COMBINATORIAL STRUCTURES IN GRAPH THEORY

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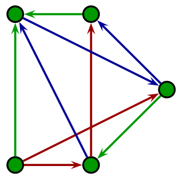
The Aim

The structure of Δ-graphs, i.e., edge coloured directed graphs.

Framework for decomposition and transformation of systems with binary relations.



- Colours are usually represented by letters $a, b, \ldots$
- The graphs will be complete because it is no restriction: a missing connection is a colour of its own.

Decompositions

There are various methods to decompose graphs and related structures.

Decompositions of combinatorial and algebraic structures (groups, rings, and, indeed, general algebras) employ the divide-and-conquer method:

(1) a large problem is partitioned into smaller parts.

(2) A method to retrace a solution of the original problem.

Clan decomposition, or modular decomposition, of graphs is closely related to the decomposition by quotients in algebra:

$$g \mapsto \begin{Bmatrix} g/\mathscr{X} \\ \mathscr{X} \end{Bmatrix} \mapsto g$$

Two topics

- The **static part**: the decomposability and indecomposability (i.e., primitivity).

  The key notion: **clan** – a subset X of elements (vertices) such that no element $y \notin X$ distinguishes elements of X by the colours.

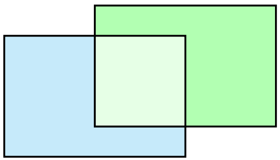
- The **dynamic part**: local transformation of **switching**.
  The colours form a group, and graphs are transformed to graphs.

Notation

- Sets of numbers: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$.
- The **cardinality** of a finite set X, denoted by $|X|$ or $\#X$, is the number of its elements.
  - A set X with k elements ($|X| = k$) is called a **k-set**.
  - Singletons $\{x\}$ are often identified with the sole member x.
- Subsets Y and Z are **comparable** if $Y \subseteq Z$ or $Z \subseteq Y$.

  Otherwise they are disjoint ($X \cap Y = \emptyset$) or they **overlap**:

$$X \cap Y \neq \emptyset, \quad Y \setminus Z \neq \emptyset, \quad Z \setminus Y \neq \emptyset.$$



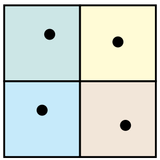
- $2^Z = \{Y \mid Y \subseteq Z\}$ is the **power set** of Z.
- For a family $\mathscr{X} \subseteq 2^X$ of sets, let

$$\begin{aligned} \cup\mathscr{X} &= \bigcup_{Y\in\mathscr{X}} Y = \{x \mid \exists Y \in \mathscr{X} :\ x \in Y\}, \\ \cap\mathscr{X} &= \bigcap_{Y\in\mathscr{X}} Y = \{x \mid \forall Y \in \mathscr{X} :\ x \in Y\}. \end{aligned}$$

- Let $\mathscr{X} = \{X_i \mid i \in I\} \subseteq 2^X$ be a **partition** of X, i.e., its sets are pairwise disjoint and $X = \cup\mathscr{X}$.

  A subset $T \subseteq X$ is a **transversal** of $\mathscr{X}$, if $|T \cap X_i| = 1$ for all $i \in I$.

  Equivalently, an injective function $\tau : \mathscr{X} \to X$ is a **transversal**, if $\tau(X_i) \in X_i$ for all i.



- ι_X is the **identity function** on X. So $\iota_X(x) = x$.

  If $\alpha : X \to Y$ is a bijection, then $\alpha^{-1}\alpha = \iota_X$ and $\alpha\alpha^{-1} = \iota_Y$.

- For $\alpha : X \to Y$ and $Z \subseteq X$, the **restriction** to Z:

$$\alpha \restriction Z(x) = \alpha(x) \text{ for all } x \in Z.$$

  The **image** and the **co-image**:

$$\alpha(Z) = \{\alpha(z) \mid z \in Z\} \text{ and } \alpha^{-1}(U) = \{x \mid x \in X,\ \alpha(x) \in U\}.$$

- The **kernel** of $\alpha : X \to Y$:

$$\ker(\alpha) = \{(x, y) \in X \times X \mid \alpha(x) = \alpha(y)\}.$$

Undirected graphs

Let

$$E_2(D) = \{(x, y) \mid x, y \in D \text{ with } x \neq y\}$$

be the set of all ordered *non-reflexive* pairs of a set D.

An **undirected graph** $g = (D, E)$ has a nonempty finite set D of **vertices** and a set of **edges** $E \subseteq E_2(D)$ satisfying

$$(u, v) \in E \iff (v, u) \in E\,.$$

for all $u, v \in D$.

Example. The undirected graph $K_D = (D, E_2(D))$ is **complete** on D. □

Characteristic function and Δ-graphs

$g = (D, E)$ can be identified with the **characteristic function** of E:

$$g : E_2(D) \to \{0,1\}, \quad g(e) = \begin{cases} 1 & \text{if } e \in E, \\ 0 & \text{if } e \notin E. \end{cases}$$

We generalize to arbitrary colour sets.

For $e = (x, y) \in E_2(D)$, let $e^{-1} = (y, x)$ be its **reverse**.

Let Δ be a set of **colours** (or **labels**).

A Δ-graph on the **domain** $D = D_g$ is a function

$$g : E_2(D) \to \Delta\,,$$

such that, for all $e_1, e_2 \in E_2(D)$,

$$g(e_1) = g(e_2) \iff g(e_1^{-1}) = g(e_2^{-1})$$

Reversibility condition

Remarks

- The reversibility condition $g(e_1) = g(e_2) \iff g(e_1^{-1}) = g(e_2^{-1})$ is adopted only for simplicity.

  Reversibility implies an **involution** $\delta : \Delta \to \Delta$ of the colours:

  $$\delta^2 = \iota \quad \text{and} \quad g(e^{-1}) = \delta(g(e)) \,.$$

- Later when Δ has a group structure δ will satisfy the condition $\delta(ab) = \delta(b)\delta(a)$ with respect to the given group operation.

- A colour $a \in \Delta$ is **symmetric**, if $a = \delta(a)$.

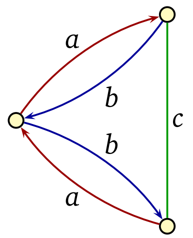
Examples

- $\Delta = \{0, 1\}$: **Undirected graph** if both colours are symmetric (with ambiguity: $0/1$). Otherwise a **tournament**.

- $\Delta = \{a, b, c\}$: One colour, say c, must be symmetric, $\delta(c) = c$, a and b *can be* reverse colours, $b = \delta(a)$ (and so $\delta(a) = b$).

  In this case a Δ-graph presents an **oriented graph**:

$$g(x, y) = \begin{cases} a & \text{if } (x, y) \text{ is an edge,} \\ b & \text{if } (y, x) \text{ is an edge,} \\ c & \text{otherwise.} \end{cases}$$



In general

- $\Delta = \{a, b, c, d\}$:

Every **directed graph** can be presented as a Δ-graph, where

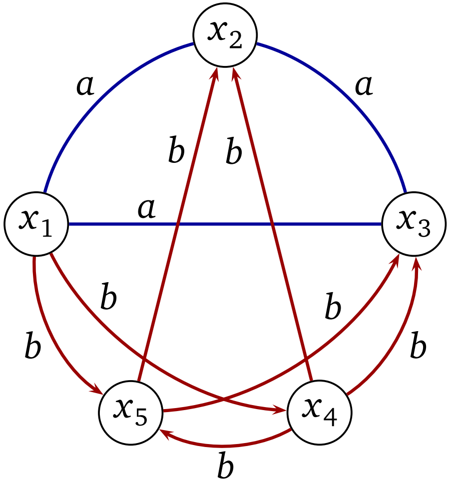
- a is symmetric: an edge in both directions,
- b is symmetric: a non-edge in both direction,
- and $c = \delta(d)$: an oriented edge.

Remarks

- **Ambiguity**: The definition distinguish edges and non-edges without an interpretation. So Δ is *up to isomorphism*.
- A Δ-graph g is drawn by representing the vertices as points in the plane and the edges as connecting arcs together with the label $g(e)$.
- The picture can be complicated, but reversibility makes things more convenient: we can
  - omit the reverse colours of the chosen non-symmetric colours,
  - draw a line without arrow heads for symmetric colours, and
  - omit one symmetric colour.

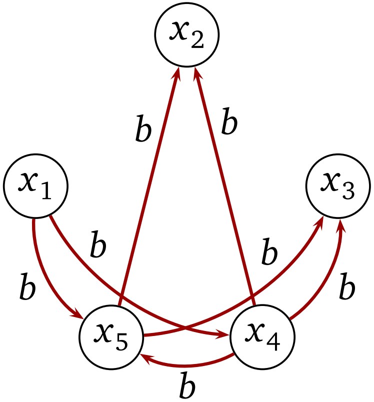
Example

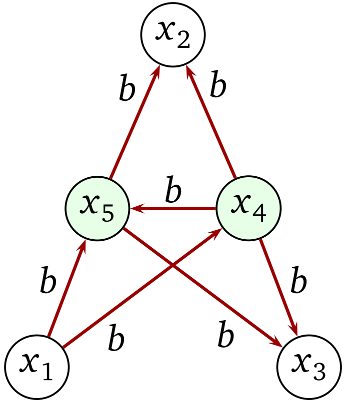
Consider the Δ-graph g on the domain $D = \{x_1, x_2, x_3, x_4, x_5\}$, where $\Delta = \{a, b, c\}$ with with $\delta(a) = a$ and $\delta(b) = c$ (and hence $\delta(c) = b$).



You do not see the reverse colour c. We could have omitted the edges having the symmetric colour a, but we didn't.

There you are

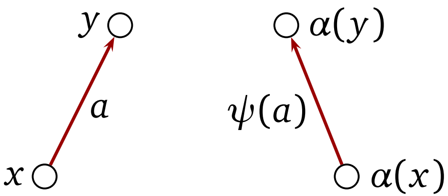




Isomorphisms and subgraphs

- g and h are **isomorphic**, $g \cong h$, if there are bijections $\alpha \colon D_g \to D_h$ and $\psi \colon \Delta_g \to \Delta_h$ such that

$$\psi(g(x,y)) = h(\alpha(x), \alpha(y)) .$$



- $h \colon E_2(A) \to \Delta_h$ with $A \subseteq D$ is a **subgraph** of $g \colon E_2(D) \to \Delta_g$, denoted

$$h = g[A]$$

if $\Delta_h \subseteq \Delta_g$, and $h = g \restriction A$.

In particular,

$$g[A](e_1) = g[A](e_2) \iff g(e_1) = g(e_2) \text{ for all } e_1, e_2 \in E_2(A) .$$