Permutations, Parenthesis Words, and Schröder Numbers*

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Abstract

A different proof for the following result due to J. West is given: the Schröder number s_{n-1} equals the number of permutations on $\{1, 2, \ldots, n\}$ that avoid the pattern $(3, 1, 4, 2)$ and its dual $(2, 4, 1, 3)$.

Keywords: Permutations, pattern, Schröder numbers, Catalan numbers, parenthesis words

1 Introduction

We give here a different and shorter proof of a result due to J. West [12], and conjectured by Shapiro and Getu: the number of permutations on $[1, n] = \{1, 2, \ldots, n\}$ avoiding the pattern $\sigma = (3, 1, 4, 2)$ and its dual $\sigma^\partial = (2, 4, 1, 3)$, is the Schröder number s_{n-1} that is known to satisfy

$$s_n = \sum_{i=0}^{n} \binom{2n-i}{i} c_{n-i}\,, \tag{1}$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. We reduce the counting problem of permutations that avoid σ and σ^∂ to Schröder's original problem from 1870 in [9] of counting parenthesis words.

Closely related results on the number of permutations that avoid a pattern, and also on the non-crossing partitions, are proved by Dershowitz and Zaks [2], [3] and Edelman [4], see also Prodinger [8].

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Schröder numbers occur in many enumeration problems, see *e.g.* Stanley [11]. Even more so do the Catalan numbers, see *e.g.* Klazar [7], Shapiro and Stephens [10] and West [12]. The connection between the permutations that avoid the pattern σ and graphs is well known in the context of P_4-free graphs, or cographs, as they are also called, see especially [1]. For a general treatment in terms of edge-coloured directed graphs (or 2-structures), see [5], and also [6].

We end this section with some notations and definitions.

Denote $[m, n] = \{m, m+1, \dots, n\}$ for the positive integers $m \le n$.

The set of all permutations on a set A is denoted by $\mathrm{Sym\,A}$, and we let

$$\mathcal{S} = \bigcup_{n \ge 1} \mathrm{Sym[1,n]}$$

be the set of all permutations on the sets $[1, n]$ for $n \ge 1$. We identify each $\delta \in \mathrm{Sym[1,n]}$ with a linear order of $[1, n]$ such that $\delta = (i_1, i_2, \dots, i_n)$, where $\delta(k) = i_k$ for all $k \in [1, n]$. In this case, the *dual* of δ is the permutation $\delta^\partial = (i_n, i_{n-1}, \dots, i_1)$.

A permutation $\delta \in \mathrm{Sym[1,n]}$ is said to *contain a pattern* $\rho \in \mathrm{Sym[1,k]}$ if there exists a mapping $\alpha \colon [1, k] \to [1, n]$ such that $\alpha(i) < \alpha(j)$ for $i < j$, and

$$\rho(i) \le \rho(j) \iff \delta(\alpha(i)) \le \delta(\alpha(j))\,.$$

Let

$$\sigma = (3, 1, 4, 2) \quad \text{and} \quad \sigma^\partial = (2, 4, 1, 3)\,.$$

If $\delta \in \mathcal{S}$ does not contain the pattern σ nor its dual σ^∂, then it is said to be σ^*-*avoiding*. Denote by

$$\mathcal{S}_{\sigma^*} = \{\delta \mid \delta \in \mathcal{S} \text{ is } \sigma^*\text{-avoiding}\}\,.$$

Example 1.1. The permutation $\delta = (1, 5, 4, 2, 6, 3) \in \mathrm{Sym}[1, 6]$ contains the pattern σ, since in δ there is a subsequence $(5, 2, 6, 3) = (\alpha(3), \alpha(1), \alpha(4), \alpha(2))$, where the mapping $\alpha \colon [1, 4] \to [1, 6]$ is defined by $\alpha(1) = 2$, $\alpha(2) = 4$, $\alpha(3) = 5$ and $\alpha(4) = 6$.

2 Sums of permutations

We define the *sum* of two permutations, $\delta_1 = (i_1, \dots, i_n) \in \mathrm{Sym[1,n]}$ and $\delta_2 = (j_1, \dots, j_m) \in \mathrm{Sym[1,m]}$, as

$$\delta_1 \oplus \delta_2 = (i_1, \dots, i_n, j_1 + n, \dots, j_m + n)\,.$$

Clearly, $\delta_1 \oplus \delta_2$ is a permutation on $[1, n+m]$. The sum of permutations in $\mathcal{S}$ is easily seen to be associative, and therefore $\mathcal{S}$ forms a (noncommutative) semigroup under this operation.

Let, for each $n \ge 1$,

$$\iota_n = (1, 2, \dots, n)\,,$$

be the *identity permutation* of $\mathrm{Sym[1,n]}$.

Example 2.1. Let $\delta = (2,1,3,5,4)$, $\iota_1 = (1)$, and $\iota_2 = (1,2) = \iota_1 \oplus \iota_1$. Then $\delta = \iota_2^\partial \oplus \iota_1 \oplus \iota_2^\partial = (\iota_1 \oplus \iota_1)^\partial \oplus \iota_1 \oplus (\iota_1 \oplus \iota_1)^\partial$.

We give here a shorter proof of the next theorem which is proved in the context of 2-structures in [5].

Theorem 2.2. *The set of σ^*-avoiding permutations is the smallest class of permutations containing $\iota_1 = (1)$ and closed under the operations of taking duals and sums.*

Proof. First of all we show that the σ^*-avoiding permutations are closed under duals and sums. For the dual the claim is trivial. For the sum, we observe that if $\delta = \delta_1 \oplus \delta_2$, where $\delta_1 \in \mathrm{Sym}[1,\mathrm{n}]$, and δ contains the pattern σ, say (i_1, i_2, i_3, i_4), then $i_4 > n$ implies that also $i_1 > n$, and in this case δ_2 contains the pattern σ. The case $i_4 \leq n$ implies that δ_1 contains the pattern σ by the definition of the sum. A similar argument is valid for σ^∂, and thus the closure properties are verified.

Let $\delta = (i_1, \ldots, i_n) \in \mathrm{Sym}[1,\mathrm{n}]$ be σ^*-avoiding, where $n \geq 2$. We prove that δ or δ^∂ is a sum of two permutations from which the claim follows by induction.

Let $i_r = 1$ and $i_s = n$, where we may suppose that $r < s$; for, otherwise, we consider δ^∂ instead of δ. If $r = 1$ then $\delta = (1) \oplus (i_2 - 1, \ldots, i_n - 1)$; and if $s = n$ then $\delta = (i_1, \ldots, i_{n-1}) \oplus (1)$. Now assume that $1 < r < s < n$. Denote

$$M_r = \max\{i_q \mid q < r\} \quad \text{and} \quad M_s = \min\{i_p \mid p > s\}\,.$$

We have $M_r < M_s$, since otherwise δ would contain the pattern σ: $(i_q, 1, n, i_p)$ for $i_q > i_p$ with $q < r$ and $p > s$. Let $t \in [1,n]$ be the last index such that $i_t < M_s$. Clearly, $r \leq t < s$ (since $i_r = 1$ and $i_s = n$), and $i_m \geq M_s$ for all $m > t$.

If there exists an index j with $r < j < t$ such that $i_j > M_s$, then δ contains the pattern σ, namely, (i_j, i_t, n, i_q) for $q > s$ with $i_q = M_s$. In conclusion, $i_j < M_s \leq i_m$ for all $j \leq t$ and $m > t$, which implies that $\delta = (i_1, \ldots, i_t) \oplus (i_{t+1} - t, \ldots, i_n - t)$. This proves the claim. $\square$

Denote by ℓ_δ the last integer in the domain of a permutation $\delta \in \mathcal{S}$, that is, $\delta \in \mathrm{Sym}[1, \ell_\delta]$. The set $\mathcal{S}_{\sigma^*}$ can be partitioned into two subsets according to whether 1 or ℓ_δ comes before the other:

$$\mathcal{S}_{\sigma^*,1} = \{\delta \mid \delta^{-1}(1) < \delta^{-1}(\ell_\delta)\} \qquad \text{and} \qquad \mathcal{S}_{\sigma^*,\ell} = \{\delta \mid \delta^{-1}(1) > \delta^{-1}(\ell_\delta)\}\,.$$

From the proof of Theorem 2.2 we obtain

Lemma 2.3. *A permutation $\delta \in \mathcal{S}_{\sigma^*}$ with $\delta \neq \iota_1$ is a sum of two permutations from $\mathcal{S}_{\sigma^*}$ if and only if $\delta \in \mathcal{S}_{\sigma^*,1}$.*

3 Parenthesis words and Schröder numbers

We shall now give an alternate description to the σ^*-avoiding permutations using parenthesis words. For this let $\imath$ be a symbol and let $A = \{\imath, (,)\}$ be an alphabet. Denote by A^* the free word monoid generated by A, that is, A^* consists of the words in the letters of A with the product of concatenation of words.

Let P be the smallest subset of A^* such that

(i) $(\imath) \in P$;

(ii) if $w_1, w_2 \in P$ then also $w_1 w_2 \in P$;

(iii) for all $w \in P$, also $(w) \in P$.

By condition (ii), P is a subsemigroup of A^*. A word $w \in P$ is said to be *reduced*, if it has no subwords in P of the form $((u))$. Hence in a reduced word we do not have 'unnecessary' parentheses. Denote the set of all reduced words in P by

$$P_{\mathrm{red}} = \{w \mid w \text{ reduced}\}\,.$$

We map the reduced words into the set of all permutations as follows. Let $\alpha\colon P_{\mathrm{red}} \to \mathcal{S}$ be defined by

$$\alpha((\imath)) = \iota_1, \quad \alpha(w_1 w_2) = \alpha(w_1) \oplus \alpha(w_2), \quad \alpha((w)) = \alpha(w)^{\partial}\,.$$

It is clear that α is a well defined function, and by the second equality, it is a semigroup homomorphism.

Example 3.1. The reduced word $w = (\imath)((\imath)((\imath)(\imath)(\imath)))(\imath)$ has the image $\alpha(w) = (1) \oplus ((1) \oplus ((1) \oplus (1) \oplus (1))^{\partial})^{\partial} \oplus (1) = (1, 3, 4, 5, 2, 6)$.

Lemma 3.2. *The mapping α is a bijection from P_{red} onto $\mathcal{S}_{\sigma^*}$.*

Proof. For this we observe (without the easy proofs) that in $\mathcal{S}$, for all $\delta_i \in \mathcal{S}$,

$$\delta_1 \oplus \delta_2 = \delta_1 \oplus \delta_3 \implies \delta_2 = \delta_3 \quad \text{and} \quad \delta_1 \oplus \delta_2 = \delta_3 \oplus \delta_2 \implies \delta_1 = \delta_3\,, \tag{2}$$

$$\delta_1 \oplus \delta_2 = \delta_3 \oplus \delta_4 \implies \delta_1 = \delta_3 \text{ or } \exists \delta \in \mathcal{S}\colon [\delta_1 = \delta_3 \oplus \delta \text{ or } \delta_3 = \delta_1 \oplus \delta]\,, \tag{3}$$

$$(\delta_1 \oplus \delta_2)^{\partial} \neq \delta_3 \oplus \delta_4\,. \tag{4}$$

The surjectivity of α is proved inductively. Let $\delta \in \mathcal{S}_{\sigma^*}$ with $\delta \neq \iota_1$. If $\delta \in \mathcal{S}_{\sigma^*,1}$ then, by Lemma 2.3, $\delta = \delta_1 \oplus \delta_2$ for some $\delta_i \in \mathcal{S}_{\sigma^*}$, and by the induction hypothesis there are words $w_1, w_2 \in P_{\mathrm{red}}$ such that $\alpha(w_i) = \delta_i$. In this case, $\alpha(w_1 w_2) = \delta_1 \oplus \delta_2 = \delta$. If, on the other hand, $\delta \in \mathcal{S}_{\sigma^*,\ell}$, then $\delta^{\partial} \in \mathcal{S}_{\sigma^*,1}$, and hence there exists a word $w \in P_{\mathrm{red}}$ such that $\alpha(w) = \delta^{\partial}$. It follows that either $w = (v)$ and $\alpha(v) = \delta$, or $(w) \in P_{\mathrm{red}}$ and $\alpha((w)) = \delta$.

We show the injectiveness of α inductively. For this let $w, v \in P_{\mathrm{red}}$ be two words such that $\alpha(w) = \alpha(v)$. Clearly, if $w = (\imath)$ or $v = (\imath)$ then $\alpha(w) = \alpha(v)$ implies $w = v$.

We have then three cases to consider:

(a) If $w = (w_1)$ and $v = (v_1)$, then from $\alpha(w) = \alpha(v)$ we obtain $\alpha(w_1) = \alpha(v_1)$ and, by the induction hypothesis, $w_1 = v_1$, from which $w = v$ follows.

(b) If $w = (w_1)$ and $v = v_1v_2$ for some $v_1, v_2 \in P_{\text{red}}$ then, by (4), $\alpha(w) = \alpha(w_1)^\partial \neq \alpha(v_1) \oplus \alpha(v_2) = \alpha(v)$.

(c) Let then $w = w_1w_2$ and $v = v_1v_2$ for words $w_1, w_2, v_1, v_2 \in P_{\text{red}}$. If $\alpha(w_1) = \alpha(v_1)$, then by (2) also $\alpha(w_2) = \alpha(v_2)$, and in this case, the induction hypothesis gives $w_1 = v_1$ and $w_2 = v_2$, and therefore also $w = v$. Suppose then that $\alpha(w_1) \neq \alpha(v_1)$. By (3), there exists a permutation δ such that $\alpha(w_1) = \alpha(v_1) \oplus \delta$ (or in the symmetric case $\alpha(v_1) = \alpha(w_1) \oplus \delta$). Now $\alpha(w) = \alpha(v)$ implies $\delta \oplus \alpha(w_2) = \alpha(v_2)$ using the property (2). Since α is surjective, $\delta = \alpha(u)$ for some $u \in P_{\text{red}}$, and therefore $\alpha(uw_2) = \alpha(v_2)$, which by the induction hypothesis, gives $uw_2 = v_2$. By these considerations, we obtain

$$\alpha(w_1) \oplus \alpha(w_2) = \alpha(w_1w_2) = \alpha(v_1v_2) = \alpha(v_1uw_2) = \alpha(v_1) \oplus \alpha(u) \oplus \alpha(w_2)\,,$$

and further, $\alpha(w_1) = \alpha(v_1u)$ by (2). The induction hypothesis gives $w_1 = v_1u$, and finally also $w = w_1w_2 = v_1uw_2 = v_1v_2 = v$. This shows that α is injective, and therefore a bijection. □

The number of words in P_{red} with n occurrences of the symbol $\imath$ is known as the Schröder number s_{n-1}, which can be shown to satisfy the equation (1). Therefore the number of words on P_{red} with n symbols $\imath$ is exactly s_{n-1}.

The following result was proved by West [12] using a somewhat different approach to the problem.

Theorem 3.3. *The number of the σ^*-avoiding permutations on $[1, n]$ equals s_{n-1}.*

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