REDUCED MIHLIN–LIZORKIN MULTIPLIER THEOREM IN VECTOR-VALUED L^p SPACES

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Dedicated to Professor Philippe Clément on the occasion of his retirement.

1. INTRODUCTION

It was a long-standing problem to generalize S. G. Mihlin's classical theorem on Fourier-multipliers on $L^p(\mathbf{R}^n)$ to the setting of $\mathscr{L}(X)$-valued multiplier functions acting on the Bôchner spaces $L^p(\mathbf{R}^n, X)$, where X is a Banach space not isomorphic to a Hilbert space. A solution was first obtained by L. Weis [11], whose work exploited decisively the notion of R-boundedness, which had been studied in detail by Ph. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet [3]. Once this breakthrough was achieved, there has been some activity towards obtaining sharper, and even optimal (in a certain sense, cf. [4]), smoothness assumptions for operator-valued multipliers on $L^p(\mathbf{R}^n, X)$ [4, 6, 8, 10]. Perhaps surprisingly, the methods developed in this connection have been able to relax the assumptions from what was known before even for the classical multipliers on $L^p(\mathbf{R}^n)$ [6].

While the sharpest known multiplier conditions are rather technical, the essence of the matter (i.e., the order of required smoothness as a function of the Fourier-type of the underlying Banach spaces) is contained in the following statement: (Recall that every UMD space indeed has some Fourier-type $t \in \,]1, 2]$.)

1.1. **Theorem** ([6]). *Let X and Y by UMD spaces with Fourier-type $t \in \,]1, 2]$. Let the function $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X, Y))$ satisfy the R-boundedness condition*

$$M_\alpha := \mathscr{R}[|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbf{R}^n \setminus \{0\}] < \infty \qquad \text{for all} \quad \alpha \in \{0, 1\}^n, \ \text{s.t.} \ |\alpha| \leq \lfloor n/t \rfloor + 1.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier multiplier for $1 < p < \infty$.

Here $\mathscr{R}(\mathscr{T})$ designates the R-bound of a set $\mathscr{T} \subset \mathscr{L}(X, Y)$; see [3, 11] for more on this notion.

The same conclusion, but assuming $M_\alpha < \infty$ for all $\alpha \in \{0, 1\}^n$ had been obtained earlier by F. Zimmermann [12] for scalar-valued and by Ž. Štrkalj and L. Weis [10] for operator-multipliers, and assuming $M_\alpha < \infty$ for all $|\alpha| \leq \lfloor n/t \rfloor + 1$ by M. Girardi and Weis [4]. They also show that the smoothness order $\lfloor n/t \rfloor + 1$ cannot be essentially relaxed. (A slight improvement is possible by considering appropriate fractional order smoothness.) The observation from [6] that one actually only needs the intersection of these two different sets of assumptions was new even in $L^p(\mathbf{R}^n)$, where it simultaneously improved the classical multiplier theorems of Mihlin ($\alpha \in \{0, 1\}^n$) and L. Hörmander ($|\alpha| \leq \lfloor n/2 \rfloor + 1$).

Whereas the Fourier-type controls the required order of smoothness of multipliers, another geometric notion is known to be related to the order of required decay and admissible blow-up of the derivatives of m. This is the property (α) of G. Pisier, which allows the following form of the multiplier theorem, obtained be Štraklj and Weis:

1.2. **Theorem** ([10]). *Let X and Y be UMD spaces with property (α). Let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X, Y))$ satisfy the R-boundedness condition*

$$N_\alpha := \mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in (\mathbf{R} \setminus \{0\})^n] < \infty \qquad \text{for all} \quad \alpha \in \{0, 1\}^n.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier multiplier for $1 < p < \infty$.

In $L^p(\mathbf{R}^n)$, this improvement of Mihlin's theorem is due to P. I. Lizorkin, and for scalar multipliers on $L^p(\mathbf{R}^n, X)$ due to Zimmermann [12]. It has also been extended to the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$, $\bar{p} = (p_1, \ldots, p_n) \in \,]1, \infty[^n$ by the author [7]. Zimmermann showed that already for scalar-multipliers, Theorem 1.2 fails to extend to all UMD spaces. Recently Weis and the author observed that its validity actually characterizes UMD spaces with property (α) [9].

If X and Y are UMD spaces with Fourier-type $t \in]1,2]$ and property (α), then one could try to check the boundedness of a multiplier $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ by either Theorem 1.1 or 1.2: the latter one requires in general more derivatives but imposes weaker size conditions on them. A natural question is whether the intersection of these assumptions would suffice. Note that this problem arises already on $L^p(\mathbf{R}^n)$, and an affirmative answer would be an improvement of the classical multiplier theorems by means of intersecting Hörmander's and Lizorkin's conditions. And indeed we are able to provide such an answer:

1.3. Theorem. *Let X and Y be UMD-spaces with property (α) and Fourier-type $t \in]1,2]$. Let $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy the R-boundedness condition*

$$N_\alpha = \mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in (\mathbf{R} \setminus \{0\})^n] < \infty \qquad \text{for all} \quad \alpha \in \{0,1\}^n \text{ s.t. } |\alpha| \leq \lfloor n/t \rfloor + 1.$$

Then m is an $(L^p(\mathbf{R}^n, X), L^p(\mathbf{R}^n, Y))$ Fourier-multiplier for $1 < p < \infty$.

In fact, Theorem 1.3 will be proved in a slightly more general form than here stated, which also covers some of the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$. There are certain limitations on our techniques in this setting, however, so that we are unable to cover all multi-exponents $\bar{p} \in]1,\infty[^n$, but only a certain subset $\mathscr{P}_t$ thereof. Nevertheless, it is strictly larger than the subset of pure exponents $p \in]1,\infty[$ appearing in the statement of Theorem 1.3 above.

The method of proof relies mainly on ideas of Girardi, Weis and the author from [4, 6, 8], appropriately modified to exploit the additional hypothesis of property (α). In particular, the proof consists of roughly the following two parts:

- obtain a criterion for the boundedness of operator-valued convolutions $f \mapsto k * f$, and then
- use a "Fourier embedding theorem" to check that the multiplier functions m appearing in Theorem 1.3 are mapped by the Fourier transform into the class of convolution kernels k for which the mentioned criterion is verified.

Just like in [6], we find that the "qualitative" UMD and (α) properties are mainly used in the first step, whereas the "quantitative" Fourier-type condition only plays a rôle in the second.

A part of Theorem 1.3, namely the multiplier property of m for $t \leq p \leq t'$, can also be obtained by a simpler approach introduced in [7], which essentially consists of an induction on the dimension n, starting from a one-dimensional result from [4]. We indicate this in more detail in the appendix. Note that while the multiplier condition of Theorem 1.1 implies the Hörmander integral condition for the associated kernel k (so that the boundedness of the multiplier on $L^p(\mathbf{R}^n, X)$ for one $p \in]1,\infty[$ already gives the boundedness for all p), the condition of Theorem 1.3 does not, and thus the simpler approach, unfortunately, appears insufficient to recover the full result. The failure of Hörmander's integral condition underlines the more delicate nature of the Lizorkin-type multipliers compared to the Mihlin or Hörmander-type ones.

2. Preliminaries, and a criterion for convolution operators

Let us begin by making some conventions. If $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$ are two vectors of same length, we define their product componentwise by $\alpha\beta := (\alpha_i\beta_i)_{i=1}^n$. A vector to the power of another vector is defined as the scalar $\alpha^\beta := \prod_{i=1}^n \alpha_i^{\beta_i}$, whereas a scalar t to the power of a vector α is the vector $t^\alpha := (t^{\alpha_i})_{i=1}^n$. A combination which often appears is $2^{-\mu}x$, where $\mu \in \mathbf{Z}^n$ and $x \in \mathbf{R}^n$. By our conventions, this is the vector $(2^{-\mu_i}x_i)_{i=1}^n \in \mathbf{R}^n$. We also record the formula $(t^\alpha)^\beta = t^{\alpha\cdot\beta}$, where $\alpha \cdot \beta := \sum_{i=1}^n \alpha_i\beta_i$ is the usual scalar product.

By ε_μ, $\mu \in \mathbf{Z}^n$, we denote a sequence of independent random variables on some probability space Ω, with distribution $P(\varepsilon_\mu = +1) = P(\varepsilon_\mu = -1)$. These are called Rademacher variables. We denote the mathematical expectation on (Ω, P) by E.

The mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$ are defined inductively as follows: For $n = 1$, $L^{(p_1)}(\mathbf{R}, X) := L^{p_1}(\mathbf{R}, X)$ is the usual Bôchner space. For $\bar{p} = (\bar{q}, p_n)$, where $\bar{q} = (p_1, \ldots, p_{n-1})$ and $n > 1$, we set $L^{\bar{p}}(\mathbf{R}^n, X) := L^{p_n}(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^{n-1}, X))$.

This recursive definition is often handy in proving n-dimensional results with a 1-dimensional version as a starting point. In this connection it is useful to note that when X is a UMD space, then so is $L^p(\mathbf{R}, X)$ for $p \in \,]1,\infty[$ and by induction also $L^{\bar{p}}(\mathbf{R}^n, X)$ for $\bar{p} \in \,]1,\infty[^n$. The same applies to property (α).

We next give a version for the mixed norm spaces of a useful lemma of J. Bourgain [2], which also lies at the heart of the multiplier theorems in [4, 6, 8]. The assumption that X be UMD is used via the validity of the $n = 1$ case of the lemma.

2.1. **Lemma.** *Let X be a UMD space and $\bar{p} \in \,]1,\infty[^n$. For a finite set of indices $\mu \in \mathbf{Z}^n$, let $f_\mu \in L^{\bar{p}}(\mathbf{R}^n, X)$ with $\hat{f}_\mu \subset 2^\mu \cdot [-1,1]$ and $h_{(\mu)} = (h^{(1)}_{\mu_1}, \ldots, h^{(n)}_{\mu_n}) \in \mathbf{R}^n$ with $|h^{(i)}_j| \le K_i 2^{-j}$, where $K_1, \ldots, K_n \ge 2$. Then*

$$E\Big\|\sum_\mu \varepsilon_\mu f_\mu(\cdot - h_{(\mu)})\Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)} \le C \cdot \prod_{i=1}^n \log K_i \cdot E\Big\|\sum_\mu \varepsilon_\mu f_\mu\Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)}.$$

Proof. The case $n = 1$ is proved in [2] for $\mathbf{T}$ in place of $\mathbf{R}$; this is transferred to $\mathbf{R}$ in [4]. (A generalization to $\mathbf{R}^n$ is also given there, but of a slightly different kind than what we want here.) Let us assume the lemma true for some n, and consider the case $n+1$. We write $\bar{p} = (\bar{q}, p_{n+1})$. Note that both sides of the estimate to be proved remain invariant if we multiply ε_μ by $\eta_{\mu_{n+1}}$, where $(\eta_j)_{j\in\mathbf{Z}}$ is an independent Rademacher sequence. For $\mu = (\nu, j) \in \mathbf{Z}^n \times \mathbf{Z}$, let us write $h_{(\nu,j)} =: (\tilde{h}_{(\nu)}, h^{(n+1)}_j) \in \mathbf{R}^n \times \mathbf{R}$.

For each $j \in \mathbf{Z}$, consider the function

$$x \in \mathbf{R}^n \mapsto F^\varepsilon_j(x) := \sum_{\nu\in\mathbf{Z}^n} \varepsilon_{(\nu,j)} f_{(\nu,j)}(\cdot - \tilde{h}_{(\nu)}, x) \in L^{\bar{q}}(\mathbf{R}^n, X).$$

Then $\operatorname{supp} \hat{F}^\varepsilon_j \subset 2^j \cdot [-1,1]$, and so we have (applying the $n = 1$ case on the UMD Banach space $L^{\bar{q}}(\mathbf{R}^n, X)$)

$$\begin{aligned} & E\Big\|\sum_\mu \varepsilon_\mu f_\mu(\cdot - h_{(\mu)})\Big\|_{L^{\bar{p}}(\mathbf{R}^{n+1},X)} \le E\Big\|\sum_j \eta_j F^\varepsilon_j(\cdot - h^{(n+1)}_j)\Big\|_{L^{p_{n+1}}(\mathbf{R},L^{\bar{q}}(\mathbf{R}^n,X))} \\ & \le C \log K_{n+1} \cdot E\Big\|\sum_j \eta_j F^\varepsilon_j\Big\|_{L^{p_{n+1}}(\mathbf{R},L^{\bar{q}}(\mathbf{R}^n,X))} \qquad (2.2) \\ & \le C \log K_{n+1} \Big(\int_{\mathbf{R}} \Big[E\Big\|\sum_\nu \eta_\nu \sum_j \varepsilon_{(\nu,j)} f_{(\nu,j)}(\cdot - \tilde{h}_{(\nu)}, x)\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)}\Big]^{p_{n+1}} \mathrm{d}x\Big)^{1/p_{n+1}}, \end{aligned}$$

where we used Kahane's inequality and the invariance of the distribution of $\varepsilon_{(\nu,j)}$ under multiplication by an independent Rademacher sequence η_ν.

Observe that the functions

$$G^{\varepsilon,x}_\nu := \sum_j \varepsilon_{(\nu,j)} f_{(\nu,j)}(\cdot, x) \in L^{\bar{q}}(\mathbf{R}^n, X)$$

satisfy $\operatorname{supp} \hat{G}^{\varepsilon,x}_\nu \subset 2^\nu \cdot [-1,1]$, while the vectors $\tilde{h}_{(\nu)} = (h^{(1)}_{\nu_1}, \ldots, h^{(n)}_{\nu_n})$ satisfy $|h^{(i)}_j| \le K_i 2^{-j}$. Thus, by the induction assumption,

$$E\Big\|\sum_\nu \eta_\nu G^{\varepsilon,x}_\nu(\cdot - h_{(\nu)})\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)} \le C \cdot \prod_{i=1}^n K_i E\Big\|\sum_\nu \eta_\nu G^{\varepsilon,x}_\nu\Big\|_{L^{\bar{q}}(\mathbf{R}^n,X)}.$$

Applying this estimate in (2.2), and using Kahane's inequality one more time, we arrive at the assertion of the lemma. □

For the next result, we introduce a partition of unity on $\mathbf{R}^n$. Let first $\hat{\varphi}^{(1)}_0 \in \mathscr{D}(\mathbf{R})$ be symmetric about the origin, have support in $[-2,2]$, be constantly 1 on $[-1,1]$ and decreasing on $[1,2]$. Let $\hat{\phi}^{(1)}_0(\xi) := \hat{\varphi}^{(1)}_0(\xi) - \hat{\varphi}^{(1)}_0(2\xi)$. We then define a function $\hat{\phi}_0 \in \mathscr{D}(\mathbf{R}^n)$ by $\hat{\phi}_0(\xi) := \prod_{i=1}^n \hat{\varphi}^{(1)}_0(\xi_i)$. Finally, for $\mu \in \mathbf{Z}^n$, let $\hat{\phi}_\mu(\xi) := \hat{\phi}_0(2^{-\mu}\xi)$. Then $\sum_{\mu\in\mathbf{Z}^n} \hat{\phi}_\mu(\xi) = 1$ for all $\xi \in (\mathbf{R} \setminus \{0\})^n =: \mathbf{R}^n_*$. Naturally, $\phi_\mu(x) = 2^{-\mu\cdot\iota}\phi_0(2^{-\mu}x)$, where $\iota := (1, \ldots, 1)$, will be the function whose Fourier transform is $\hat{\phi}_\mu$. It is useful to introduce $\chi_\mu := \phi_{\mu-1} + \phi_\mu + \phi_{\mu+1}$, so that $\hat{\chi}_\mu$ is constantly 1 on $\operatorname{supp}\hat{\phi}_\mu$.

The assumption that a certain space X is UMD with property (α) will mainly be used through the following Littlewood–Paley-type theorem on $L^{\bar{p}}(\mathbf{R}^n, X)$, and the similar result on $L^{\bar{p}}(\mathbf{R}^n, X')$, which follows from the known fact that X', too, is a UMD space with property (α) when X is. (Note that unlike the UMD condition, the property (α) alone is not self-dual; nevertheless, the joint property "UMD and (α)" is.)

2.3. **Proposition.** *Let X be a UMD space with property (α). Then*

$$E\Big\| \sum_\mu \varepsilon_\mu \chi_\mu * f \Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)} \leq C \, \|f\|_{L^{\bar{p}}(\mathbf{R}^n,X)}$$

for all $f \in L^{\bar{p}}(\mathbf{R}^n, X)$ and $\bar{p} \in \,]1,\infty[^n$, with C depending only on X and $\bar{p}$.

For $\bar{p} = p \cdot \iota$, this follows from the results of Zimmermann [12], while the general case can easily be obtained from [7].

The $n = 1$ case holds in every UMD space, and there are also variants of the Littlewood–Paley decomposition which remain bounded in the multi-dimensional setting for arbitrary UMD spaces. Results analogous to the following have been proved in that setting in [4, 8]. The proof given below is not very different from that in [8], but it is hoped that we have managed to make the argument a little more transparent.

2.4. **Proposition.** *Let X and Y be UMD spaces with property (α), and let $\bar{p} \in \,]1,\infty[^n$. Let the distribution $k \in \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfy*

$$\int_{\mathbf{R}^n} E\Big\| \sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y) f_\mu \Big\|_{L^{\bar{p}}(\mathbf{R}^n,Y)} \prod_{i=1}^n \log(2+|y_i|)\,\mathrm{d}y \leq CE\Big\| \sum_\mu \varepsilon_\mu f_\mu \Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)} \tag{2.5}$$

for all finitely non-zero sequences of $g_\mu \in L^{\bar{p}}(\mathbf{R}^n, X)$. Then $f \mapsto k * f$ is a bounded map from $L^{\bar{p}}(\mathbf{R}^n, X)$ to $L^{\bar{p}}(\mathbf{R}^n, Y)$.

Proof. For $f \in X \otimes [\hat{\mathscr{D}}_0(\mathbf{R})]^n$, $g \in Y' \otimes [\hat{\mathscr{D}}_0(\mathbf{R})]^n$, we have

$$\langle g, k * f\rangle = \sum_{\mu\in\mathbf{Z}^n} \langle \chi_\mu * g, (\phi_\mu * k) * (\chi_\mu * f)\rangle = \sum_{\mu\in\mathbf{Z}^n} \Big\langle (\phi_\mu * \tilde{k})' * (\chi_\mu * g), \chi_\mu * f \Big\rangle$$

where only finitely many of the $\chi_\mu * g$ and $\chi_\mu * f$ are zero, and $\tilde{k}$ denotes the reflection of k about the origin. A change of variables gives

$$(\phi_\mu * \tilde{k})' * (\chi_\mu * g)(x) = \int_{\mathbf{R}^n} 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)'(\chi_\mu * g)(x + 2^{-\mu}y)\,\mathrm{d}y,$$

and then

$$\begin{aligned}
&\sum_\mu \Big\langle (\phi_\mu * \tilde{k})' * (\chi_\mu * g), \chi_\mu * f \Big\rangle \\
&= \sum_\mu \int_{\mathbf{R}^n} \big\langle 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)'(\chi_\mu * g)(\cdot + 2^{-\mu}y), \chi_\mu * f \big\rangle \,\mathrm{d}y \\
&= \int_{\mathbf{R}^n} E\left\langle \sum_\mu \varepsilon_\mu (\chi_\mu * g)(\cdot + 2^{-\mu}y), \sum_\nu \varepsilon_\nu 2^{-\nu\cdot\iota}(\phi_\nu * k)(2^{-\nu}y)(\chi_\nu * f) \right\rangle \mathrm{d}y.
\end{aligned}$$

By Hölder's and Kahane's inequalities, the absolute value of this is bounded by

$$C \int_{\mathbf{R}^n} E\Big\| \sum_\mu \varepsilon_\mu (\chi_\mu * g)(\cdot + 2^{-\mu}y) \Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} \cdot E\Big\| \sum_\nu \varepsilon_\nu 2^{-\nu\cdot\iota}(\phi_\nu * k)(2^{-\nu}y)(\chi_\nu * f) \Big\|_{L^{\bar{p}}(\mathbf{R}^n,Y)} \mathrm{d}y.$$

By the previous lemma and the Littlewood–Paley theorem on $L^{\bar{p}'}(\mathbf{R}^n, Y')$, we have

$$\begin{aligned} E\Big\| \sum_\nu \varepsilon_\nu (\chi_\nu * g)(\cdot + 2^{-\nu} y)\Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} &\leq C \prod_{i=1}^n \log(2+|y_i|) \cdot \Big\| \sum_\nu \varepsilon_\nu \chi_\nu * g\Big\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} \\ &\leq C \prod_{i=1}^n \log(2+|y_i|) \, \|g\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} . \end{aligned}$$

By the assumption and the Littlewood–Paley theorem on $L^{\bar{p}}(\mathbf{R}^n, X)$,

$$\begin{aligned} &\int_{\mathbf{R}^n} E\Big\| \sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota} (\phi_\mu * k)(2^{-\mu} y)(\chi_\mu * f)\Big\|_{L^{\bar{p}}(\mathbf{R}^n,Y)} \cdot \prod_{i=1}^n \log(2+|y_i|)\,\mathrm{d}y \\ &\leq CE\Big\| \sum_\mu \varepsilon_\mu \chi_\mu * f\Big\|_{L^{\bar{p}}(\mathbf{R}^n,X)} \leq C\,\|f\|_{L^{\bar{p}}(\mathbf{R}^n,X)} . \end{aligned}$$

Everything combined, we have shown that

$$|\langle g, k * f\rangle| \leq C\,\|g\|_{L^{\bar{p}'}(\mathbf{R}^n,Y')} \|f\|_{L^{\bar{p}}(\mathbf{R}^n,X)} ,$$

which obviously gives the assertion. □

3. Multiplier theorems via embeddings

The next step on the road to multiplier theorems is to find efficient conditions for checking the assumption of Prop. 2.4 in terms of smoothness and size of the multiplier $m = \hat{k}$. Note that the mentioned assumption is the requirement of membership of a certain function in the logarithmically weighted L^1 space. Thus the following result about an embedding into such a space is not completely unrelated. The proof is similar to that of Lemma 8.1 in [6], but we give the details for the convenience of the reader.

Below, the assumption that X have Fourier-type $t \in \,]1,2]$ enters the scene. Recall that this means the boundedness of the Fourier transform from $L^t(\mathbf{R}^n, X)$ to $L^{t'}(\mathbf{R}^n, X)$.

We also employ the product symbol in connection with multiple integrals in a rather formal way, which is best understood by thinking of "$\prod_{i:\alpha_i=1} \mathscr{E}_i$" simply as "write the expressions $\mathscr{E}_i$, for all i such that $\alpha_i = 1$, in a row, and then interpret what you have in the usual way"; cf. [6].

3.1. Proposition. *Let X be a Banach space with Fourier-type $t \in \,]1,2]$. Let $w(x) = \prod_{i=1}^n w_i(x_i)$, where each w_i is even, positive and non-decreasing on $\mathbf{R}_+$. Then*

$$\int_{\mathbf{R}^n} |\hat{f}(x)|_X w(x)\,\mathrm{d}x \leq C \sum_{\alpha\in\{0,1\}^n} \prod_{i:\alpha_i=1} \int_0^{1/4} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} w_i(h_i^{-1}) \, \|\delta_h^\alpha f\|_{L^t(\mathbf{R}^n,X)} .$$

The $\alpha = 0$ term on the right means simply $\|f\|_{L^t(\mathbf{R}^n,X)}$.

Proof. We make use of the decomposition of $\mathbf{R}^n$ introduced in [6]. For $\rho \in \,]0,\infty[^n$, $\alpha \in \{0,1\}^n$ and $j \in \mathbf{N}^\alpha := \{\mu \in \mathbf{N}^n : \mu_i = 0 \text{ if } \alpha_i = 0\}$, let

$$\begin{aligned} E(\alpha,\rho) &:= \{x \in \mathbf{R}^n : |x_i| \leq \rho_i \text{ if } \alpha_i = 0,\ |x_i| > \rho_i \text{ if } \alpha_i = 1\} \\ E(\alpha,\rho,j) &:= \{x \in E(\alpha,\rho) : 2^{j_i}\rho_i < |x_i| \leq 2^{j_i+1}\rho_i \text{ if } \alpha_i = 1\}. \end{aligned}$$

The two key observations are the the facts that

$$|1 - e^{\mathrm{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i}| \geq c \qquad \text{for} \qquad x \in E(\alpha,\rho,j),\ \alpha_i = 1,$$

and that $(1-e^{\mathbf{i}2\pi x\cdot h})\hat{f}(x)$ is the Fourier transform of $f-f(\cdot-h)=:f-\tau_h f=:\Delta_h f$ at x. Then

$$\begin{aligned}
&\int_{E(\alpha,\rho)}|\hat{f}(x)|_X\cdot w(x)\,\mathrm{d}x=\sum_{j\in\mathbf{N}^\alpha}\int_{E(\alpha,\rho,j)}|\hat{f}(x)|_X\cdot w(x)\,\mathrm{d}x\\
&\leq C\sum_{j\in\mathbf{N}^\alpha}\int_{E(\alpha,\rho,j)}\Big|\prod_{i:\alpha_i=1}(1-e^{\mathbf{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i})\cdot\hat{f}(x)\Big|_X w(x)\,\mathrm{d}x\\
&\leq C\sum_{j\in\mathbf{N}^\alpha}\Big\|x\mapsto\prod_{i:\alpha_i=1}(1-e^{\mathbf{i}2\pi x\cdot e_i/2^{j_i+2}\rho_i})\cdot\hat{f}(x)\Big\|_{L^{t'}(\mathbf{R}^n,X)}\Big(\int_{E(\alpha,\rho,j)}w^t(x)\,\mathrm{d}x\Big)^{1/t}\\
&\leq\sum_{j\in\mathbf{N}^\alpha}C\Big\|\prod_{i:\alpha_i=1}\Delta_{e_i/2^{j_i+2}\rho_i}f\Big\|_{L^t(\mathbf{R}^n,X)}\prod_{i:\alpha_i=0}w_i(2\rho_i)(2\rho_i)^{1/t}\cdot\prod_{i:\alpha_i=1}w_i(2^{j_i+1}\rho_i)(2^{j_i+1}\rho_i)^{1/t}.
\end{aligned}$$

We integrate with respect to $\mathrm{d}\rho_i/\rho_i$ from r to $2r$ for every i:

$$\begin{aligned}
&\prod_{i=1}^n\int_r^{2r}\frac{\mathrm{d}\rho_i}{\rho_i}\int_{E(\alpha,\rho)}|\hat{f}(x)|_X\cdot w(x)\,\mathrm{d}x\\
&\leq Cr^{(n-|\alpha|)/t}\prod_{i:\alpha_i=0}w_i(4r)\sum_{j\in\mathbf{N}^\alpha}\prod_{i:\alpha_i=1}\int_r^{2r}\frac{\mathrm{d}\rho_i}{\rho_i}w_i(2^{j_i+1}\rho_i)(2^{j_i+1}\rho_i)^{1/t}\Big\|\prod_{i:\alpha_i=1}\Delta_{e_i/2^{j_i+2}\rho_i}f\Big\|_{L^t_X}\\
&=Cr^{(n-|\alpha|)/t}\prod_{i:\alpha_i=0}w_i(4r)\sum_{j\in\mathbf{N}^\alpha}\prod_{i:\alpha_i=1}\int_{1/2^{j_i+3}r}^{1/2^{j_i+2}r}\frac{\mathrm{d}h_i}{h_i}w_i(1/2h_i)(2h_i)^{-1/t}\Big\|\delta_h^\alpha f\Big\|_{L^t(\mathbf{R}^n,X)}\\
&\leq Cr^{(n-|\alpha|)/t}\prod_{i:\alpha_i=0}w_i(4r)\times\prod_{i:\alpha_i=1}\int_0^{1/4r}\frac{\mathrm{d}h_i}{h_i}h_i^{-1/t}w_i(h_i^{-1})\,\|\delta_h^\alpha f\|_{L^t(\mathbf{R}^n,X)},
\end{aligned}$$

where we have adopted the notation $\delta_h^\alpha:=\prod_{i:\alpha_i=1}\Delta_{h_ie_i}$.

Summing over all $\alpha\in\{0,1\}^n$ and taking $r=1$, we get the assertion. □

Again, we introduce some more notation. Since in the interpolation of $L^{\bar{p}}$ spaces a key rôle is played by the fact that the reciprocal of a certain multi-exponent can be expressed as a linear combination of others, we define

$$\operatorname{conv}_{-1}\mathscr{A}:=(\operatorname{conv}\mathscr{A}^{-1})^{-1}=\{\bar{p}:1/\bar{p}=\sum_{j=1}^N\sigma_j/\bar{p}_i^{(j)},\ \bar{p}^{(j)}\in\mathscr{A},\ \sigma_j\geq 0,\ \sum_{j=1}^N\sigma_j=1\}$$

Let us denote $\iota_k:=(1,\ldots,1)\in\mathbf{R}^k$, and define

$$\mathscr{A}_t:=\operatorname{conv}_{-1}\bigcup_{k=0}^n[t,t']^k\times\{\iota_{n-k}\}.$$

We also recall that $\operatorname{Rad}X$ is the completion of all finitely non-zero sums $\sum_\mu\varepsilon_\mu x_\mu$ in $L^2(\Omega,X)$. By Kahane's inequality it follows that we could equally well define this as a completion in $L^r(\Omega,X)$ for any $r\in[1,\infty[$, and then by Fubini's theorem $\operatorname{Rad}(L^p(\mathbf{R},X))\approx L^p(\mathbf{R},\operatorname{Rad}X)$, and by induction $\operatorname{Rad}(L^{\bar{p}}(\mathbf{R}^n,X))\approx L^{\bar{p}}(\mathbf{R}^n,\operatorname{Rad}X)$ for all $p\in[1,\infty[$, resp. $\bar{p}\in[1,\infty[^n$. See e.g. [4] for more on this space and its use in the present kind of connection.

3.2. Lemma. *Let* Y *have Fourier-type* $t\in\,]1,2]$, *and let* $k\in\mathscr{S}'(\mathbf{R}^n,\mathscr{L}(X,Y))$ *with* $m:=\hat{k}\in L^\infty(\mathbf{R}^n,\mathscr{L}(X,Y))$. *Let*

$$M:=\sum_{\alpha\in\{0,1\}^n}\prod_{i:\alpha_i=1}\int_0^{1/4}\frac{\mathrm{d}h_i}{h_i}h_i^{-1/t}\log(h_i^{-1})\,\Big\|\mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\cdot)m(2^\mu\cdot)):\mu\in\mathbf{N}^n|\mathscr{L}(X,Y)]\Big\|_t \tag{3.3}$$

be finite. Then (2.5) *holds with* $C=c(\bar{p},X)M<\infty$ *for all* $f_\mu\in L^{\bar{p}}(\mathbf{R}^n,X)$ *and* $\bar{p}\in\mathscr{A}_t$.

Proof. Let first $f_\mu\in L^{\bar{q}}(\mathbf{R}^k,X)$, where $\bar{q}\in[t,t']^k$ and $k\in\{0,1,\ldots,n\}$; for $k=0$, we understand this simply as $f_\mu\in X$. For a fixed y, we have $2^{\mu\cdot\iota}(\phi_\mu*k)(2^{-\mu}y)\in\mathscr{L}(X,Y)$, and this operator has a canonical extension to $\mathscr{L}(L^{\bar{q}}(\mathbf{R}^k,X),L^{\bar{q}}(\mathbf{R}^k,Y)$. For $\bar{q}$ in the described range, the space $L^{\bar{q}}(\mathbf{R}^k,Y)$

has Fourier-type t, and the same is still true of the space $\mathrm{Rad}(L^{\bar{q}}(\mathbf{R}^k,Y)) \approx L^{\bar{q}}(\mathbf{R}^k,\mathrm{Rad}\,Y)$. It follows from Prop. 3.1 that

$$\begin{aligned}
&\int_{\mathbf{R}^n}\Big\|\sum_\mu \varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu * k)(2^{-\mu}y)f_\mu\Big\|_{L^{\bar{q}}(\mathbf{R}^k,\mathrm{Rad}\,Y)}\prod_{i=1}^n \log(2+|y_i|)\,\mathrm{d}y\\
&\leq C\sum_{\alpha\in\{0,1\}^n}\prod_{i:\alpha_i=1}\int_0^{1/4}\frac{\mathrm{d}h_i}{h_i}h_i^{-1/t}\log(2+h_i^{-1})\Big\|\delta_h^\alpha\sum_\mu\varepsilon_\mu(\hat{\phi}_\mu\hat{k})(2^\mu\cdot)f_\mu\Big\|_{L^t(\mathbf{R}^n,L^{\bar{q}}(\mathbf{R}^k,\mathrm{Rad}\,Y))}\\
&\leq C\sum_{\alpha\in\{0,1\}^n}\prod_{i:\alpha_i=1}\int_0^{1/4}\frac{\mathrm{d}h_i}{h_i}h_i^{-1/t}\log(h_i^{-1})\,\Big\|\mathscr{R}\{\delta_h^\alpha(\hat{\phi}_0(\cdot)m(2^\mu\cdot))\}_{\mu\in\mathbf{N}^n}\Big\|_t\,\Big\|\sum_\mu\varepsilon_\mu f_\mu\Big\|_{L^{\bar{q}}(\mathbf{R}^k,\mathrm{Rad}\,X)}\\
&= CM\Big\|\sum_\mu\varepsilon_\mu f_\mu\Big\|_{L^{\bar{q}}(\mathbf{R}^k,\mathrm{Rad}\,X)}.
\end{aligned}$$

Next, consider a multi-exponent $\bar{p}=(\bar{q},\iota_{n-k})$, where $\bar{q}$ is as above. Let $F_\mu\in L^{\bar{p}}(\mathbf{R}^n,X)=L^{\iota_{n-k}}(\mathbf{R}^{n-k},L^{\bar{q}}(\mathbf{R}^k,X))$. Then the partial point-evaluations $F_\mu(\cdot,x)$, $x\in\mathbf{R}^{n-k}$, belong to $L^{\bar{q}}(\mathbf{R}^k,X)$, so the estimate just established applies to $f_\mu=F_\mu(\cdot,x)$. If we integrate the resulting inequality with respect to $\mathrm{d}x$ on $\mathbf{R}^{n-k}$, we obtain

$$\int_{\mathbf{R}^n}\Big\|\sum_\mu\varepsilon_\mu 2^{-\mu\cdot\iota}(\phi_\mu*k)(2^{-\mu}y)F_\mu\Big\|_{L^{\bar{p}}(\mathbf{R}^n,\mathrm{Rad}\,Y)}\prod_{i=1}^n\log(2+|y_i|)\,\mathrm{d}y\leq CM\Big\|\sum_\mu\varepsilon_\mu F_\mu\Big\|_{L^{\bar{p}}(\mathbf{R}^n,\mathrm{Rad}\,X)}.$$

Thus we have established the desired boundedness

$$L^{\bar{p}}(\mathbf{R}^n,\mathrm{Rad}\,X)\to L^1(\mathbf{R}^n,\prod_{i=1}^n\log(2+|y_i|)\,\mathrm{d}y;L^{\bar{p}}(\mathbf{R}^n,\mathrm{Rad}\,Y))$$

for all $\bar{p}\in\bigcup_{k=0}^n[t,t']^k\times\{\iota_{n-k}\}$. It suffices to apply, say, the complex interpolation method (more precisely, see [1], 5.1.2) to extend this to all the exponents in the assertion. □

Now we are ready for a multiplier theorem. The version given below is the most general of this paper, and just like the most general versions in [4, 6] has rather technical assumptions. Hopefully more attractive versions are given as Corollaries below, where it is shown, in particular, that Theorem 1.3 follows from this abstract version.

3.4. **Theorem.** *Let X and Y be UMD spaces with property (α), each of Fourier-type $t\in\,]1,2]$. Let $m\in L^\infty(\mathbf{R}^n,\mathscr{L}(X,Y))$ be such that the quantity M in (3.3) is finite. Let us denote*

$$\mathscr{B}_t:=\mathscr{A}_t\cap\,]1,\infty[^n,\qquad \mathscr{B}_t':=\{\bar{p}':\bar{p}\in\mathscr{B}_t\},\qquad \mathscr{P}_t:=\mathrm{conv}_{-1}(\mathscr{B}_t\cup\mathscr{B}_t').$$

Then m is an $(L^{\bar{p}}(\mathbf{R}^n,X),L^{\bar{p}}(\mathbf{R}^n,Y))$ Fourier-multiplier for all $\bar{p}\in\mathscr{P}_t$, in particular for all $\bar{p}=p\cdot\iota$, $p\in\,]1,\infty[$.

Proof. The assertion for $\bar{p}\in\mathscr{B}_t$ is an immediate consequence of Prop. 2.4 and Lemma 3.2. On the other hand, the assumptions of the Proposition remain invariant on replacing X by Y', Y by X' and m by $\xi\mapsto m(\xi)'\in\mathscr{L}(Y',X')$. Hence it also follows that $m(\cdot)'$ is an $(L^{\bar{p}}(\mathbf{R}^n,Y'),L^{\bar{p}}(\mathbf{R}^n,X'))$ Fourier-multiplier for $\bar{p}\in\mathscr{B}_t$, and this implies by duality that m is an $(L^{\bar{p}}(\mathbf{R}^n,X),L^{\bar{p}}(\mathbf{R}^n,Y))$ for $\bar{p}\in\mathscr{B}_t'$. The full assertion now follows by interpolation. □

3.5. **Corollary.** *Let X and Y be UMD spaces with property (α) and Fourier-type $t\in\,]1,2]$. Let $t^{-1}<\gamma\leq 1$, and let $m\in L^\infty(\mathbf{R}^n,\mathscr{L}(X,Y))$ satisfy the following R-boundedness condition:*

$$L_\alpha:=\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi):\alpha\in\{0,1\}^n,\ |\xi_i|>2\,|\eta_i|>0]<\infty$$

for all $\alpha\in\{0,1\}^n$. Then m is an $(L^{\bar{p}}(\mathbf{R}^n,X),L^{\bar{p}}(\mathbf{R}^n,Y))$ Fourier-multiplier for all $\bar{p}\in\mathscr{P}_t$.

The case $n=1$, which holds even without property (α), is due to Girardi and Weis [4].

Proof. Obviously, it suffices to show that the quantity M in (3.3) is bounded by the sum of L_α's. It is readily verified that

$$\delta_h^\alpha(f\cdot g)=\sum_{\theta\leq\alpha}\tau_{\theta h}\delta_h^{\alpha-\theta}f\cdot\delta_h^\theta g.$$

Thus, for $h \in]0,1/4]^n$,

$$\mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\xi)m(2^\mu\xi)) : \mu \in \mathbf{N}] \le \sum_{\theta\le\alpha} |\tau_{\theta h}\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi)| \cdot \mathscr{R}[(\delta^\theta_{2^\mu h}m)(2^\mu\xi) : \mu \in \mathbf{N}].$$

Note that the condition that $\tau_{\theta h}\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi) \neq 0$ forces $|\xi_i| \approx 1$ for all $i = 1,\ldots,n$. The R-bound above is dominated by $L_\theta h^{\theta\gamma}$. On the other hand, since $\hat{\phi}_0$ is a smooth function, we have $|\delta_h^{\alpha-\theta}\hat{\phi}_0(\xi)| \le Ch^{\alpha-\theta} \le Ch^{(\alpha-\theta)\gamma}$. Taking moreover into account the finite support of $\hat{\phi}_0$, we find that

$$\left\|\mathscr{R}[\delta_h^\alpha(\hat{\phi}_0(\cdot)m(2^\mu\cdot)) : \mu \in \mathbf{N}]\right\|_t \le Ch^{\alpha\gamma}\sum_{\theta\le\alpha} L_\theta,$$

and then

$$M \le C \sum_{\alpha\in\{0,1\}^n}\prod_{i:\alpha_i=1}\int_0^{1/4}\frac{\mathrm{d}h_i}{h_i}h_i^{-1/t+\gamma}\log(h_i^{-1}) \times \sum_{\theta\le\alpha} L_\theta,$$

and all the integrals are convergent, since $\gamma > 1/t$. □

The specialization of the following Corollary to the case of a pure exponent $\bar{p} = p\cdot\iota$, $p \in]1,\infty[$, is Theorem 1.3, which was stated in the introduction.

3.6. Corollary. *Let X and Y be a UMD space with property (α) and Fourier-type $t \in]1,2]$. If $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ satisfies*

$$\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*,\ \alpha \in \{0,1\}^n,\ |\alpha| \le \lfloor n/t\rfloor + 1] < \infty,$$

then m is an $(L^{\bar{p}}(\mathbf{R}^n,X), L^{\bar{p}}(\mathbf{R}^n,Y))$ Fourier multiplier for all $\bar{p} \in \mathscr{P}_t$.

Proof. It suffices to pick $1/t < \gamma \le 1$ and show that the R-bounds assumed in the previous Proposition can be estimated by the one assumed now. Note that $n/t < \lfloor n/t\rfloor + 1 \le n$, so that we may choose γ in such a way that $n/t < n\gamma \le \lfloor n/t\rfloor + 1$.

Let $\alpha \in \{0,1\}^n$. If $|\alpha| \le \lfloor n/t\rfloor + 1$, then we obtain at once that

$$\begin{aligned}\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) &= \xi^{\alpha\gamma}\eta^{-\alpha\gamma}\int_{[0,1]^\alpha}\eta^\alpha D^\alpha m(\xi - u\eta)\,\mathrm{d}u \\ &= \eta^{\alpha(1-\gamma)}\xi^{\alpha(\gamma-1)}\int_{[0,1]^\alpha}\frac{\xi^\alpha}{(\xi-u\eta)^\alpha}(\xi-u\eta)^\alpha D^\alpha m(\xi-u\eta)\,\mathrm{d}u,\end{aligned}$$

so that

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma} : |\xi_i| > 2\,|\eta_i| > 0] \le 2^{|\alpha|\gamma}\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}^n_*].$$

If $\alpha > \lfloor n/t\rfloor + 1$, then a somewhat more careful argument is required. Consider a splitting $\alpha = \beta + \theta$; $\beta,\theta \ge 0$, where $|\beta| = \lfloor n/t\rfloor + 1 > n/t$, and then $|\theta| < n/t'$. For each of the finite number of such splittings, we consider

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : 2 < |\xi_i/\eta_i| \le |\xi_j/\eta_j| \text{ for } \beta_j = 1,\ \theta_i = 1].$$

Note that the sets $\{2 < |\xi_i/\eta_i| \le |\xi_j/\eta_j| \text{ for } \beta_j = 1,\ \theta_i = 1\}$, when (β,θ) ranges over all splittings of α as above, cover the set $\{|\xi_i| > 2\,|\eta_i| > 0\}$, for among the $|\alpha|$ indices i with $\alpha_i = 1$, there always exists a collection of $\lfloor n/t\rfloor + 1$ indices j for which the ratio $|\xi_j/\eta_j|$ is at least as large as for the remaining indices.

Then we have

$$\begin{aligned}&\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) = \xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\theta\int_{[0,1]^\beta}\eta^\beta D^\beta m(\xi-u\eta)\,\mathrm{d}u \\ &= \xi^{\theta\gamma}\eta^{-\theta\gamma}\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}\sum_{\kappa\le\theta}(-1)^{|\kappa|}\int_{[0,1]^\beta}\frac{\xi^\beta}{(\xi-(u+\kappa)\eta)^\beta}(\xi-(u+\kappa)\eta)^\beta D^\beta m(\xi-(u+\kappa)\eta)\,\mathrm{d}u,\end{aligned}$$

so that

$$\begin{aligned}&\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : 2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \text{ for } \beta_j = 1,\ \theta_i = 1]\\ &\qquad\leq \sup\Big[\big|\xi^{\theta\gamma}\eta^{-\theta\gamma}\big| \cdot |\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}| : 2 < |\xi_i/\eta_i| \leq |\xi_j/\eta_j| \text{ for } \beta_j = 1,\ \theta_i = 1\Big]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times \mathscr{R}[\xi^\beta D^\beta m(\xi) : \xi \in \mathbf{R}^n_*].\end{aligned}$$

In the set under consideration, we have

$$|\xi^{\theta\gamma}\eta^{-\theta\gamma}| \cdot |\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}| \leq |\xi^{\beta\gamma}\eta^{-\beta\gamma}|^{|\theta|/|\beta|} \cdot |\xi^{\beta(\gamma-1)}\eta^{\beta(1-\gamma)}| = |\xi^\beta\eta^{-\beta}|^{\gamma(|\theta|/|\beta|+1)-1}.$$

Since $\left|\xi^\beta\eta^{-\beta}\right| > 2^{|\beta|}$, it remains to show that $\gamma(|\theta|\,/\,|\beta| + 1) - 1 \leq 0$. But

$$\gamma(|\theta| + |\beta|) = \gamma\,|\alpha| \leq \gamma n \leq \lfloor n/t\rfloor + 1 = |\beta|\,,$$

which is equivalent to this claim. □

One more corollary is in order: it says that if a family of multipliers satisfies the assumptions of the previous Corollaries in a uniform way, not only are the individual operators bounded, but in fact the whole family is again R-bounded. The proof is omitted, since it is an immediate consequence of the previous results by using the general bootstrapping method for operator-valued multiplier theorems on spaces with property (α), which was invented by Girardi and Weis in [5].

3.7. **Corollary.** *Let X and Y be UMD spaces with property (α) and Fourier-type $t \in\ ]1,2]$. Let $\mathscr{M} \subset L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ be a collection of multipliers which satisfies*

$$\mathscr{R}[\xi^\alpha D^\alpha m(\xi) : \alpha \in \{0,1\}^n,\ |\alpha| \leq \lfloor n/t\rfloor + 1,\ m \in \mathscr{M}] < \infty,$$

or more generally, with $1/t < \gamma \leq 1$,

$$\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^n,\ |\xi_i| > 2\,|\eta_i| > 0,\ m \in \mathscr{M}] < \infty.$$

Then the collection $\mathscr{T}$ of all $(L^{\bar p}(\mathbf{R}^n, X), L^{\bar p}(\mathbf{R}^n, Y))$ Fourier-multipliers associated with $m \in \mathscr{M}$ is an R-bounded set for all $\bar p \in \mathscr{P}_t$, and $\mathscr{R}(\mathscr{T})$ is estimated in terms of the R-bounds in the assumptions.

4. Appendix: Another approach to a special case of the main theorem

In [7] it was shown that certain multiplier theorems in UMD spaces with property (α) admit a relatively simple proof by induction on the dimension n, starting from the one-dimensional result. In particular, a new proof of Theorem 1.2 was given by this method. It is natural to ask whether this inductive approach could also yield the improved Theorem 1.3.

The answer is in part "yes", but with certain limitations. This approach has the problem that while $L^p(\mathbf{R}, X)$ (and then $L^{\bar p}(\mathbf{R}^n, X)$) inherits the UMD and (α) properties of X for all $p \in\ ]1,\infty[$ (resp. $\bar p \in\ ]1,\infty[^n$), this is only true for the Fourier-type t property for $p \in [t, t']$ (resp. $\bar p \in [t,t']^n$). This restricts the multi-exponents we are able to cover with such approach.

But to see what can be done, we give the following proposition, whose statement is just a specialization of Cor. 3.7 but for which we can give a simpler proof. (Of course, simplicity is a conditional property depending on our prior knowledge, and in the present case our "simpler proof" relies essentially on the $n = 1$ case from [4], whose proof is not in any substantial way easier than that of the general result above.)

4.1. **Proposition.** *Let X and Y be UMD spaces with property (α) and Fourier-type $t \in\ ]1,2]$. For an $(L^{\bar p}(\mathbf{R}^n, X), L^{\bar p}(\mathbf{R}^n, Y))$ Fourier-multiplier m, denote by T_m the corresponding operator between these spaces. Then for $\mathscr{M} \subset L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$,*

$$\mathscr{R}[T_m : m \in \mathscr{M}] \leq C\mathscr{R}[\xi^{\alpha\gamma}\eta^{-\alpha\gamma}\delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^n,\ |\xi_i| > 2\,|\eta_i| > 0,\ m \in \mathscr{M}]$$

for all $\bar p \in\ ]1,\infty[\ \times [t,t']^{n-1}$ and $1/t < \gamma \leq 1$, where C depends only on X, $\bar p$ and γ.

Proof. For $n = 1$ and $\mathscr{M} = \{m\}$, this was proved by Girardi and Weis [4] (without property (α)), and for a general $\mathscr{M}$ (and assuming (α)) this follows from their bootstrapping method [5].

If we assume the theorem valid for some n, we can use the induction method from [7] to prove it for $n+1$. In fact, the $(n+1)$-dimensional multiplier operator T_m, with $m \in L^\infty(\mathbf{R}^n, \mathscr{L}(X,Y))$ acting on $L^{(\bar{q},p)}(\mathbf{R}^{n+1}, X) = L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, X))$ is naturally identified (at least on a suitable test function class) with the one-dimensional multiplier operator $T_{\tilde{m}}$, where $\tilde{m}(x) := T_{m(\cdot,x)}$. Thus

$$\begin{aligned}
&\mathscr{R}[T_m : m \in \mathscr{M} | \mathscr{L}(L^{(\bar{q},p)}(\mathbf{R}^{n+1}, X), L^{(\bar{q},p)}(\mathbf{R}^{n+1}, Y))] \\
&= \mathscr{R}[T_{\tilde{m}} : m \in \mathscr{M} | \mathscr{L}(L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, X)), L^p(\mathbf{R}, L^{\bar{q}}(\mathbf{R}^n, Y)))] \\
&\leq C\mathscr{R}[x^{\alpha\gamma} y^{-\alpha\gamma} \delta_y^\alpha \tilde{m}(x) : \alpha \in \{0,1\},\ |x| > 2\,|y| > 0,\ m \in \mathscr{M} | \mathscr{L}(L^{\bar{q}}(\mathbf{R}^n, X), L^{\bar{q}}(\mathbf{R}^n, Y))] \\
&= C\mathscr{R}[T_{x^{\alpha\gamma} y^{-\alpha\gamma} \Delta^\alpha_{y e_{n+1}} m(\cdot,x)} : \alpha \in \{0,1\},\ |x| > 2\,|y| > 0,\ m \in \mathscr{M}] \\
&\leq C\mathscr{R}[\xi^{\alpha\gamma} \eta^{-\alpha\gamma} \delta_\eta^\alpha m(\xi) : \alpha \in \{0,1\}^{n+1},\ |\xi_i| > 2\,|\eta_i| > 0, m \in \mathscr{M} | \mathscr{L}(X,Y)],
\end{aligned}$$

where the first inequality was an application of Girardi and Weis' $n = 1$ case on the spaces $L^{\bar{q}}(\mathbf{R}^n, X)$ and $L^{\bar{q}}(\mathbf{R}^n, Y)$, both of which are UMD with (α) and Fourier-type t, and the second used the induction assumption. □

By repeating the reasoning in the proof of Cor. 3.6, one sees that for $\bar{p} \in \,]1,\infty[\, \times [t,t']^{n-1}$, the mentioned Corollary also follows from the previous Proposition, and in particular we get Theorem 1.3 for $p \in [t,t']$, as we claimed.

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