Combinatorial Structures 2019 — Problem set 3 Feb 7

Exercise 3.1. By definition an undirected graph g is a **cograph** (**complement reducible graph**) if it does not have any induced subgraph that is a path P_4 on four vertices.

Let g be an undirected graph. The following conditions are equivalent for g to be a cograph:

(1) Every non-singleton induced subgraph has a clan of cardinality two.

(2) The graph g has only complete special quotients $g[P]/\mathscr{P}_{\max}(g[P])$ for $P \in \mathscr{P}(g)$.

Solution. If g is primitive with $|D_g| \geq 4$, then it has a subgraph P_4, since there are no primitive undirected graphs of three vertices, and P_4 is the only primitive undirected graph of four vertices.

(1) Assume g is a cograph. Then it is non-primitive. Consider a smallest nontrivial clan X of g, and suppose $|X| > 2$. Then $g[X]$ contains P_4; a contradiction.

On the other hand, if g is not a cograph, it has P_4, and that is primitive.

(2) For undirected graphs the special quotients are either complete or primitive.

Each quotient is isomorphic to a subgraph, and the subgraphs of cographs are cographs. The special quotients $g[P]/\mathscr{P}_{\max}(g[P])$ miss P_4, i.e., $g[P]/\mathscr{P}_{\max}(g[P])$ is not primitive. Then they are complete.

For the converse, suppose $g[P]/\mathscr{P}_{\max}(g[P])$ is always complete for all prime clans P. If g has a P_4, it must be inside some $P \in \mathscr{P}(g)$. Let P be minimal with this property. Then the P_4 is in a maximal prime clan $Q \in \mathscr{P}_{\max}(g[P])$; a contradiction to minimality.

Exercise 3.2. Let Δ be a group.

(a) Assume that the order $|\Delta|$ is even. Then, by Cauchy's Theorem, Δ has an element a of order two: $a^2 = \varepsilon$. Show that the mapping $\delta_a(x) = ax^{-1}a$ $(x \in \Delta)$ is an involution.

(b) The group Δ is abelian if and only if the identity function $\iota_\Delta \colon \Delta \to \Delta$ is an involution.

Solution. (a) Calculation.

(b) True, since involutions are anti-homomorphisms.

Exercise 3.3. Let Δ^δ be a group with an involution δ.

(a) Show that either there exists a fixed point $a^\delta = a$ with $a \neq \varepsilon$, or Δ is of odd order and δ is the inversion of Δ.

(b) Show that either there exists an element $a \neq \varepsilon$ such that $a^\delta = a^{-1}$, or Δ is an abelian group of odd order and δ is the identity function.

Solution. (a) We have $(aa^\delta)^\delta = aa^\delta$, and hence aa^δ is a fixed point. Suppose $aa^\delta = \varepsilon$ for all a. Then $a^\delta = a^{-1}$ for all a. Hence δ is the inversion of Δ.

If $|\Delta|$ is even, it contains an element a of order two, $a = a^{-1}$ by Cauchy's theorem. In this case, we have a fixed point $a = a^\delta$.

(b) First $aa^\delta = bb^\delta$ implies

$$(a^{-1}b)^\delta = b^\delta(a^{-1})^\delta = b^{-1}aa^\delta(a^\delta)^{-1} = b^{-1}a = (a^{-1}b)^{-1},$$

and, in this case, there exists an element $c = a^{-1}b$ such that $c^\delta = c^{-1}$.

Suppose then that $aa^\delta \neq bb^\delta$ for all $a \neq b$. Let $\alpha\colon \Delta \to \Delta$ be defined by $\alpha(a) = aa^\delta$. It is a bijection, and hence $\Delta = \{aa^\delta \mid a \in \Delta\}$. Thus for each $b \in \Delta$ there exists an a such that $b = aa^\delta$. So for all b, $b^\delta = b$, and δ is the identity function. In this case, Δ is abelian. If $a^\delta \neq a^{-1}$ for all $a \neq \varepsilon$, then $|\Delta|$ is odd, since $\varepsilon^\delta = \varepsilon$ and the other elements come in pairs (a^δ, a^{-1}), where $a^\delta \neq a^{-1}$.

Exercise 3.4. Show that the centre $Z(\Delta) = \{a \mid ax = xa \text{ for all } x \in \Delta\}$ of a group Δ is closed under every involution of Δ: if $a \in Z(\Delta)$ and δ is an involution, then also $a^\delta \in Z(\Delta)$.

Solution. Let $a \in Z(\Delta)$, and let δ be an involution. Then

$$a^\delta x = (x^\delta a)^\delta \overset{Z}{=} (ax^\delta)^\delta = xa^\delta.$$

Exercise 3.5. (Cournier, Ille 1998) Let $g\colon E_2(D) \to \{0,1\}$ be a primitive undirected graph with $|D| \geq 4$, and let $x \in D$ be a vertex that is not contained in any induced path P_4.

(a) Show that $g[N_i(x)]$ is complete of colour i where $N_i(x) = \{y \mid g(x,y) = i\}$ for $i = 0,1$.

(b) Show that x is the only vertex with the property that it belongs to no induced P_4.

Solution. (a) Assume there exists a connected component X of colour 1 in $g[N_0(x)]$ with $|X| \geq 2$. By primitivity there exists a $y \in N_1(x)$ and $x_1, x_2 \in X$ such that $g(y, x_1) = 1$ and $g(y, x_2) = 0$. By looking at the connecting path between x_1 and x_2, we can assume without restriction that $g(x_1, x_2) = 1$. But now $\{x, y, x_1, x_2\}$ forms a P_4; a contradiction.

Hence $g[N_0(x)]$ is complete of colour 0. While considering the complement graph, we deduce that the connected components of $g[N_1(x)]$ of colour 0 are trivial, i.e., $g[N_1(x)]$ is complete of colour 1.

(b) Assume that both x and y elude all P_4's. By symmetry we can suppose that $g(x,y) = 1$. Since $g[N_1(x)]$ is complete for the colour 1 and $y \in N_1(x)$, $N_1(x) \subseteq N_1(y)$ (with x and y included there). By symmetry, also the other way round, and so $N_1(x) = N_1(y)$, which means that $\{x, y\}$ is a clan (twin); a contradiction.