Combinatorial Structures 2019 **Problem set 5** Feb 19

Exercise 5.1. Let $g\colon E_2(D) \to \mathbb{Z}_2$ and let $\alpha\colon D \to D$ be a permutation. Show that $\alpha([g]) = [\alpha(g)]$. In particular, if $\alpha(h) \in [g]$ for some $h \in [g]$, then α is an automorphism of $[g]$.

Solution. By Lemma 8.4, we have $\alpha(g^\sigma) = \alpha(g)^{\sigma\alpha^{-1}} \in [\alpha(g)]$ for each σ, and so $\alpha([g]) \subseteq [\alpha(g)]$. Now if $h \in [\alpha(g)]$, say $h = \alpha(g)^\tau$, then, again by Lemma 8.4, $h = \alpha(g^{\tau\alpha}) \in \alpha([g])$, and the claim follows.

Let Δ be a set of colours containing the special *symmetric* **zero label** 0. We do not assume that 0 occurs in a Δ-graph g, and so there is no essential restriction to the g's. Denote

$$E_g = \{e \mid g(e) \neq 0\}.$$

For a fixed edge $e \in E_g$, let $g{-}e$ be obtained by recolouring the edges e and e^{-1} by 0:

$$(g{-}e)(e') = \begin{cases} g(e') & \text{if } e' \notin \{e, e^{-1}\}, \\ 0 & \text{if } e' = e \text{ or } e' = e^{-1}. \end{cases}$$

So $E_{g-e} = E_g \setminus \{e, e^{-1}\}$. A truly primitive g is said to be **unstable** if $g{-}e$ is not primitive for all $e \in E_g$.

Exercise 5.2. Characterize all n-vertex unstable Δ-graphs for $n = 3$ and 4.

Solution. Systematic search.

Exercise 5.3. Let a Δ-graph g be **triangle-free**, i.e., every 3-subset $X = \{x_1, x_2, x_3\}$ has a zero edge $g(x_i, x_j) = 0$ for some $i \neq j$. Let

$$N_g(x) = \{y \mid (x, y) \in E_g\}.$$

Show that

(i) for each vertex x, either $N_g(x) = \emptyset$ or $g[N_g(x)]$ is discrete (all zero edges);
(ii) each $g[X]$, with $X \in \mathscr{C}(g)$, is discrete, or X is a union of connected components of g.

Solution. For (i), suppose that $N_g(x) \neq \emptyset$. If $y, z \in N_g(x)$, then (x, y) and (x, z) are non-zero edges, and thus $g(y, z) = 0$, since g is triangle-free.
For (ii), assume that X is a proper clan of g, which is not a union of connected components. This means that $N_g(X) \neq \emptyset$. Let $z \in N_g(X)$. Since $X \in \mathscr{C}(g)$, we have $X \subseteq N_g(z)$. By (i), $g[N_g(z)]$ is discrete and thus $g[X]$, as a subgraph of $g[N_g(x)]$, is also discrete.

Exercise 5.4. Assume that g is a triangle-free Δ-graph such that the associated undirected graph (D, E_g) is connected. Show that the maximal proper clans of g are disjoint. In particular, the quotient $g/\mathscr{P}_{\max}(g)$ is primitive.

Solution. Let X_1 and X_2 be two maximal proper clans. If they intersect, then $X_1 \cup X_2 \in \mathscr{C}(g)$, and hence $X_1 \cup X_2 = D$. Let $y \in X_1 \cap X_2 \in \mathscr{C}(g)$. By connectivity, there exists $x \in X_1 \setminus X_2$ (or symmetrically $x \in X_2$) such that $g(x, y) = a \neq 0$, and hence $g(x, z) = a$ for all $z \in X_2$. Since g is triangle free, $g[X_2]$ is discrete, and so $|X_2| = 2$ and $Y = \{z\}$. Now $g(z, y) = 0$ implies that $g(z, x) = 0$ for all $x \in X_1$ contradicting the connectivity.

The proof of the next result is bit complicated.

Theorem 5.1. *Each unstable Δ-graph is triangle-free.*

Exercise 5.5. Let $|D| \geq 3$ be a finite set.

(i) Show that for a family of 3-subsets Ω of D, Ω is a two-graph if and only if there exists $g: E_2(D) \to \mathbb{Z}_2$ such that $\Omega = \Omega(g)$.

(ii) Show that for $g, h: E_2(D) \to \mathbb{Z}_2$, $\Omega(g) = \Omega(h)$ if and only if $[g] = [h]$.

Solution. Let $g: E_2(D) \to \mathbb{Z}_2$, and consider the 4-sets with elements $\{1, 2, 3, 4\} \subseteq D$. Then do a systematic search of triangles in these vertices: $\Omega(g)$ is a two-graph.

Let then Ω be a two-graph on D. Fix $x \in D$. We construct a graph g_Ω as follows. Let $g_\Omega(x, y) = 0$ for all $y \neq x$, and let for all $y, z \neq x$,

$$g_\Omega(y, z) = 1 \iff \{x, y, z\} \in \Omega\,.$$

Hence, for each 3-subset $\{x, y, z\}$ of D, $g(\langle x, y, z\rangle) = 1$ if and only if $\{x, y, z\} \in \Omega$. Moreover, for all 3-subsets $\{y, z, u\}$ of $D \setminus \{x\}$,

$$g(\langle x, y, z\rangle) + g(\langle x, y, u\rangle) + g(\langle x, z, u\rangle) = g(\langle y, z, u\rangle)\,,$$

from which it follows that $g(\langle y, z, u\rangle) = 1$ if and only if $\{y, z, u\} \in \Omega$. Therefore $\Omega = \Omega(g_\Omega)$.

For (ii), suppose $\Omega(g) = \Omega(h)$, i.e., $g(\langle x, y, z\rangle) = h(\langle x, y, z\rangle)$ for all 3-subsets $\{x, y, z\}$ of D (same parity). Hence $[g] = [h]$ by Theorem 8.4 (and the fact that g and h are undirected). The converse claim follows similarly from Theorem 8.4.