~~PANCYCLIC~~ HAMILTONIAN SWITCHING CLASSES

Theorem

Let g be a graph of $n \geq 3$ vertices. Then $[g]$ contains a hamiltonian graph $\iff$ $[g] \neq [\mathbb{O}|$ with n odd.

For odd n, $\mathbb{O}_{AB} \in [\mathbb{O}]$ is never hamiltonian because you have to visit both sides A and B equally many times.

For even n, $\mathbb{O}$ has a switch $\mathbb{O}_{AB}$ with $|A| = |B|$, and this is hamiltonian.

Choose g so that it has the maximum number of edges in its switching class.

Hence

$$d_g(x) \geq (n-1)/2$$

for all x, since otherwise you switch at x.

(1) If n is even, then $d_g(x) \geq n/2$ for all x, and then g is hamiltonian by:

Theorem (Dirac-Ore 1952,1962)

Let g be a graph of $n \geq 3$ vertices. If

$$d_g(x) + d_g(y) \geq n$$

for all x, y with $g(x, y) = 0$, then g is hamiltonian.

Recall

$$N_g(x) = \{y \mid g(x,y) = 1\}.$$

(2) Assume n is odd, and choose any $x \in D$.
Then $g[D \setminus \{x\}]$ is even and it has a Hamilton cycle:

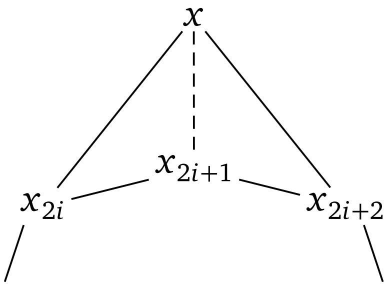
$$x_1 \to x_2 \to \ldots \to x_{n-1}.$$

If, for some i (modulo $n-1$), $g(x, x_i) = 1 = g(x, x_{i+1})$, then

$$x_1 \to \ldots \to x_i \to x \to x_{i+1} \to \ldots \to x_{n-1}.$$

is a Hamilton cycle of g. The same holds, if $g(x, x_i) = 0 = g(x, x_{i+1})$, since we can make a switch σ_x at x.
Suppose not.

Then $N_g(x) = \{x_{2i} \mid i = 1, 2, \ldots\}$
(or $N_g(x) = \{x_{2i+1} \mid i = 1, 2, \ldots\}$).

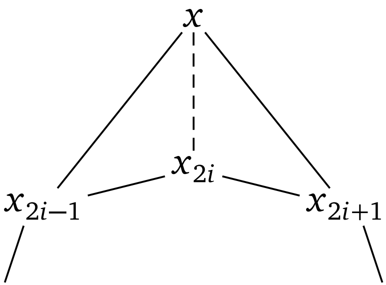


Replacing each x_{2i+1} by x,

$$x_1 \to \ldots x_{2i} \to x \to x_{2i+2} \to \ldots \to x_{n-1}$$

we obtain

$$N_g(x_{2i+1}) = \{x_{2i} \mid i = 1, 2, \ldots\} \; (= N_g(x))$$



In g^{σ_x} the vertex x is adjacent to all odd vertices. Now,

$$N_g(x_{2i}) = \{x_{2i+1} \mid i = 0, 1, \ldots\} \; (= N_{g^{\sigma_x}}(x))$$

But now, g is an odd complete bipartite graph, with $[g] = [\mathbb{O}]$. $\square$