Compactness of Systems of Equations on Completely Regular Semigroups*

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Abstract. A semigroup S is said to have the compactness property, or CP for short, if each system of equations over a finite set of variables has an equivalent finite subsystem, that is, having exactly the same solutions in S. We prove that a completely 0-simple semigroup S satisfies CP if and only if the group G in a Rees matrix representation $S = \mathcal{M}^0(I, G, \Lambda; P)$ satisfies this property. Further, a variety of completely regular semigroups in which the finitely generated members have finite matrices is shown to satisfy CP if and only if the corresponding variety of groups satisfies CP. It is then shown that the varieties of bands of groups satisfy the condition on finite matrices.

1 Introduction and Summary

A topological space is compact if every open cover of it has a finite subcover. A most natural analogue of this statement in the context of semigroup equations is the following. A semigroup S has the *compactness property* if any system of semigroup equations $\mathcal{E}$ over a finite set of variables is equivalent to a finite subsystem $\mathcal{F}$ of $\mathcal{E}$, that is, $\mathcal{E}$ and $\mathcal{F}$ have the same set of solutions in S. There is an alternative, nonequivalent, concept which also claims to be a rightful analogue of compactness in topology for the semigroup setting, namely: a semigroup S is *equationally compact*, see [6], if every system of equations $\mathcal{E}$ has a solution in S whenever all of its finite subsystems have a solution in S. This latter notion is actually an analogue of the contrapositive of the definition of compactness in topology: a collection of open sets none of whose finite subcollections covers a set, does not cover the set either.

In order to avoid confusion, we retain the above definition of the compactness property even though in view of the above arguments, it represents a closer analogue of compactness in topology than the concept of equational compactness.

In this article we study the compactness property for completely regular and completely (0-)simple semigroups. This is motivated not only by the rich and well-developed theory of these semigroups, but partly also by an immediate corollary to results of [8] and Andersen's theorem, see [4], stating that if a

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simple semigroup satisfies CP, then it is completely simple (and thus completely regular).

We conclude this introduction with a short synopsis of our results. In Sect. 2 we give a formal definition of the compactness property, and present some earlier results needed in later sections. In Sect. 3 we state the basic definitions in completely regular semigroups.

Section 4 is devoted to the general study of the compactness property in the varieties of completely regular semigroups. We generalize a result from [8] for these varieties: a variety $\mathcal{V}$ of completely regular semigroups satisfies CP if and only if the finitely generated members in $\mathcal{V}$ satisfy the ascending chain condition for congruences.

In Sect. 5 it is shown that if a semigroup S satisfies CP, so do its Rees matrix semigroups $\mathcal{M}^0(I, S, \Lambda; P)$. The converse of this result holds in the case when S is a group, and in particular it follows that a completely simple semigroup satisfies CP just in case its subgroups do. This line of thought is continued in Sect. 6, but now for the varieties of completely regular semigroups. It is shown that if $\mathcal{V}$ is a variety of completely regular semigroups, where the finitely generated members have finite matrices, then $\mathcal{V}$ satisfies CP if and only if the variety $\mathcal{V} \cap \mathcal{G}$ satisfies CP. As an example the variety $\mathcal{BG}$ of bands of groups satisfies the condition on finite matrices.

2 Systems of Equations

A *semigroup equation* $u = v$ is a pair of words over a finite alphabet $X = \{x_1, \ldots, x_n\}$, say, the elements of which are called *variables*. A *solution* of an equation $u = v$ *in a semigroup* S is an n-tuple $a_1, a_2, \ldots, a_n$ of elements of S such that $u(a_1, \ldots, a_n) = v(a_1, \ldots, a_n)$. Equivalently, a solution of $u = v$ is a morphism $\alpha \colon X^+ \to S$ from the free semigroup X^+ generated by X into S such that $u\alpha = v\alpha$. Hence an equation generalizes the notion of an *identity*; indeed, an identity $u = v$ is satisfied by S if for all morphisms $\alpha \colon X^+ \to S$, we have $u\alpha = v\alpha$, that is, the solution set of the equation $u = v$ is $S^n = S \times \ldots \times S$.

Let S be a semigroup, X a finite set of variables and $\mathcal{E}$ a system of equations $\{u_i = v_i\}_{i \in I}$ over X. We say that $\mathcal{E}$ is *equivalent (in S) to its subsystem* $\mathcal{F} \subseteq \mathcal{E}$, if $\mathcal{E}$ and $\mathcal{F}$ have the same set of solutions, *i.e.*, if $\alpha \colon X^+ \to S$ is a solution of all $u = v \in \mathcal{F}$, then α is a solution of all $u = v \in \mathcal{E}$.

A semigroup S is said to satisfy the *compactness property*, or *CP* for short, if for all finite sets of variables X every system $\mathcal{E}$ over X is equivalent in S to one of its finite subsystems $\mathcal{F} \subseteq \mathcal{E}$.

The set X of variables is always supposed to be finite. Indeed, for infinite sets of variables the compactness property, as defined above, reduces to a triviality, since any finite (sub)system of equations contains only finitely many variables. From the finiteness of X it also follows that each system of equations over X contains denumerably many equations, and therefore each (infinite) system $\mathcal{E}$ can be enumerated, $\{u_i = v_i\}_{i \geq 1}$.

A class $\mathcal{C}$ of semigroups is said to satisfy the *compactness property* if every semigroup $S \in \mathcal{C}$ satisfies it. Further, the class $\mathcal{C}$ satisfies the compactness property *uniformly* if for each system $\mathcal{E}$ of equations there exists a finite subsystem $\mathcal{F}$ such that $\mathcal{F}$ is equivalent to $\mathcal{E}$ in all $S \in \mathcal{C}$, that is, if the semigroups $S \in \mathcal{C}$ have a common equivalent finite subsystem for each $\mathcal{E}$.

As an example, all finite semigroups satisfy CP, but as shown in [8], the class of finite semigroups does not satisfy CP uniformly. The variety of all semigroups does not satisfy CP, but all free semigroups do. Indeed, this is the most striking example, known as the *Ehrenfeucht's compactness theorem*, which was proved independently by Albert and Lawrence [2] and Guba [7]. In [7], see also [5], it was also shown that free groups satisfy CP.

As examples of individual semigroups that do not satisfy CP, we mention the bicyclic semigroup and the free inverse semigroups. Both of these examples follow from a result in [8] stating that if a finitely generated semigroup S satisfies CP, then every chain of idempotents of S is finite.

As it is easy to see, the subsemigroups and finite direct products of semigroups satisfying CP satisfy CP. On the other hand, CP is not closed under taking quotients since the free semigroups do satisfy CP, but *e.g.* the bicyclic semigroup does not. Indeed, as the following example shows, CP is not preserved even for taking Rees quotients.

Example 1. We give an example in which a finitely generated semigroup S and its ideal I both satisfy CP but the Rees quotient S/I does not. For this let A^+ be the free semigroup generated by $A = \{a, b\}$ and define

$$L = \{ab^k ab^j a \mid 1 \le j \le k\} \quad \text{and} \quad I = A^* L A^* \ .$$

Clearly, I is an ideal of A^+. The free semigroup A^+ satisfies CP and I satisfies CP since it is a subsemigroup of A^+. Let $X = \{x, y, z\}$ be a set of variables and

$$\mathcal{E} = \{xy^k xz^j x = xy^k xz^j xx \mid k, j \ge 1\} \ .$$

Let $\mathcal{F} \subseteq \mathcal{E}$ be any finite subsystem. Set

$$m = \max\{j \mid xy^k xz^j x = xy^k xz^j xx \in \mathcal{F}\} \ ,$$

and define $\alpha \colon X^+ \to A^+/I$ by $x\alpha = a$, $y\alpha = b^m$, $z\alpha = b$. Now, by the definition of L, we have $(xy^k xz^j x)\alpha = (xy^k xz^j xx)\alpha$ in S/I for each $xy^k xz^j x = xy^k xz^j xx$ from $\mathcal{F}$, but $(xyxz^{m+1}x)\alpha \ne (xyxz^{m+1}xx)\alpha$, and therefore $\mathcal{F}$ is not equivalent to $\mathcal{E}$. This proves the claim. $\square$

Other examples of semigroups in which CP is known to fail are the nonhopfian semigroups and groups such as the Baumslag-Solitar group, see [8].

We conclude this section with a few technical comments on the compactness property. For this let $u \in X^+$ be a word. The *content* $c(u)$ of u is the set of variables that occur in u, that is, $x \in c(u)$ if and only if $u = u_1 x u_2$ for some words $u_1, u_2 \in X^*$. We denote by $h(u)$ and $t(u)$ the *head* and the *tail* of u, that is, $h(u), t(u) \in X$ and $u \in h(u)X^* \cap X^*t(u)$. Also denote by

$$f(u) = \{xy \mid x, y \in X,\ xy \text{ occurs in } u\}$$

the set of all *2-factors* of u. Define a relation $\sim$ on X^+ by

$$u \sim u' \iff h(u) = h(u'),\ f(u) = f(u'),\ t(u) = t(u') \ .$$

Lemma 1. *Let X be a finite set of variables. The relation $\sim$ is a congruence on X^+ and the quotient $X^+/\sim$ is a finite semigroup.*

Proof. The congruence property of $\sim$ is obvious. For the finiteness of $X^+/\sim$ it is enough to observe that there are only finitely many triples (x, F, y), where $x, y \in X$ and $F \subseteq X^2$, and each such triple (x, F, y) defines uniquely an equivalence class of the relation $\sim$. □

We say that a family $U = \{u_i\}_{i \geq 1}$ of words over X is *locally uniform* if $u_i \sim u_j$ for all i and j. Clearly, if U is locally uniform, then also $c(u_i) = c(u_j)$ for all i, j. A system $\{u_i = v_i\}_{i \geq 1}$ of equations over X is *locally uniform* if both families $\{u_i\}_{i \geq 1}$ and $\{v_i\}_{i \geq 1}$ are locally uniform.

Lemma 2. *A semigroup S satisfies CP if and only if each locally uniform system of equations has a finite equivalent subsystem.*

Proof. The direct part of the lemma is trivial. For the converse statement consider any system $\mathcal{E} = \{u_i = v_i\}_{i \geq 1}$ of equations over a finite set X. By Lemma 1, the sets $U = \{u_i\}_{i \geq 1}$ and $V = \{v_i\}_{i \geq 1}$ are partitioned into finitely many congruence classes $U = U_1 \cup \ldots \cup U_r$ and $V = V_1 \cup \ldots \cup V_s$ by the relation $\sim$, and therefore $\mathcal{E}$ has a finite partition into $\mathcal{E} = \cup_{i,j} \mathcal{E}_{ij}$, where $\mathcal{E}_{i,j} = \mathcal{E} \cap U_i \times V_j$. By the construction each $\mathcal{E}_{ij}$ is locally uniform, and by assumption $\mathcal{E}_{ij}$ has a finite equivalent subsystem, say $\mathcal{F}_{ij}$. It is immediate that $\mathcal{F} = \cup_{i,j} \mathcal{F}_{ij}$ is equivalent to $\mathcal{E}$ in S, and since this union is finite, the claim follows. □

For a semigroup S let S^0 be the semigroup where a new zero element is adjoined.

Lemma 3. *A semigroup S satisfies CP if and only if S^0 satisfies CP.*

Proof. Suppose that S satisfies CP. Let $\mathcal{E} = \{u_i = v_i\}_{i \geq 1}$ be a locally uniform system of equations over X as in Lemma 2, and let $\mathcal{F} \subseteq \mathcal{E}$ be a finite equivalent subsystem in S. Let $\alpha: X^+ \to S^0$ be a morphism. Since $c(u_i) = c(u_j)$ for all i and j, $u_i\alpha = 0$ for some i, then $u_i\alpha = 0$ for all i. The case for the right hand sides v_i is similar. Therefore if $u_1\alpha = v_1\alpha$, then either $u_i\alpha, v_i\alpha \in S$ for all i or $u_i\alpha = 0 = v_i\alpha$ for all i. Thus $\mathcal{F} \cup \{u_1 = v_1\}$ is equivalent to $\mathcal{E}$ in S^0.

The converse part is immediate since S is a subsemigroup of S^0. □

The corresponding result for S^1, where an identity element 1 is adjoined to S (if it is not a monoid), was shown in [8]: A semigroup S satisfies CP if and only if the monoid S^1 satisfies CP.

3 Some Terminology on Semigroups

For the results in theory of semigroups stated in this section, including completely (0-)simple and completely regular semigroups, we refer to [4] and [10].

A semigroup S is *completely regular*, *c.r.* for short, if S is a union of its subgroups. In particular, each element $a \in S$ belongs to a maximal subgroup G of S, where there exists a unique element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ with e being the identity element of G. The element e is clearly an idempotent of S, and the set $E(S)$ of idempotents of S consists of the identity elements of its maximal subgroups.

A semigroup S is *completely simple* (respectively, *completely* 0-*simple*) if it is simple (respectively, 0-simple) and it contains a *primitive idempotent*, that is, an idempotent which is minimal (respectively, minimal among the nonzero idempotents) in the partially ordered set $E(S)$ of idempotents.

It can be easily shown that any $\mathcal{H}$-class H of a completely simple semigroup is a group. Further, each completely simple semigroup is a regular semigroup. Indeed, we have a stronger result which states that a semigroup S is completely simple if and only if it is simple and completely regular.

Let T be a semigroup to which we adjoin a zero element to obtain $T^0 = T \cup \{0\}$. Let I and Λ be two nonempty index sets and $P = (p_{\lambda i}): \Lambda \times I \to T^0$ a $\Lambda \times I$-matrix with entries in T^0 such that each row and column of P has at least one nonzero entry. Denote by $\mathcal{M}^0(I, T, \Lambda; P)$ the *Rees matrix semigroup (with zero) over* T defined on $(I \times T \times \Lambda) \cup \{0\}$, where 0 is the zero element and the operation is given by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \ , \\ 0 & \text{if } p_{\lambda j} = 0 \ . \end{cases}$$

Similarly, we define the *Rees matrix semigroup (without zero) over* T as the semigroup $\mathcal{M}(I, T, \Lambda; P)$ on $I \times T \times \Lambda$, where $P: \Lambda \times I \to T$.

It is well known that a semigroup is completely $(0-)$simple if and only if it is isomorphic to a Rees matrix semigroup (with zero) over a group.

It can be shown, see the proof of Theorem IV.2.7 in [10], that each Rees matrix semigroup without zero is isomorphic to a semigroup $S = \mathcal{M}(I, G, \Lambda; P)$, where there exists an element $j \in I \cap \Lambda$ such that $p_{\lambda j} = 1 = p_{ji}$ for all $\lambda \in \Lambda$ and $i \in I$. Here 1 denotes the identity element of the group G. Such a matrix P is said to be *normalized* at j. We shall assume throughout this paper that the matrices are normalized at some j in each Rees matrix semigroup.

A semigroup B is a *band* if all its elements are idempotent, that is, $B = E(B)$. If μ is a congruence on a semigroup S such that $B = S/\mu$ is a band, then the congruence classes $S_{a\mu} = a\mu$ are subsemigroups of S. In this case S is called a *band of semigroups* S_α for $\alpha = a\mu$, $a \in S$. In particular, if each S_α is a group, then S is a *band of groups*. By definition each band of groups is a c.r. semigroup, and for each band of groups S, $\mathcal{H}$ is a congruence on S such that $S/\mathcal{H}$ is a band. Here, as in all c.r. semigroups, the $\mathcal{H}$-classes H_a are the maximal subgroups of S.

A semigroup Y is a *semilattice* if it is a commutative band. If ρ is a congruence on a semigroup S such that the quotient $Y = S/\rho$ is a semilattice, then we write $S = (Y; S_\alpha)$ and say that S is a *semilattice* Y *of semigroups* S_α for $\alpha \in Y$, where S_α is the ρ-class corresponding to α. Note that each semigroup S has a least congruence η for which the quotient S/η is a semilattice.

By an early result of Clifford c.r. semigroups can be characterized as those semigroups that are semilattices $(Y; S_\alpha)$ of completely simple semigroups.

4 Compactness in Varieties of Completely Regular Semigroups

The family of c.r. semigroups is not a variety of semigroups because a subsemigroup of a c.r. semigroup need not be completely regular. A subsemigroup T of a c.r. semigroup S is a completely regular subsemigroup if for all $a \in T$ also $a^{-1} \in T$. However, as it is easily seen, epimorphic images and direct products of c.r. semigroups are completely regular.

A c.r. semigroup S can be regarded as a *unary semigroup* $(S, \cdot, ^{-1})$ with an additional unary operation $a \mapsto a^{-1}$; c.r. semigroups form a variety of these algebras. We denote by $\mathcal{CR}$ the variety of c.r. semigroups (as unary semigroups). The variety of groups $\mathcal{G}$ is a subvariety of $\mathcal{CR}$, and as it is well known, if $\mathcal{V}$ is a variety of groups, then the c.r. semigroups with subgroups in $\mathcal{V}$ form a variety of c.r. semigroups.

Recall that a semigroup S satisfies the *ascending chain condition* (a.c.c.) for congruences if each properly ascending chain $\rho_1 \subset \rho_2 \subset \ldots$ of congruences ρ_i on S is finite.

It was shown in [8] that if a variety $\mathcal{V}$ of semigroups satisfies CP, then $\mathcal{V}$ satisfies CP uniformly, and, moreover, that a variety $\mathcal{V}$ of *monoids* satisfies CP if and only if each finitely generated member of $\mathcal{V}$ satisfies the a.c.c. for congruences.

We argue that the equivalence of CP and the a.c.c. condition is valid also for the varieties of c.r. semigroups (which need not be varieties of semigroups).

Theorem 4. *The following conditions are equivalent for a variety $\mathcal{V}$ of c.r. semigroups.*

(1) *$\mathcal{V}$ satisfies CP.*

(2) *The finitely generated semigroups in $\mathcal{V}$ satisfy the a.c.c. for congruences.*

(3) *The finitely generated free c.r. semigroups of $\mathcal{V}$ satisfy the a.c.c. for congruences.*

Proof. For the equivalence of (1) and (2) we rely on the proof of the corresponding result for monoids in [8] except for one modification. The identity elements of the monoids are needed in the original proof only to ensure the existence of a natural embedding $\beta_j : S_j \to \Pi_{i \geq 1} S_i$ such that, with the projections $\pi_k : \Pi_{i \geq 1} S_i \to S_k$, the mapping $\beta_j \pi_j : S_j \to S_j$ is the identity morphism and $\beta_j \pi_k : S_j \to S_k$ is the constant morphism (images equal to the identity element of S_k) for all $k \neq j$. The latter condition can be substituted in c.r. semigroups by setting $\pi_k \beta_j(s) = e_k$ for each $k \neq j$, where $e_k \in E(S_k)$ is a fixed idempotent.

Further, the proof in [8] does not make use of the property that varieties are closed under taking subsemigroups. Since morphic images and direct products of c.r. semigroups are completely regular, it follows that the proof in [8] applies as such to c.r. semigroups with the above modification.

The nontrivial part of the equivalence of (2) and (3) follows from the observation that if a semigroup S satisfies the a.c.c. for congruences, then so do all its morphic images. □

Note that if a c.r. semigroup S satisfies CP, then so do all its subsemigroups and not only those that are completely regular.

Groups form a subvariety of c.r. semigroups, and thus as a corollary to Theorem 4 we have the following result of Albert and Lawrence [1].

Corollary 5. *A variety $\mathcal{V}$ of groups satisfies CP if and only if the finitely generated groups in $\mathcal{V}$ satisfy the a.c.c. for normal subgroups.*

Since *e.g.* the Baumslag-Solitar group, which has a group presentation $G = \langle a, b \mid b^2a = ab^3\rangle$, does not satisfy CP (see [8]), we have by Theorem 4 that no variety of c.r. semigroups containing all groups satisfies CP.

5 Compactness in Rees Matrix Semigroups

As the next result states, the completely simple semigroups are natural objects to study in the context of the compactness property.

Lemma 6. *A (0-)simple semigroup S satisfying CP is completely (0-)simple.*

Proof. If S is a (0-)simple semigroup but not completely (0-)simple, then S contains a bicyclic semigroup as its subsemigroup, see [4, Theorem 2.54]. As mentioned above the bicyclic semigroup does not satisfy CP and the claim follows. □

Lemma 7. *Let X be a finite set of variables and $S = \mathcal{M}^0(I, T, \Lambda; P)$ be a Rees matrix semigroup over T. For a locally uniform family $\{u_i\}_{i\geq 1}$ of words over X and a morphism $\alpha: X^+ \to S$, $u_1\alpha = 0$ if and only if $u_i\alpha = 0$ for all $i \geq 0$.*

Proof. Assume $u_1\alpha = 0$ in the notation of the lemma. It is immediate that there exists a 2-factor $xy \in f(u_1)$ such that $x\alpha \cdot y\alpha = 0$. Since $\{u_i\}_{i\geq 1}$ is locally uniform, $xy \in f(u_i)$ and thus $u_i\alpha = 0$ for all i. □

Theorem 8. *Let $S = \mathcal{M}^0(I, T, \Lambda; P)$ be a Rees matrix semigroup over T. If T satisfies CP, so does S. The converse holds if T is a monoid and P has an invertible element.*

Proof. Assume that the semigroup T satisfies CP. Let X be a finite set of variables and $\mathcal{E} = \{u_i = v_i\}_{i\geq 1}$ a locally uniform system of equations over X as allowed by Lemma 2. Further, let $\alpha: X^+ \to S$ be any morphism.

If $u_i\alpha = 0$ for some i, then $u_i\alpha = 0$ for all i (since $f(u_i) = f(u_j)$ for all i, j), and hence α is a solution of $\mathcal{E}$ in S if and only if $u_i\alpha = 0 = v_i\alpha$ for all i. By Lemma 7 it follows that $u_i\alpha = v_i\alpha$ for all i if $v_1\alpha = 0$, and thus also if $u_1\alpha = v_1\alpha$.

The case when $v_i\alpha = 0$ for some i is clearly symmetric to the one above.

Suppose then that $u_i\alpha, v_i\alpha \neq 0$ for all i. Write

$$x\alpha = (i_x, a_x, \lambda_x) \in I \times T \times \Lambda$$

for each $x \in X$ and let

$$Y = X \cup \{z_{xy} \mid x, y \in X\}$$

be an extended set of variables. Define a morphism $\alpha' : Y^+ \to T^0$ by

$$\begin{cases} x\alpha' = a_x \in T, \\ z_{xy}\alpha' = p_{\lambda_x i_y} \in T^0 \end{cases} .$$

For each word $u = x_1 x_2 \dots x_n$ with $x_i \in X$ set

$$u' = x_1 z_{x_1 x_2} x_2 z_{x_2 x_3} \dots z_{x_{n-1} x_n} x_n \in Y^+ \quad .$$

If $u \in X^+$ with $u\alpha \neq 0$, then

$$u\alpha = (i_{x_1}, u'\alpha', \lambda_{x_n}) \tag{1}$$

since $p_{\lambda_k i_{k+1}} \neq 0$ for all k.

Consider the system $\mathcal{E}' = \{u_i' = v_i'\}_{i\geq 1}$ of equations over Y. By assumption $\mathcal{E}'$ has a finite equivalent subsystem $\mathcal{F}' = \{u_i' = v_i'\}_{i\in F}$ in T. Now for a solution $\alpha : X^+ \to S$ of $\mathcal{F} = \{u_i = v_i\}_{i\in F}$ in S, we have, by (1), that α' is a solution of $\mathcal{F}'$ and hence of $\mathcal{E}'$. Again by (1) and the fact that $\mathcal{E}$ is locally uniform (and so $h(u_i) = h(u_j)$, $t(u_i) = t(u_j)$ and $h(v_i) = h(v_j)$, $t(v_i) = t(v_j)$), α is a solution of $\mathcal{E}$. We conclude that $\mathcal{F} \cup \{u_1 = v_1\}$ is a finite equivalent subsystem of $\mathcal{E}$ in S.

For the latter claim we observe that if T has an identity element and the matrix P has an invertible element $p_{\lambda i}$, then the mapping $t \mapsto (i, tp_{\lambda i}^{-1}, \lambda)$ is an embedding of T into S. Indeed, it is plain that this mapping is injective, and that for any two elements $t_1, t_2 \in T$ we have $t_1 t_2 \mapsto (i, t_1 p_{\lambda i}^{-1} p_{\lambda i} t_2 p_{\lambda i}^{-1}, \lambda) = (i, t_1 t_2 p_{\lambda i}^{-1}, \lambda)$. □

The (nonzero) maximal subgroups of a completely (0−)simple semigroup are isomorphic with one another and therefore we have the following

Corollary 9. *Let S be completely (0−)simple. Then S satisfies CP if and only if any of its maximal subgroups does.*

Corollary 10. *A variety $\mathcal{V}$ of completely simple semigroups satisfies CP if and only if the variety $\mathcal{V} \cap \mathcal{G}$ does.*

Using a construction of free completely simple semigroups from [3] we obtain

Corollary 11. *Every free completely simple semigroup satisfies CP.*

6 Compactness in Completely Regular Semigroups

For a variety $\mathcal{V}$ of c.r. semigroups the intersection $\mathcal{V} \cap \mathcal{G}$ consists of subgroups of semigroups in $\mathcal{V}$, and therefore if $\mathcal{V}$ satisfies CP, then so does $\mathcal{V} \cap \mathcal{G}$. Our aim in this section is to consider the converse of this statement. Using Theorem 4 in Sect. 4 we shall find a nontrivial special case when this holds.

We shall first relate the a.c.c. for congruences on a completely simple semigroup to the a.c.c. for its maximal subgroups.

Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a Rees matrix semigroup and let ρ be a congruence on S. We associate with ρ a triple $(r_\rho, N_\rho, \pi_\rho)$ as follows. Define the relations r_ρ on I and π_ρ on Λ by

$$
\begin{aligned}
i\, r_\rho\, j &\iff \forall \lambda \in \Lambda: \ (i, p_{\lambda i}^{-1}, \lambda)\, \rho\, (j, p_{\lambda j}^{-1}, \lambda) \ , \\
\lambda\, \pi_\rho\, \mu &\iff \forall i \in I: \ (i, p_{\lambda i}^{-1}, \lambda)\, \rho\, (i, p_{\mu i}^{-1}, \mu) \ .
\end{aligned}
$$

Further, let

$$N_\rho = \{ g \mid (1, g, 1) \rho (1, 1_G, 1) \} \ ,$$

where P is normalized at 1 and 1_G is the identity of G. One can now show that r_ρ is an equivalence relation on I, π_ρ is an equivalence relation on Λ, and N_ρ is a normal subgroup of G.

For the next two lemmas we refer to [9, Sect. IV.4].

Lemma 12. *Let ρ and σ be congruences on $S = \mathcal{M}(I, G, \Lambda; P)$. Then $\rho \subseteq \sigma$ if and only if $r_\rho \subseteq r_\sigma$, $N_\rho \subseteq N_\sigma$ and $\pi_\rho \subseteq \pi_\sigma$. In particular, $\rho = \sigma$ if and only if $(r_\rho, N_\rho, \pi_\rho) = (r_\sigma, N_\sigma, \pi_\sigma)$.*

A triple (r, N, π) is said to be *admissible* if r is an equivalence relation on I, N a normal subgroup of G, and π is an equivalence relation on Λ such that

$$\forall i, j \in I, \ \lambda, \mu \in \Lambda \text{ and either } i\, r\, j \text{ or } \lambda\, \pi\, \mu, \text{ then } p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1} \in N \ .$$

Lemma 13. *To each admissible triple (r, N, π) there corresponds a unique congruence ρ on S such that $(r_\rho, N_\rho, \pi_\rho) = (r, N, \pi)$.*

For the relation between the congruences on a completely simple semigroup and the normal subgroups of its maximal subgroups, we have

Lemma 14. *A Rees matrix semigroup $S = \mathcal{M}(I, G, \Lambda; P)$ satisfies the a.c.c. for congruences if and only if I and Λ are finite and G satisfies the a.c.c. for normal subgroups.*

Proof. If I is infinite, let $r_1 \subset r_2 \subset \ldots$ be a strictly ascending chain of equivalence relations on I, and let $\omega = \Lambda \times \Lambda$. First, for each $i \geq 1$, (r_i, G, ω) is an admissible triple for S, and hence the corresponding congruences ρ_i form a strictly ascending chain by Lemmas 12 and 13. The case for Λ is dual.

Next assume that $N_1 \subset N_2 \subset \ldots$ is a strictly ascending chain of normal subgroups of G. Let ε denote the identity relation on I and Λ. Clearly, $(\varepsilon, N_i, \varepsilon)$

is an admissible triple for S, and again the corresponding congruences ρ_i on S form a strictly ascending chain.

In the other direction, assume $\rho_1 \subset \rho_2 \subset \dots$ is an ascending chain of congruences on S, and let the corresponding admissible triples be (r_i, N_i, π_i). By Lemma 12, we have

$$r_1 \subseteq r_2 \subseteq \dots , \quad N_1 \subseteq N_2 \subseteq \dots , \quad \pi_1 \subseteq \pi_2 \subseteq \dots ,$$

and since I and Λ are finite and G satisfies the a.c.c. for normal subgroups, these chains are finite, which implies that for $i \geq n_0$, $\rho_i = \rho_{i+k}$ for all $k \geq 0$. The assertion follows. □

For a finitely generated c.r. semigroup $S = (Y; S_\alpha)$ the semilattice Y is finite. However, the completely simple subsemigroups S_α need not be finitely generated. An example of such a situation is the 2-generator free c.r. semigroup as constructed by Clifford [3].

We say that a c.r. semigroup $S = (Y; S_\alpha)$, where $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$, has *finite matrices* if for each α the index sets I_α and Λ_α are finite. Note that S has finite matrices if and only if every left and every right zero subsemigroup of S is finite.

Lemma 15. *If $S = (Y; S_\alpha)$ is a finitely generated c.r. semigroup with finite matrices, then the Rees matrix semigroups S_α are finitely generated.*

Proof. Let $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$ for $\alpha \in Y$, where the matrices P_α are normalized at an element which is denoted simply by α. We use (a part of) the standard representation of c.r. semigroup of [11]: there are three sets of functions

$$\begin{gathered} \langle\ ,\ \rangle\colon S_\alpha \times I_\alpha \to I_\beta \ , \\ [\ ,\ ]\colon \Lambda_\alpha \times S_\alpha \to \Lambda_\beta \ , \\ \tau\colon S_\alpha \to G_\beta \ (\text{denoted} \ \ a \mapsto a_\beta) \end{gathered}$$

defined for all $\alpha \geq \beta$ such that for all $a \in S_\alpha$ and $b \in S_\beta$,

$$ab = (\langle a, \langle b, \alpha\beta\rangle\rangle, a_{\alpha\beta}\, p_{[\alpha\beta,a]\langle b,\alpha\beta\rangle}\, b_{\alpha\beta}, [[\alpha\beta, a], b]) \ , \tag{2}$$

where the function $a \mapsto a_\beta$ satisfies

$$a = (\langle a, i\rangle, a_\alpha, [\lambda, a]) \tag{3}$$

for all $a \in S_\alpha$, and

$$(ab)_\gamma = a_\gamma\, p_{[\gamma,a]\langle b,\gamma\rangle}\, b_\gamma \tag{4}$$

for all $a \in S_\alpha$, $b \in S_\beta$ and $\gamma \leq \alpha\beta$.

Assume that S is generated by a finite set A of elements and let $\gamma \in Y$. Let $c \in S_\gamma$ be such that $c = (i, g, \lambda)$. Now c is a product $c = a^{(1)}a^{(2)} \dots a^{(n)}$ of some

$a^{(j)} \in A$, where $a^{(j)} \in S_{\alpha_j}$ with $\alpha_j \geq \gamma$. From (3) it follows that $g = c_\gamma$, and thus, by (4),

$$g = (a^{(1)}a^{(2)}\ldots a^{(n)})_\gamma = a_\gamma^{(1)}p_1 a_\gamma^{(2)}p_2 \ldots p_{n-1}a_\gamma^{(n)}$$

for some entries p_j of P_γ. We conclude that G_γ is generated by the finite set

$$A_\gamma = \{a_\gamma \mid a \in A\} \cup \{p \mid p \in P_\gamma\} \ .$$

Since the matrix P_γ is finite, the group G_γ is finitely generated. Moreover the index sets I_γ and Λ_γ are finite, and therefore the finite set $I_\gamma \times A_\gamma \times \Lambda_\gamma$ generates S_γ. □

Recall that for a congruence η on a regular semigroup S

$$\ker \eta = \{a \mid a\eta e \ \text{ for some } e \in E(S)\}$$

is its *kernel*.

Lemma 16. *Let $S = (Y; S_\alpha)$ be a finitely generated c.r. semigroup. If each of the Rees matrix semigroups S_α satisfies the a.c.c. for congruences, then so does S.*

Proof. To prove the claim we use (a part of) the characterization of the congruences on a c.r. semigroup from [12]. This result states that each congruence ρ on S can be represented uniquely by a congruence aggregate $(\xi; \eta_\alpha)$, in notation $\rho = \rho_{(\xi;\eta_\alpha)}$, where ξ is a congruence on Y and η_α is a congruence on S_α such that for all $a \in S_\alpha$ and $b \in S_\beta$,

$$a\rho_{(\xi;\eta_\alpha)}b \iff \alpha\xi\beta \ \ , ab\,\eta_{\alpha\beta}\,ba, \ \ ab^{-1} \in \ker \eta_{\alpha\beta} \ .$$

Further, η_α is the restriction of ρ on S_α and

$$\alpha\xi\beta \iff a\rho c, \ b\rho d \ \text{ for some } a \in S_\alpha, \ b \in S_\beta, \ c, d \in S_{\alpha\beta} \ .$$

One obtains from these, see again [12], that for all congruences $\rho_1 = \rho_{(\xi^{(1)};\eta_\alpha^{(1)})}$ and $\rho_2 = \rho_{(\xi^{(2)};\eta_\alpha^{(2)})}$,

$$\rho_1 \subseteq \rho_2 \iff \xi^{(1)} \subseteq \xi^{(2)} \ \text{ and } \ \eta_\alpha^{(1)} \subseteq \eta_\alpha^{(2)} \text{ for all } \alpha \in Y \ . \tag{5}$$

Let then $\rho_i = \rho_{(\xi^{(i)};\eta_\alpha^{(i)})}$, for $i \geq 1$, be congruences on S such that $\rho_i \subseteq \rho_{i+1}$. By (5) we have $\xi^{(i)} \subseteq \xi^{(i+1)}$ and $\eta_\alpha^{(i)} \subseteq \eta_\alpha^{(i+1)}$ for all i. Since S is finitely generated, Y is a finite semilattice, and therefore there exists an integer n_0 such that $\xi^{(i)} = \xi^{(i+k)}$ for all $i \geq n_0$ and $k \geq 0$. On the other hand, if each S_α satisfies the a.c.c. for congruences, then there exists an integer n_1 such that $\eta_\alpha^{(i)} = \eta_\alpha^{(i+k)}$ for all $i \geq n_1$ and $k \geq 0$ since Y is finite. This shows, by (5), that $\rho_i = \rho_{i+k}$ for all $i \geq \max(n_0, n_1)$ and $k \geq 0$, and hence S satisfies the a.c.c. for congruences. □

In conclusion we have the following result.

Theorem 17. *Let $S = (Y; S_\alpha)$ be a finitely generated c.r. semigroup with finite matrices. If each of its finitely generated subgroups satisfies the a.c.c. for normal subgroups, then S satisfies the a.c.c. for congruences.*

Proof. Suppose the finitely generated subgroups of S satisfy a.c.c. on normal subgroups. Then by Lemma 15 each S_α is finitely generated, and by Lemma 14 each S_α satisfies the a.c.c. for congruences. Finally, by Lemma 16, S itself satisfies the a.c.c. for congruences. $\square$

Corollary 18. *Let $\mathcal{V}$ be a variety of c.r. semigroups whose finitely generated members have finite matrices. Then $\mathcal{V}$ satisfies CP if and only if $\mathcal{V} \cap \mathcal{G}$ does.*

Proof. The direct part of the claim is clear since $\mathcal{V} \cap \mathcal{G} \subseteq \mathcal{V}$.

In the reverse direction, suppose that the groups in $\mathcal{V}$ satisfy CP, and thus, since $\mathcal{V} \cap \mathcal{G}$ is a variety of groups, the finitely generated groups in $\mathcal{V}$ satisfy the a.c.c. for normal subgroups by Corollary 5. By Theorem 17 each finitely generated $S \in \mathcal{V}$ satisfies the a.c.c. for congruences, and thus the claim follows from Theorem 4. $\square$

We can now show that the condition of Theorem 17 is satisfied by finitely generated bands of groups.

Lemma 19. *Let $S = (Y; S_\alpha)$ be a finitely generated c.r. semigroup that is a band of groups. Then S has finite matrices.*

Proof. Let $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$ for all $\alpha \in Y$. Now $S/\mathcal{H}$ is a band, and since S is finitely generated, so is $S/\mathcal{H}$. Every finitely generated band is finite, see *e.g.* [9, Theorem IV.4.9], and thus $S/\mathcal{H}$ is a finite band. Therefore S is a finite union of groups and so $E(S)$ is finite. The claim follows since the idempotents of S are exactly the identity elements of its maximal subgroups. $\square$

Bands of groups form a subvariety $\mathcal{BG}$ of the variety $\mathcal{CR}$ of c.r. semigroups and thus we have the following generalization of Corollary 5.

Corollary 20. *A variety $\mathcal{V}$ of bands of groups satisfies CP if and only if the variety $\mathcal{V} \cap \mathcal{G}$ does.*

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