left]\frac{1}{1/q' + \ell/n}, 1\right].$$

*Remark* 5.10. Observe that $1/q' + \ell/n > 1/q' + 1/q = 1$ under the assumptions, so that the asserted range of $p$ is non-empty. Also note that only the Fourier-type of the image space $Y$ is relevant, and moreover the theorem always contains the case $q = 1$, without any geometric conditions on the Banach spaces in question.

All the results stated so far require the *a priori* knowledge of the boundedness of our operator on some $L^{\tilde{p}}(\mathbf{R}^n; X)$. When $X$ is a UMD-space, there exist various sets of conditions for checking this boundedness, e.g. in [8, 13, 15, 21].

We state the following result which can be obtained as a corollary of Theorem 5.9 and any of the multiplier theorems from the above-cited papers. The assumptions are actually unnecessarily strong (in particular, it would suffice to assume $R$-boundedness only for a finite number of the $\alpha$'s and uniform boundedness for the rest); nevertheless, they are satisfied by the multipliers occurring in our application to maximal regularity.

**Corollary 5.11.** *Let $X$ and $Y$ be UMD-spaces and $m \in \mathscr{C}^\infty(\mathbf{R}^n \setminus \{0\}; \mathscr{L}(X,Y))$ be a multiplier such that $\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \neq 0\}$ is $R$-bounded for every $\alpha \in \mathbf{N}^n$. Then $f \in X \otimes \hat{\mathscr{D}}_0(\mathbf{R}^n) \mapsto \mathscr{F}^{-1}[m\hat{f}]$ extends to a bounded mapping from $L^p(\mathbf{R}^n;X)$ to $L^p(\mathbf{R}^n;Y)$ for all $p \in ]1,\infty[$ and from $H^p(\mathbf{R}^n;X)$ to $H^p(\mathbf{R}^n;Y)$ for all $p \in ]0,1]$.*

We conclude this section with a bounded extension result of a somewhat different kind:

**Lemma 5.12.** *Let $X$, $Y$ be UMD-spaces, and $T_m : f \mapsto \mathscr{F}^{-1}[m\hat{f}]$ be bounded from $H^1(\mathbf{R};X)$ to $L^1(\mathbf{R}^n;Y)$. Then $T_m$ is also bounded from $H^1(\mathbf{R};X)$ to $H^1(\mathbf{R};Y)$.*

*Proof.* The proof depends on the fact (cf. [3]) that for a UMD-space $X$, we have $H^1(\mathbf{R};X) = \{f \in L^1(\mathbf{R};X) : \mathscr{H}f \in L^1(\mathbf{R};X)\}$, and $\|f\|_{L^1(\mathbf{R};X)} + \|\mathscr{H}f\|_{L^1(\mathbf{R};X)}$ gives an equivalent norm, which we denote by $\|f\|_{H^1(\mathbf{R};X)}$ through this proof.

With $C := \|T_m\|_{\mathscr{L}(H^1(\mathbf{R};X),L^1(\mathbf{R};Y))} < \infty$, we can estimate $\|T_m f\|_{L^1(\mathbf{R};Y)} \leq C\|f\|_{H^1(\mathbf{R};X)}$ by assumption, and

$$\|\mathscr{H}T_m f\|_{L^1(\mathbf{R};Y)} = \|T_m \mathscr{H} f\|_{L^1(\mathbf{R};Y)} \leq C\|\mathscr{H}f\|_{H^1(\mathbf{R};X)} = C\|f\|_{H^1(\mathbf{R};X)},$$

where the commutativity of the multiplier operators $\mathscr{H}$ and $T_m$ is clear when investigated in terms of the Fourier transforms, and we used the fact that

$$\|\mathscr{H}f\|_{H^1(\mathbf{R};X)} = \|\mathscr{H}f\|_{L^1(\mathbf{R};X)} + \|-f\|_{L^1(\mathbf{R};X)} = \|f\|_{H^1(\mathbf{R};X)},$$

since $\mathscr{H}^2 = -1$, which is also clear from the Fourier transforms. □

# 6 Maximal $H^p$-regularity of the ACP

We have now developed the necessary tools to attack the maximal regularity problem for the ACP discussed in the Introduction. Everything will be clear once we check that the conditions $K(q,\ell;X)$ and $M(q,\ell;X)$ are verified by the convolution kernel and the multiplier, respectively, related to the ACP. In fact, it would suffice to consider just one of them, but we give both the short proofs for purposes of illustration. We begin with the following lemma:

**Lemma 6.1.** *Let $-A$ be the generator of a bounded analytic semigroup $(T^t)$, and*

$$k(t) := AT^t \chi_{\mathbf{R}_+}(t), \qquad m(\xi) := A(\mathbf{i}2\pi\xi + A)^{-1}, \tag{6.2}$$

*for $t \in \mathbf{R}$, $\xi \in \mathbf{R} \setminus \{0\}$. Then $k \in K(q,\ell;X)$ and $m \in M(q,\ell;X)$ for any $q \in [1,\infty]$, $\ell > 0$.*

*If, moreover, $\{m(\xi)|\ \xi \neq 0\}$ is $R$-bounded, then also $\{\xi^\nu D^\nu m(\xi)|\ \xi \neq 0\}$ is $R$-bounded for all $\nu \in \mathbf{N}$.*

*Proof.* By Remarks 5.3 and 5.7, it suffices to check $\|D^\nu k(t)\|_{\mathscr{L}(X)} \leq C_\nu |t|^{-1-\nu}$ and $\|D^\nu m(\xi)\|_{\mathscr{L}(X)} \leq C_\nu |\xi|^{-\nu}$ for all $\nu \in \mathbf{N}$, i.e., we need the estimates

$$\left\|A^{1+\nu}T^t\right\|_{\mathscr{L}(X)} \leq C_\nu |t|^{-1-\nu} \qquad \text{and}$$
$$\nu!(2\pi)^\nu \left\|A(\mathbf{i}2\pi\xi + A)^{-1-\nu}\right\|_{\mathscr{L}(X,Y)} \leq C_\nu |\xi|^{-\nu},$$

but the first estimate is well-known and follows easily from $\|tAT^t\|_{\mathscr{L}(X)} \leq C$ and the semigroup property, whereas for the second we only need recall that $\left\|(\lambda + A)^{-1}\right\|_{\mathscr{L}(X)} \leq C|\lambda|^{-1}$ for all $\lambda$ with $|\arg(\lambda)| < \vartheta$, where $\vartheta > \pi/2$, in paricular, for $\lambda = \mathbf{i}2\pi\xi$.

Finally, the $R$-boundedness of $\xi^\nu D^\nu m(\xi) = \nu!(2\pi\xi)^\nu A(\mathbf{i}2\pi\xi + A)^{-1-\nu}$ follows from the $R$-boundedness of $m(\xi)$ in exactly the same way as the norm boundedness of the derivatives followed from the norm boundedness of $m(\xi)$. □

Now the proof of Theorem 1.4 is a matter of collecting the pieces together.

*Proof of Theorem 1.4.* If the ACP has maximal $L^p$-regularity, $p \in ]1,\infty[$, then by Sobolevskiĭ's classical result, $-A$ generates a bounded analytic semigroup, and by Weis' Theorem 1.3, the collection $\{A(\mathbf{i}2\pi\xi + A)^{-1}|\ \xi \neq 0\}$ is $R$-bounded. Then by Lemma 6.1, the related convolution kernel and multiplier in (6.2) satisfy infinitely many of the conditions required to apply our extension results, and we obtain the boundedness of $f \mapsto k * f$ from Theorem 5.4, or equally well the boundedness of $f \mapsto \mathscr{F}^{-1}[m\hat{f}]$ from either Theorem 5.9 or Corollary 5.11. Thus we have $C_1 \Rightarrow C_2, C_4, C_5$.

This did not require UMD, since the operator extension theorems work for general Banach spaces, as soon as the boundedness on one $L^{\tilde{p}}(\mathbf{R};X)$ is known *a priori*; also the $R$-boundedness can be deduced from the result of Clément and Prüss [7] which holds for general $X$.

Clearly $C_2 \Rightarrow C_3$, but we also have $C_1 \Rightarrow C_3$ directly from the classical result of Benedek, Calderón and Panzone [2]; thus the condition $C_1$ implies all the other conditions. (Still no UMD required!)

If the ACP has maximal $(H^1, L^1)$-regularity, then by Theorem 3.1, $-A$ generates a bounded analytic semigroup and by Theorem 4.1, $\{A(\mathbf{i}2\pi\xi + A)^{-1}|\xi \neq 0\}$ is $R$-bounded. Thus $C_3 \Rightarrow C_4$. Moreover, $C_4 \Rightarrow C_1$ by Weis' Theorem 1.3, and here we need the UMD assumption.

Summarizing, we have $C_2 \Rightarrow C_3 \Rightarrow C_4 \Rightarrow C_1 \Rightarrow C_2, C_3, C_4, C_5$, and this is the theorem. □

*Remark* 6.3. That one can manage by investigating only the kernel $k(t) = AT^t\chi_{\mathbf{R}_+}(t)$ is worth emphasizing, since this means that the technical Lemma 5.8 from [22] can be avoided, as long as only the Cauchy problem is concerned.

If one is only interested in $H^1$ and not in $p < 1$, then probably the easiest argument runs as follows: $C_1 \Rightarrow C_3$ by the classical theorem of Benedek *et al.* [2]. $C_3 \Rightarrow C_2$ by Lemma 5.12, and the converse is trivial. The implications $C_3 \Rightarrow C_4 \Rightarrow C_1$ are proved as in the proof above. (This simplification for this particular case was pointed out to the author by L. Weis.)

# 7 Regularity results for other equations

The applicability of the multiplier method is by no means restricted to the treatment of the ACP. We illustrate this fact by two other model problems: the fractional order equation

$$D^\alpha u(t) + Au(t) = f(t) \text{ for } t \geq 0, \qquad u(0) = 0, \qquad (\dot{u}(0) = 0 \text{ if } \alpha > 1), \quad (7.1)$$

where $0 < \alpha < 2$, and the abstract Laplace equation

$$-\triangle u(t) + Au(t) = f(t) \text{ for } t \in \mathbf{R}^n. \quad (7.2)$$

(For the notion of the fractional derivative $D^\alpha$, see [24], §12.8, for the classical [scalar-valued] setting, or [5] for the vector-valued context.) We here consider a stronger notion of maximal regularity, in which it is also asked that $u$ itself have the same regularity as required for $Au$.

Since the treatment of these results by the multiplier method is very similar to that of the ACP, we omit most of the details. One can show that the multiplier $m$ related to the map $f \mapsto Au$ is $m(\xi) = A((\mathbf{i}2\pi\xi)^\alpha + A)^{-1}$ (the principal branch of the logarithm is used in defining complex powers) for (7.1) and $m(\xi) = A(4\pi^2 |\xi|^2 + A)^{-1}$ for (7.2). Both of these share with the multiplier of the ACP the property that their ($R$-)boundedness already implies the ($R$-)boundedness of all the functions $|\xi|^\beta D^\beta m(\xi)$. These remarks made, we state the following results:

**Proposition 7.3.** *Let $\alpha \in \,]0,2[$, let $X$ be a UMD-space and $A$ a boundedly invertible sectorial operator of angle $\omega < \pi(1-\alpha/2)$. Then the following are equivalent:*

($F_1$) (7.1) *has strong maximal regularity on $L^p(\mathbf{R}^n;X)$ for $p \in \,]1,\infty[$.*

($F_2$) (7.1) *has strong maximal regularity on $H^1(\mathbf{R}^n;X)$.*

($F_3$) (7.1) *has strong regularity from $H^1(\mathbf{R}^n;X)$ to $L^1(\mathbf{R}^n;X)$.*

($F_4$) $\{A((\mathbf{i}2\pi\xi)^\alpha + A)^{-1} |\ \xi \in \mathbf{R}\}$ *is R-bounded.*

*Moreover, any of the above conditions implies*

($F_5$) *For all $p \in \,]0,1[$, for all $f \in H^p \cap L^q(\bar{\mathbf{R}}_+;X)$ and all $f \in H^p \cap H^1(\bar{\mathbf{R}}_+;X)$, where $q > 1$ is a Fourier-type for $X$, the solution $u$ of* (7.1) *satisfies*

$$\|D^\alpha u\|_{H^p(\bar{\mathbf{R}}_+;X)} + \|u\|_{H^p(\bar{\mathbf{R}}_+;X)} + \|Au\|_{H^p(\bar{\mathbf{R}}_+;X)} \leq C \|f\|_{H^p(\bar{\mathbf{R}}_+;X)}.$$

We note that the $L^p$-theory for this problem has been treated by Clément and Prüss [7], and the equivalence of $(F_1)$ and $(F_4)$ is due to them.

**Proposition 7.4.** *Let $A$ be an invertible sectorial operator on a UMD-space $X$. Then the following are equivalent:*

$(L_1)$ (7.2) *has strong maximal regularity on $L^p(\mathbf{R}^n; X)$ for all $p \in ]1, \infty[$.*

$(L_2)$ (7.2) *has strong maximal regularity on $H^1(\mathbf{R}^n; X)$.*

$(L_3)$ (7.2) *has strong regularity from $H^1(\mathbf{R}^n; X)$ to $L^1(\mathbf{R}^n; X)$.*

$(L_4)$ $\{A(t + A)^{-1} |\ t \geq 0\}$ *is R-bounded.*

*Moreover, these imply*

$(L_5)$ *For all $p \in ]0, 1[$, $q > 1$ a Fourier-type for $X$ and $f \in H^p \cap L^q(\mathbf{R}^n; X)$ or $f \in H^p \cap H^1(\mathbf{R}^n; X)$, the unique solution $u$ of* (7.2)*, together with $\triangle u$ and $Au$, also belong to this same space, and moreover*

$$\|\triangle u\|_{H^p(\mathbf{R}^n;X)} + \|u\|_{H^p(\mathbf{R}^n;X)} + \|Au\|_{H^p(\mathbf{R}^n;X)} \leq C \|f\|_{H^p(\mathbf{R}^n;X)} .$$

# 8 Proof of the Density Lemma 2.4

*Proof.* It follows from the atomic definition of the $H^p$ norm that finite linear combinations of atoms are dense in $H^p(\mathbf{R}^n; X)$. The $H^p$ norm of an atom can be controlled if its $L^q$ norm can be controlled, preserving the moment and support conditions. Simple functions being dense in $L^q(\bar{B}; X)$ for $q < \infty$, given a $(p, q)$-atom $a$ supported on the ball $\bar{B}$, we can find a simple function $s = \sum x_k \chi_{E_k}$ with $E_k \subset \bar{B}$ measurable and $\|s - a\|_{L^q} < \epsilon$. Clearly, if we replace the $x_k$ by $z_k \in Z$ taken sufficiently close to the respective $x_k$, we get a new simple function, still denoted by $s$, which approximates $a$ as closely as desired in the $L^q$ norm and belongs to $Z \otimes L^q(\bar{B})$.

For $g \in L^q(\bar{B}; X)$, let $Pg$ denote the unique polynomial (with $X$-coefficients) of degree at most $N$ and satisfying $\int_{\bar{B}}(g(t) - Pg(t))t^\alpha \, dt = 0$ for $|\alpha| \leq N$. It is easy to see that $P$ is a bounded operator on $L^q(\bar{B}; X)$, and moreover, it maps $Z \otimes L^q(\bar{B})$ to itself.

Now $s - Ps$ is supported on the same ball $\bar{B}$ as $a$, it has the appropriate number of vanishing moments provided $N$ is chosen large enough, and finally

$$\begin{aligned} \|(s - Ps) - a\|_{L^q(\bar{B};X)} &\leq \|s - a\|_{L^q} + \|Ps - Pa\|_{L^q} \\ &\leq (1 + \|P\|_{\mathscr{L}(L^q(\bar{B};X))}) \|s - a\|_{L^q(\bar{B};X)} , \end{aligned}$$

where the first estimate exploits the fact that $Pa = 0$, since $a$ already has the appropriate vanishing moments. Since $s$ can be chosen as close to $a$ as desired, the same will be true of $b := s - Ps$.

Replacing each of the finite number of atoms $a_i$ in the truncated atomic series of a given $f \in H^p(\mathbf{R}^n; X)$ by the corresponding $b_i$ constructed as above, we can estimate $f$ as closely as desired by a finite sum

$$\tilde{f} = \sum_{i=1}^{M} \lambda_i b_i = \sum_{i=1}^{M} \lambda_i \sum_{j=1}^{m_i} z_{i,j} f_{i,j}, \qquad z_{i,j} \in Z,\ f_{i,j} \in L^q_c(\mathbf{R}^n).$$

Since there are only finitely many of the $z_{i,j}$, all of them belong to some finite-dimensional subspace $E$ of $Z$, for which we can find a basis $e_1, \ldots, e_m$. Expressing each $z_{i,j}$ as a linear combination of the $e_k$, the sum above gets the form $\tilde{f} = \sum_{k=1}^{m} e_k f_k$, where the $f_k$ are compactly supported scalar $L^q$ functions. Integrating this equality multiplied by $t^\alpha$ and using the linear independence of the $e_k$, we find that the $f_k$ have (at least) the same vanishing moments as $\tilde{f}$. A compactly supported $L^q$ function with appropriate moment conditions is clearly an atom, up to scaling, thus in particular an element of $H^p(\mathbf{R}^n)$.

Thus, an arbitrary $f \in H^p(\mathbf{R}^n; X)$ has been approximated by $\sum_{k=1}^{m} e_k f_k$, with $e_k \in Z$ and $f_k \in H^p(\mathbf{R}^n)$. It is clear that if the $f_k$ are now replaced by suitable $g_k$ in the dense subspace $G$ of $H^p(\mathbf{R}^n)$, we can retain arbitrarily good approximation, and clearly $\sum_{k=1}^{m} e_k g_k \in Z \otimes G$, as desired.

To see the density of $\mathscr{D}(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$ in $H^p(\mathbf{R}^n)$, or of $\mathscr{C}_c^\infty(\mathbf{R}_+) \cap H^p(\bar{\mathbf{R}}_+)$ in $H^p(\bar{\mathbf{R}}_+)$, it suffices to convolute a truncated atomic series by a smooth, compactly supported approximation of the identity, observing that the amount in which this disturbes the supports of the atoms can be made as small as desired. In the case of a half-line, we can first shift the atoms in the truncated series slightly to the right, noting that this transformation preserves atoms and is continuous in the $L^q$ norm.

The fact that $\hat{\mathscr{D}}_0(\mathbf{R}^n)$ is a dense subspace of $H^p(\mathbf{R}^n)$ can be found in [22]. □

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