Mathematics and Statistics
University of Turku

Combinatorial Structures 2019 **Problem set 4** Feb 14

Exercise 4.1. Let δ be an inversion of Δ. Show that, for all Δ^δ-graphs g,

$$g \in [\mathbb{O}] \iff g(x,y)g(y,z) = g(x,z) \text{ for all different vertices } x,y,z \in D\,.$$

Solution. Suppose first that $g = \mathbb{O}^\sigma$ for some σ. Then

$$\begin{aligned} g(x,y) &= \sigma(x)\mathbb{O}(x,y)\sigma(y)^{-1} = \sigma(x)\sigma(y)^{-1} \\ g(y,z) &= \sigma(y)\sigma(z)^{-1}, \text{ and so} \\ g(x,y)g(y,z) &= \sigma(x)\sigma(z)^{-1} = g(x,z). \end{aligned}$$

Assume then that g satisfies the condition. Let $x \in D$ be fixed with $\sigma(x) = \varepsilon$ (horizon), and $\sigma(y) = g(x,y)$ for all $y \neq x$. Now,

$$g^\sigma(x,y) = \varepsilon \cdot g(x,y) \cdot g(x,y)^{-1} = \varepsilon.$$

Moreover, given $y,z \in D \setminus \{x\}$, using the condition,

$$g^\sigma(y,z) = \sigma(y)g(y,z)\sigma(z)^{-1} = g(x,y)g(y,z)g(x,z)^{-1} = g(x,z)g(x,z)^{-1} = \varepsilon.$$

Exercise 4.2. Let Δ be a finite group of order $k \geq 2$. Show that if g is a Δ^δ-graph having a complete factor $g[X]$ for some X with $|X| \geq k+1$, then the switching class $[g]$ has no primitive Δ^δ-graphs.

Solution. Let σ be a selector. Since $|X| \geq k+1$, there are two elements $x_1, x_2 \in X$ such that $\sigma(x_1) = \sigma(x_2)$. For all $y \notin \{x_1, x_2\}$ (also for $y \in X \setminus \{x_1, x_2\}$), we have

$$g^\sigma(y,x_1) = \sigma(y)g(y,x_1)\sigma(x_1)^\delta = \sigma(y)g(y,x_2)\sigma(x_2)^\delta = g^\sigma(y,x_2).$$

Therefore g^σ has a clan $\{x_1, x_2\}$ and it is not primitive.

Exercise 4.3. Prove the following claims.

(a) The set $S(D)$ of the selectors $D \to \Delta$ forms an abelian group under addition; see page 42.

(b) Let $\sigma\colon D \to \mathbb{Z}_2$ be a selector where $\sigma^{-1}(1) = \{x_0, x_1, \dots, x_k\}$ for some $k \geq 0$. Denote by σ_i the elementary selector at x_i. Then

$$\sigma(x) = \sum_{i=0}^{k} \sigma_i(x)\,. \tag{4.1}$$

Therefore every selector is a sum of elementary selectors.

Solution. (a) In this group $\sigma(x) + \sigma(x) = 0$ for all $x \in D$. The zero element of $S(D)$ is the selector satisfying

$$\zeta(x) = 0$$

for all $x \in D$. Each selector is its own inverse.

(b) OK.

Exercise 4.4. An undirected graph $g\colon E_2(D) \to \mathbb{Z}_2$ is **even** (in the *eulerian sense*), if for all $x \in D$, $n_g(x) = |\{y \mid g(x,y) = 1\}|$ is even. Show that if both $g, h\colon E_2(D) \to \mathbb{Z}_2$ are even, so is their sum $g + h$.

> **Solution.** Suppose g and h are even, and let $x \in D$. Then $(g+h)(x,y) = g(x,y)+h(x,y)$ for the neighbours y of x. Now, $(g+h)(x,y) = 1$ if and only if $g(x,y) \neq h(x,y)$. We count: let $r = |\{y \mid g(x,y) = 1 = h(x,y)\}|$. Then $n_{g+h} = n_g - r + n_h - r$, which is even.

Exercise 4.5. We say that two subsets X and Y of a set D **cross** if they overlap and $X \cup Y \neq D$. Let g be a Δ-graph with $X, Y \in \mathscr{C}[g]$ be two crossing clans of the switching class $[g]$. Show that $X \cup Y, X \cap Y, X \setminus Y \in \mathscr{C}[g]$.

> **Solution.** By assumption, there exists a node $x \notin X \cup Y$. Let $h \in [g]$ be such that x is a horizon of h. By Theorem 7.5, $X, Y \in \mathscr{C}(h)$, and the claim follows from this. by a basic lemma.