# Permutable Transformation Semigroups

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Let $\Delta$ be a finite set and $\mathcal{T}_\Delta$ the full transformation semigroup on $\Delta$. In this note we shall prove that if $S$ and $R$ are two permutable transitive subsemigroups of $\mathcal{T}_\Delta$, *i.e.*, they commute elementwise,

$$\alpha\beta = \beta\alpha \qquad \text{for all } \alpha \in S,\ \beta \in R\ ,$$

then they are simply transitive groups of permutations and centralizers of each other.

The problem under consideration has its origin in the theory of networks, [2], where permutability of two transformation semigroups (with respect to a network) refers to an independence relation and hence to concurrency of events in the network.

For the results and definitions needed here for the permutation groups we refer to [3].

We shall consider transformation semigroups on a *finite* set $\Delta$. A subsemigroup $S$ of $\mathcal{T}_\Delta$ is said to be *transitive* if for all $a, b \in \Delta$ there exists an $\alpha \in S$ such that $\alpha(a) = b$. A subsemigroup $S$ of $\mathcal{T}_\Delta$ is a *permutation group*, if $S$ is a subgroup of the symmetric group $Sym(\Delta)$.

Let

$$C(S) = \{\beta \in \mathcal{T}_\Delta \mid \alpha\beta = \beta\alpha \text{ for all } \alpha \in S\}$$

be the *centralizer* of the transformation semigroup $S$. It is immediate that $C(S)$ is a submonoid of $\mathcal{T}_\Delta$. The identity transformation is denoted by $\iota$.

**Lemma 1.** *Let $S$ be a transitive transformation semigroup on $\Delta$. An element $\beta \in C(S)$ has a fixed point if and only if $\beta = \iota$.*

**Proof.** Let $\beta \in C(S)$ have a fixed point, say $\beta(a) = a$. Now, for each $\alpha \in S$, $\beta\alpha(a) = \alpha\beta(a) = \alpha(a)$, and hence $\beta$ fixes the elements of $\{\alpha(a) \mid \alpha \in S\}$. The claim follows from this by the assumption of transitivity for $S$. ■

In general if $S$ is a subgroup of $\mathcal{T}_\Delta$, it does not follow that $S$ is a permutation group. Moreover, if $S$ is a permutation group, then the centralizer $C(S)$ of $S$ in $\mathcal{T}_\Delta$ need not coincide with the group centralizer of $S$ in $Sym(\Delta)$. Indeed, the centralizer of the trivial subgroup $\langle\iota\rangle$ is the whole of $\mathcal{T}_\Delta$. However, a transitive subgroup of $\mathcal{T}_\Delta$ is a permutation group and the two centralizers of it are equal.

**Lemma 2.** *Let $S$ be a transitive subgroup of $\mathcal{T}_\Delta$. Then $S$ is a subgroup of $Sym(\Delta)$ and $C(S)$ coincides with the group centralizer of $S$ in $Sym(\Delta)$.*

**Proof.** It is immediate that the identity element $\epsilon$ of $S$ is an idempotent of $\mathcal{T}_\Delta$ and, moreover, $\epsilon \in C(S)$. Hence $\epsilon^2(a) = \epsilon(a)$ for all $a \in \Delta$ and, consequently, $\epsilon = \iota$ by Lemma 1. It follows from this that $S$ is a permutation group.

Let then $\beta \in C(S)$. For each $a \in \Delta$ there exists an $\alpha \in S$ such that $\alpha\beta(a) = a$, because $S$ is transitive. Hence $\beta(\alpha(a)) = a$, which shows that $\beta$ maps $\Delta$ onto $\Delta$. Since $\Delta$ is finite it follows that $\beta$ is a permutation on $\Delta$. As a submonoid of the finite group $Sym(\Delta)$, the centralizer $C(S)$ is a permutation group, which proves the claim. ■

Let $S$ be a transformation semigroup on $\Delta$. For each $a \in \Delta$ let

$$S_a = \{\alpha \in S \mid \alpha(a) = a\}$$

be the *stabilizer* of $a$ in $S$.

A permutation subgroup $S$ of $Sym(\Delta)$ is said to be *semiregular* if $S_a$ is the trivial group $\langle \iota \rangle$ for each $a \in \Delta$. Further, the group $S$ is *simply transitive* (or *regular*) if it is a transitive semiregular group.

From the definition it follows easily that a permutation group $S$ is simply transitive if and only if for all $a, b \in \Delta$ there exists a unique $\alpha \in S$ such that $\alpha(a) = b$.

**Theorem 1.** *If $S$ is a transformation semigroup on $\Delta$ with a transitive centralizer $C(S)$, then $S$ is a semiregular group.*

**Proof.** Let $\alpha \in S$. To show that $\alpha$ is onto $\Delta$, let $a \in \Delta$. By transitivity of $C(S)$ there exists a $\beta \in C(S)$ such that $\beta(\alpha(a)) = a$. Since $\beta\alpha = \alpha\beta$, we have $\alpha(\beta(a)) = a$ as required. Just like in the previous proof it follows from this that $S$ is a permutation group.

Since by assumption $C(S)$ is transitive and $S \subseteq C(C(S))$, Lemma 1 implies that $S_a = \langle \iota \rangle$ for all $a \in \Delta$. This shows that $S$ is a semiregular group. ■

Suppose $S$ is a simply transitive group on $\Delta$ and fix an element $a \in \Delta$. It easily follows, see [3], that $S$ can be written in the form

$$S = \{\alpha_b \mid b \in \Delta,\ \alpha_b(a) = b\}\ ,$$

where $\alpha_b \neq \alpha_c$ for all distinct $b, c \in \Delta$. In particular, the order of $S$ equals $|\Delta|$, the cardinality of $\Delta$. From the previous result we have thus obtained

**Theorem 2.** *If $S$ is a transitive transformation semigroup such that $C(S)$ is transitive, then $S$ is a simply transitive group of order $|\Delta|$.* ■

In [1] (p.174, Theorem II) it was shown that if $S$ is a transitive permutation group on $\Delta$ of order $|\Delta|$, then $C(S)$ is also transitive and isomorphic with $S$. Indeed, if $S = \{\alpha_b \mid b \in \Delta\}$ and $C(S) = \{\beta_b \mid b \in \Delta\}$ are simply transitive, where $\alpha_b(a) = b = \beta_b(a)$ for a fixed $a \in \Delta$, then the mapping $\varphi \colon S \to C(S)$ defined by

$$\varphi(\alpha_b) = \beta_b^{-1}$$

is a desired isomorphism.

**Theorem 3.** *Two transitive transformation semigroups $S$ and $R$ on $\Delta$ are permutable if and only if they are isomorphic simply transitive groups and centralizers of each other.*

**Proof.** If $S$ and $R$ are transitive and $\alpha\beta = \beta\alpha$ for all $\alpha \in S$ and $\beta \in R$, then $R \subseteq C(S)$ and $C(S)$ is thus transitive. By Theorem 2, $R$ is a simply transitive group of order $|\Delta|$. Similarly, $C(S)$ is a simply transitive group of the same order. Hence $R = C(S)$, which proves the claim.

The proof in the other direction is trivial. ■

As is well known one cannot pose any restrictions on the structure of the simply transitive groups, since every group $G$ has a simply transitive representation, see [3]. Indeed, take $\Delta = G$ and define for each $g \in G$ the permutations $\varphi_g$ and $\psi_g$ by right and left multiplication:

$$\varphi_g(a) = ga \quad , \qquad \psi_g(a) = ag^{-1}\ .$$

Now, $S = \{\varphi_g \mid g \in \Delta\}$ and $R = \{\psi_g \mid g \in \Delta\}$ are two permutable simply transitive groups and they are centralizers of each other in the symmetric group $Sym(\Delta)$.

## References


[1] Burnside, W., "Theory of Groups", (2nd ed.) 1911, Dover, New York, 1955.

[2] Ehrenfeucht, A., and G. Rozenberg, Dynamic labeled 2-structures, in preparation.

[3] Wielandt, H., "Finite Permutation Groups", Academic Press, New York, 1964.



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