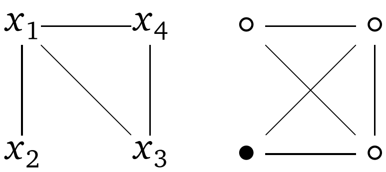
CLOSURES OF CLANS

The closure properties of clans of the switching classes differ from those of single Δ-graphs.

Example. The intersection of two clans $X, Y \in \mathscr{C}(g)$ is always a clan.
This is **not so** for switching classes.

Indeed, consider $\Delta = \mathbb{Z}_2$, and the next g together with a switch g^σ:



- $X = \{x_2, x_3, x_4\} \in \mathscr{C}(g)$,
  $Y = \{x_1, x_2, x_3\} \in \mathscr{C}(g^\sigma$.
  Both in $\mathscr{C}[g]$.
- But the intersection $X \cap Y = \{x_2, x_3\} \notin \mathscr{C}[g]$.

On the other hand, say that two subsets X and Y of a set D **cross** if they overlap and $X \cup Y \neq D$.

Exercise. Let g be a Δ-graph with $X, Y \in \mathscr{C}[g]$ be two crossing clans of $[g]$.
Show that $X \cup Y, X \cap Y, X \setminus Y \in \mathscr{C}[g]$.

TOWARDS DECOMPOSITION – AGAIN

In analogy to individual Δ-graphs, let, for $[g]$,

$$G[g] = (D, E)$$

where

$$E = \{(x, y) \in E_2(D \mid \{x, y\} \in \mathscr{C}[g]\}.$$

Lemma:

> Let $\{x, y_1\}, \{x, y_2\}, \{x, y_3\} \in \mathscr{C}[g]$ be different 2-node clans, then $\{x, y_1, y_2, y_3\}$ induces a complete subgraph of $G[g]$.

Proof. **1**. If $D = \{x, y_1, y_2, y_3\}$, then $\{y_1, y_2\} = D \setminus \{x, y_3\} \in \mathscr{C}[g]$. Similarly, $\{y_2, y_3\}, \{y_1, y_3\} \in \mathscr{C}[g]$ and so $G[g]$ is a complete graph.

2. Suppose then $|D| \geq 5$.

The clans $\{x, y_1\}$ and $\{x, y_2\}$ cross $\Longrightarrow$ $\{x, y_1, y_2\} \in \mathscr{C}[g]$.

The clans $\{x, y_1, y_2\}$ and $\{x, y_3\}$ cross $\Longrightarrow$ $\{y_1, y_2\} \in \mathscr{C}[g]$.

Symmetrically, $\{y_1, y_3\}, \{y_2, y_3\} \in \mathscr{C}[g]$, and thus the claim. ☐

Theorem

> Let g be a Δ^δ-graph. If $G[g]$ is connected, then it is complete or a cycle.

Proof. Suppose $|D| \geq 4$.

By the lemma, if $G[g]$ has a vertex of degree ≥ 3, then $G[g]$ is complete.

Suppose that the degrees are ≤ 2.

By connectivity, $G[g]$ is either a cycle or a path.

In both cases, $D = \{x_1, \ldots, x_n\}$ so that (x_i, x_{i+1}) are the edges of $G[g]$.

Now $\{x_2, \ldots, x_{n-1}\} \in \mathscr{C}[g]$, since $\{x_2, \ldots, x_k\}$ crosses with $\{x_k, x_{k+1}\}$ for each $k = 3, \ldots, n-1$.

Hence $\{x_1, x_n\} = D \setminus \{x_2, \ldots, x_{n-1}\} \in \mathscr{C}[g]$.

Therefore $G[g]$ is a (hamiltonian) cycle. □

QUOTIENTS OF SWITCHING CLASSES

A **factorization** of $[g]$ is a partition D into clans $\mathscr{X} \subseteq \mathscr{C}[g]$.

These are not quite as straightforward as for the individual Δ-graphs, because the clans in $\mathscr{X}$ may come from different switches g^σ of g.

The **quotient** of $[g]$ **by** $\mathscr{X}$ is

$$[g]/\mathscr{X} = \{h/\mathscr{X} \mid h \in [g],\ \mathscr{X} \text{ a factorization of } h\}.$$

We shown that $[g]/\mathscr{X}$ is, **indeed**, a switching class $[h/\mathscr{X}]$ for some $h \in [g]$.

If σ is a constant selector, then $g \cong g^\sigma$. Hence

Lemma

Let g be a Δ^δ-graph and σ a **constant selector**.

- For each nonempty subset $X \subseteq D$, the subgraphs $g[X]$ and $g^\sigma[X]$ are isomorphic.
- For each factorization $\mathscr{X}$ of g, the quotients $g/\mathscr{X}$ and $g^\sigma/\mathscr{X}$, are isomorphic.

Assume $\mathscr{X}$ is a factorization of g.

For a selector $\sigma : D \to \Delta$, let $\widehat{\sigma} : \mathscr{X} \to D$ be determined by

$$\widehat{\sigma}(X) = \sigma(x) \ \text{ if } x \in X \ \text{ in } \mathscr{X}.$$

It should be clear that

$$\boxed{g^{\sigma}/\mathscr{X} = (g/\mathscr{X})^{\widehat{\sigma}}}$$

simply because $g^{\sigma}/\mathscr{X}$ is isomorphic to the substructure $g[T]$, where T is a transversal of $\mathscr{X}$.

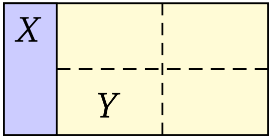
Also, the reverse holds: given $\widehat{\sigma}$, the selector σ can be recovered.

THEOREM

Let $\mathscr{X}$ be a (proper) factorization of the switching class $[g]$. There exists a selector σ such that $\mathscr{X}$ is a factorization of g^σ, and

$$[g]/\mathscr{X} = [g^\sigma/\mathscr{X}].$$

Proof. Denote $\overline{X} = D \setminus X$ for a $X \in \mathscr{X}$.



Isolate X: there exists σ with $X, \overline{X} \in \mathscr{C}(g^\sigma)$.

Let $Y \in \mathscr{X} \setminus \{X\}$. Hence $Y \subseteq \overline{X}$.

Lemma 7.5(ii): since $\overline{X} \in \mathscr{C}(g^\sigma)$ and $Y \in \mathscr{C}((g^\sigma)^\sigma) = \mathscr{C}(g)$ then $Y \in \mathscr{C}(g^\sigma)$.

So: $\mathscr{X}$ **is a factorization of** g^σ and $g^\sigma/\mathscr{X}$ is well defined.

Assume that $\mathscr{X}$ is a factorization of some $h \in [g]$ (among with g^σ).

Since $[g] = [g^\sigma]$, there is a τ with $h = (g^\sigma)^\tau$.

- Now, each $X \in \mathscr{X}$ is a clan in both g^σ and $(g^\sigma)^\tau$.
  By Lemma 7.5(i), τ is **constant on each** $X \in \mathscr{X}$.
- Hence $h/\mathscr{X} = (g^\sigma/\mathscr{X})^{\widehat{\tau}}$. **Consequently**, $h/\mathscr{X} \in [g^\sigma/\mathscr{X}]$.
- Finally, let $\widehat{\tau} \colon \mathscr{X} \to \Delta$ be given.
  By Lemma 7.5(ii), $\mathscr{X}$ is a factorization of $g^{\tau\sigma}$.
  Since $g^{\tau\sigma}/\mathscr{X} = (g^\sigma/\mathscr{X})^{\widehat{\tau}}$,

$$[g]/\mathscr{X} = \{(g^\sigma/\mathscr{X})^{\widehat{\tau}} \mid \widehat{\tau} \text{ a selector}\}$$

  as required.

$\square$