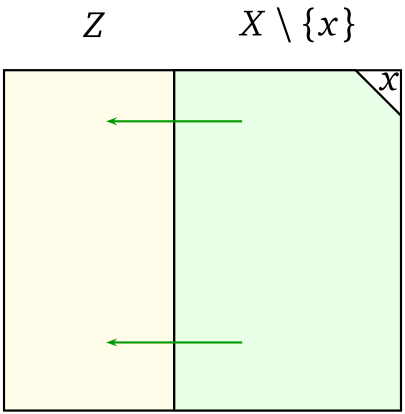
Thm 5.1 Every truly primitive Δ-graph g has a primitive subgraph of 3 or 4 vertices.

Suppose not: Assume that g has no proper, truly primitive subgraphs. (Contradiction + reduction on the size proves the claim.)

We may assume that $|D_g| \geq 5$.

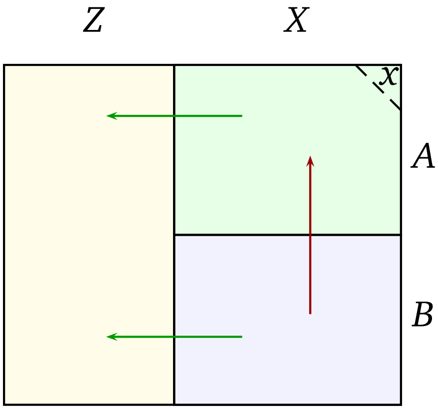
UNP= uniformly non-primitive

$$h = g[D \setminus \{x\}]$$



- h UNP: partition to Z, $X \setminus \{x\} \in \mathscr{C}(h)$. Here $x \in X$.

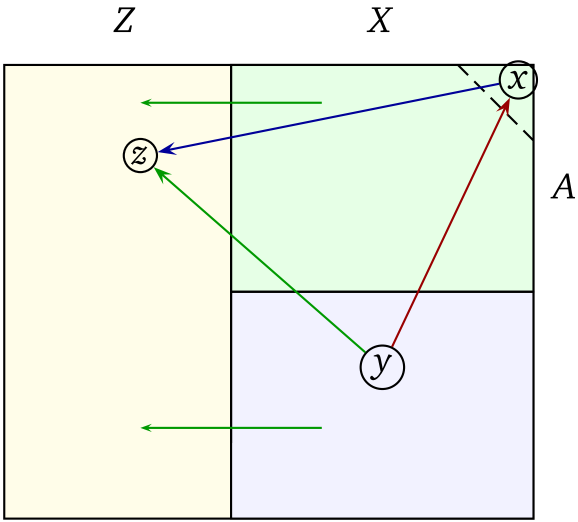
  One of them has at least two vertices, say $X \setminus \{x\}$.



- $x \in X$ and $|X| \geq 3$

  $\therefore g[X]$ UNP: partition $A, B \in \mathscr{C}(g[X])$.
  Assume $x \in A$.

  $\therefore B \in \mathscr{C}(g)$
  $\therefore B = \{y\}$ since g is primitive.



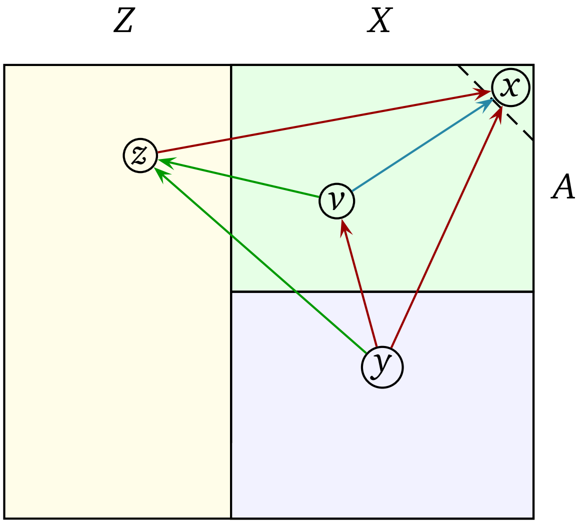
- $X \notin \mathscr{C}(g)$
  $\therefore\ \exists z \in Z : g(z,x) \neq g(z,y)$, i.e.,

  Blue $\neq$ Green

  $\therefore\ \{z,y\} \in \mathscr{C}\{g[x,z,y]\}$

  and so Blue=Red^{-1}

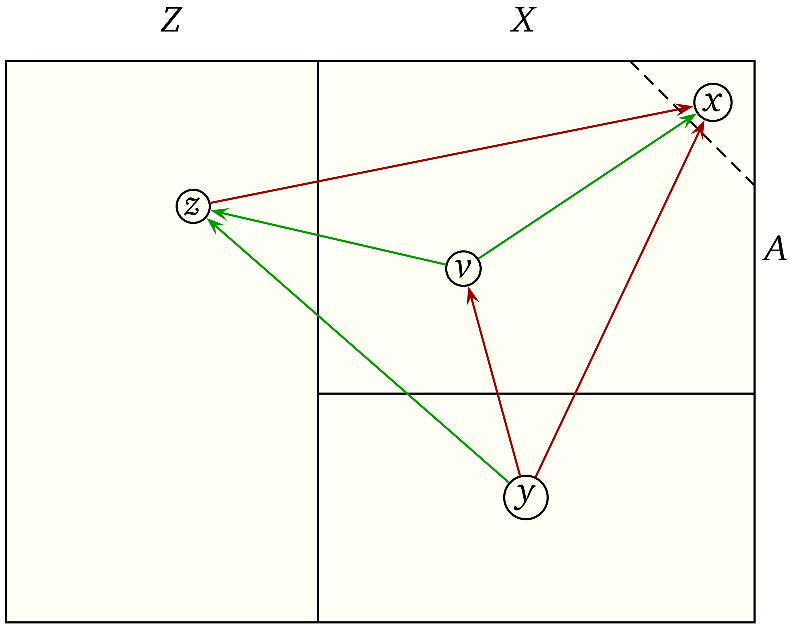
- Note: $A \notin \mathscr{C}(g)$ and so
  $\{z,x\} \notin \mathscr{C}\{g[x,z,y]\}$.



$X \setminus \{x\} \notin \mathscr{C}(g)$

$\therefore\ \exists v \in A : \text{Red} \neq \text{Cyan}$

$\therefore\ \text{Cyan} = \text{Green (by } \{x, z, v\})$



But now $g[x, y, z, v]$ is primitive