THE NUMBER OF EULERIAN GRAPHS

$g : E_2(D) \to \mathbb{Z}_2$ is **eulerian** if it has a closed walk $x_0 \to x_1 \to \ldots \to x_m = x_0$ containing every edge exactly once.

Theorem

A graph g is eulerian $\iff$ it is connected and **even**: the degree $d_g(x) = |\{y \mid g(x,y) = 1\}|$ is even for all x.

HOW MANY?

For *even graphs, up to isomorphism, that need not be connected*:

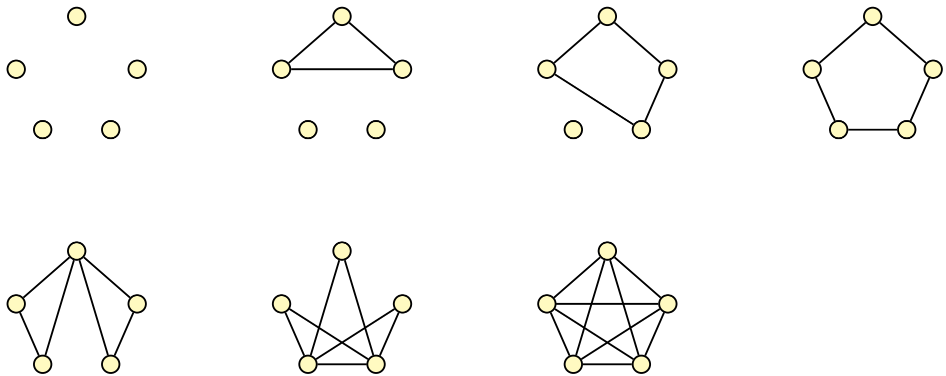
3	4	5	6	7	8	...
2	3	7	16	54	243	...

WHAT?

$$\sum_{\alpha} \frac{2^{\nu(\alpha)-\lambda(\alpha)}}{\prod_i i^{\alpha_i} \alpha_i!}$$

where α's go through all n-tuples and ν and λ are ..., well, complicated; see R. W. Robinson, Enumeration of Euler graphs (1969):

- OK, but much of the complications come from “up to isomorphisms"
- $n = 5$: The first 3 correspond to the case $n = 4$.
  The last 4 are true eulerian graphs.
- **These belong to different switching classes!**



Theorem 8.7

For odd $|D|$, each $[g]$ has a unique even graph.

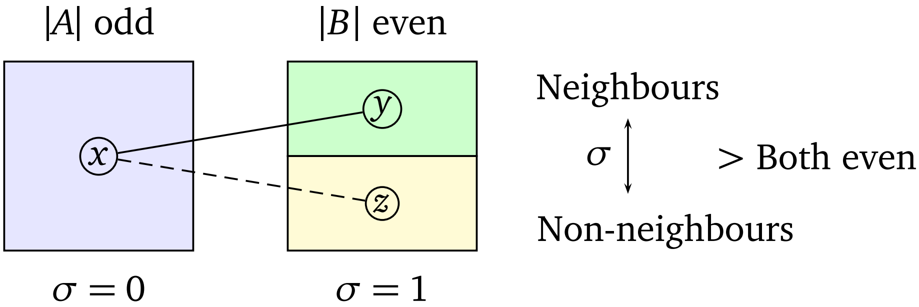
Proof. Let

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in B = \{x \mid d_g(x) \text{ odd}\}, \\ 0 & \text{if } x \in A = \{x \mid d_g(x) \text{ even}\}. \end{cases}$$

Claim. g^σ is even.

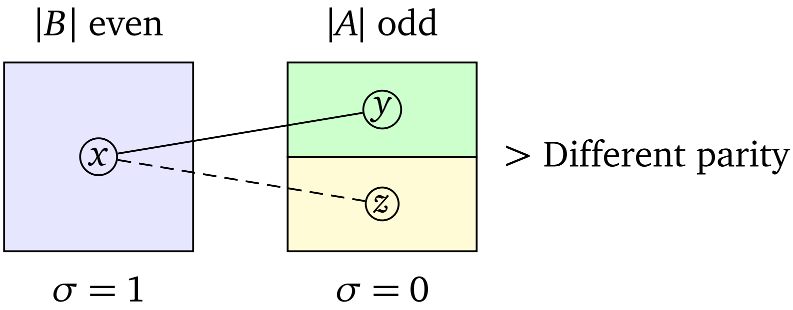
By Handshaking Lemma, $|B|$ is even, and so $|A|$ is odd.

Case $x \in A$ (even):



The parity of $x \in A$ does not change while switching.

Case $x \in B$ (odd):



The parity of $x \in A$ does change.

Uniqueness: Let $g^\sigma \neq g$.

- The even graphs are closed under sum:

  If both g and g^σ are even, so is $g + g^\sigma = \mathbb{O}_{OI}$
  (where $O = \sigma^{-1}(0)$ and $I = \sigma^{-1}(1)$).

- $\mathbb{O}_{OI}$ (complete bip) is even
  $\implies$ $|O|$ and $|I|$ are both even (as neighbourhoods)
  $\implies$ $|D| = |O| + |I|$ is even.

  Hence $[g]$ has a unique even graph, whenever $|D|$ is odd.

□

A graph g is **odd**, if the degrees are all odd.

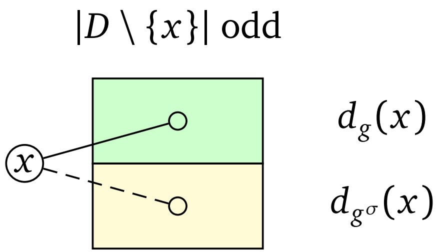
Theorem 8.8

> Let $|D| = n$ be **even**. Then either $[g]$ has no even and no odd graphs, or exactly half of its graphs are even and half odd.

Proof. Write $x \sim_g y$ if x and y have degrees of the same parity in g.

Let $\sigma = \sigma_x$ be elementary. Then

$$d_{g^\sigma}(y) = \begin{cases} (n-1) - d_g(y) & \text{if } y = x \\ d_g(y) + 1 & \text{if } g(x,y) = 0 \\ d_g(y) - 1 & \text{if } g(x,y) = 1 \end{cases}$$



- Every vertex changes its parity (including x). Thus $\sim_g$ and $\sim_{g^\sigma}$ agree.
- Let $\sigma = \sum_{i=0}^{k} \sigma_i$ for elementary σ_i.
- $\sim_g$ and $\sim_{g^\sigma}$ agree for all σ_i and thus for all σ.

  So $d_g(x)$ and $d_{g^\sigma}(x)$ have the same parity $\iff$ k is even.

$\square$