2-STRUCTURES
A FRAMEWORK FOR DECOMPOSITION AND TRANSFORMATION OF GRAPHS †

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1 Introduction

This paper is a tutorial on the theory of 2-structures as initiated in Ehrenfeucht and Rozenberg [16]. We have included only the most illustrative proofs from the literature together with new proofs for some of the older results. We assume that the reader is familiar with the basic concepts of graph theory, see Bondy and Murty [1].

A 2-structure $g = (D, R)$ consists of a finite domain D together with an equivalence relation R on the ordered pairs $(x, y) \in D \times D$ with $x \neq y$. Hence a 2-structure can be considered as a complete directed graph with an 'abstract colouring' of the edges.

To make a colouring of the edges concrete one adds a labeling function to a 2-structure obtaining in this way a labeled 2-structure. In the literature the term '2-structure' is used both as a technical term as defined above and as a generic term referring to the theory of labeled and 'unlabeled' 2-structures.

These structures can be used to study various combinatorial properties of systems such as graphs, partially ordered sets and communication networks, where all kinds of (binary) relationships hold between the objects of a system.

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We refer to Buer and Möhring [3], Ehrenfeucht and Rozenberg [18] and Schmerl and Trotter [41] for some of the intuition and motivation.

In this paper we present the basic theory of 2-structures. We cover both the 'static' part of the theory concerned mainly with the hierarchical representations of 2-structures, and the 'dynamic' part of the theory concerned with the local transformations of 2-structures.

The general results on the static properties of 2-structures are presented in Sections 2 through 6, where the main interest is in the *clan decomposition* of these structures. A clan X of a 2-structure g is a subset of the domain of g such that an element $y \notin X$ cannot make a distinction between the elements of X by using the equivalence relation R of g, *i.e.*, for all $y \notin X$ and $x_1, x_2 \in X$ we have that $(y, x_1)R(y, x_2)$ and $(x_1, y)R(x_2, y)$.

Decompositions of combinatorial and algebraic structures are special cases of the *divide-and-conquer method*, where a large problem is partitioned into smaller problems, and a method is given how to link the solutions of the subproblems into a solution of the original problem. The *clan decomposition* of 2-structures is an example of such a method, which provides a generalization of the *modular decomposition* (or *substitution decomposition*) of graphs and directed graphs, see Gallai [26], Habib and Maurer [28], Cunningham [4] and Muller and Spinrad [38]. In the clan decomposition a 2-structure g is divided into disjoint substructures $g_1, \ldots, g_k$, and a quotient 2-structure is constructed to indicate the relationships between the substructures g_i. For another kind of a general approach to the clan decomposition theorem we refer to Möhring and Radermacher [37] and Möhring [36].

The results on the local transformations are presented in Sections 7 through 10. Here one assumes that the labels used by a labeled 2-structure form a group. One uses then the group operations, applied locally in the nodes, to induce transformations of the labels of the edges. The domain, *i.e.*, the set of nodes, does not change in these transformations. In this way a single labeled 2-structure generates a whole set of labeled 2-structures, which is referred to as a dynamic labeled 2-structure. Since the transformations are based on groups, one obtains in this way an elegant way to define graph transformations.

Also here the notion of a clan remains central. Now a clan of a dynamic labeled 2-structure is a clan of any of its component labeled 2-structures. As a matter of fact one of the main themes investigated in the theory of dynamic labeled 2-structures is the behaviour of the clan decomposition under the group induced by the local transformations.

It is interesting to notice that the notion of a clan has been rediscovered quite many times under various names. In particular, a clan corresponds to the domain of the right hand side of a production in node-rewriting graph grammars with hereditary connection relation, see Engelfriet and Rozenberg [25]. For other appearances and applications of the notion of a clan, see Buer and Möhring [3] or Muller and Spinrad [38].

2 2-Structures and Their Clans

2.1 Definition of a 2-structure

Let D be a finite nonempty set, and let

$$E_2(D) = \{(x, y) \mid x \in D, y \in D, x \neq y\}$$

be the set of all (*directed*) *edges* between the elements of D. For an edge $e = (x, y)$, we denote by $e^{-1} = (y, x)$ the *reverse edge* of e.

A *2-structure* $g = (D, R)$ consists of a finite nonempty set D, called its *domain*, and an equivalence relation $R \subseteq E_2(D) \times E_2(D)$ on its *edges*. The domain of g is denoted by $\mathrm{dom}(g)$. The elements of the domain D of g are called *nodes*. The equivalence relation of a 2-structure g is sometimes written as R_g.

Equivalently, a 2-structure $g = (D, R)$ can be defined as a triple $G = (D, E, R')$, where the pair (D, E) is a directed graph, *i.e.*, $E \subseteq E_2(D)$, and R' is an equivalence relation defined on E such that the equivalence relation R of g is obtained as an extension of R' by additionally setting $e_1 R e_2$ for all 'nonedges' $e_1, e_2 \notin E$.

For a 2-structure $g = (D, R)$ and an edge $e \in E_2(D)$, we let $eR = \{e' \mid eRe'\}$ denote the equivalence class of R containing $e \in E_2(D)$, and we call eR an *edge class* of g. Hence $e_1 R e_2$ if and only if $e_1 R = e_2 R$ for the edges e_1 and e_2 of g.

A 2-structure $g = (D, R)$ can be represented as a complete directed graph with nodes D and (directed) edges $E_2(D)$ together with a colouring of the edges corresponding to the edge classes. Here a *colouring* is a mapping $\alpha: E_2(D) \to K$ of the edges into a set K of colours such that $e_1 R e_2$ if and only if $\alpha(e_1) = \alpha(e_2)$. Notice, however, that such a colouring α for a 2-structure g is by no means unique, since the choice of the colours is *arbitrary* in the representation. The function α that maps each edge e to its edge class eR is just one possible colouring of g.

Example 2.1 Let $D = \{x_1, x_2, x_3, x_4\}$ be a domain of four nodes, and let $e_{ij} = (x_i, x_j)$ for $i \neq j$. Define a 2-structure $g = (D, R)$ by its edge classes:

$$\begin{aligned} e_{12}R &= \{e_{12}, e_{21}, e_{13}, e_{23}, e_{24}, e_{42}, e_{41}\}, \\ e_{14}R &= \{e_{14}\}, \end{aligned}$$

$$e_{31}R \quad = \quad \{e_{31}, e_{32}, e_{34}, e_{43}\}.$$

Clearly, these edge classes are disjoint, and $E_2(D) = e_{12}R \cup e_{14}R \cup e_{31}R$.

Both of the complete coloured directed graphs below represent g. In these representations the edge classes have received the colours a,b,c and b,a,d, respectively. □

Figure 1: Two colourings of a 2-structure

We notice that a 2-structure $g = (D, R)$ can also be considered as a *packing* of the directed graphs $g_{eR} = (D, eR)$ with the edge sets eR for $e \in E_2(D)$. For an edge class $a = eR$ we call g_a a *packed component* of g. See, *e.g.*, Yap [44] for more information on the topic of packing (of trees).

Example 2.2 In Figure 2 we have unpacked the 2-structure of Example 2.1. Here the edge classes of g are $(x_1, x_2)R, (x_3, x_1)R$ and $(x_1, x_4)R$. □

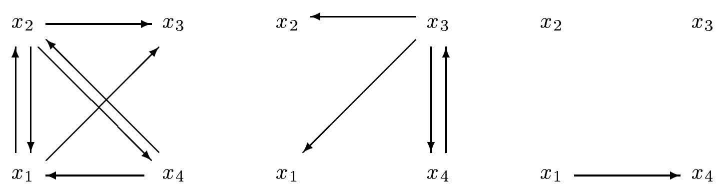


Figure 2: The packed components of g

An edge e of g is said to be *symmetric*, if eRe^{-1} (*i.e.*, $eR = e^{-1}R$), and an edge class a is *symmetric*, if all edges $e \in a$ are symmetric, *i.e.*, $e \in a$ if and only if $e^{-1} \in a$. If a is a symmetric edge class then the packed component g_a can be thought of as an undirected graph.

In Example 2.1, the edge (x_1, x_2) is symmetric, but the edge class $(x_1, x_2)R$ is not symmetric, because $(x_1, x_2)R = (x_4, x_1)R$ and $(x_1, x_4)R \neq (x_4, x_1)R$.

In fact, the 2-structure g from Example 2.1 does not have any symmetric edge classes.

Example 2.3 One can represent various properties of graphs using 2-structures. Let $G = (D, E)$ be an undirected graph. Here E is a subset of the set $S_2(D)$ of 2-element subsets $\{x, y\}$ of D with $x \neq y$.

(1) The graph G can be *represented* as a 2-structure $g(G) = (D, R)$ with two symmetric edge classes by defining $e_1 R e_2$ for all $e_1, e_2 \in E$ and for all $e_1, e_2 \notin E$. Notice, however, that the representing 2-structure $g(G)$ cannot make a distinction between a graph G and the complement graph $(D, S_2(D) \setminus E)$ of G.

(2) We can define a 2-structure $g = (D, R)$ for G by setting $e_1 R e_2$ if and only if the edges $e_1, e_2 \in E_2(D)$ belong to a common cycle of G. (Recall that a cycle in a graph is a sequence $e_1 e_2 \dots e_k$ of edges $e_i = \{x_i, x_{i+1}\}$ for $i = 1, 2, \dots, k$, where $x_1 = x_{k+1}$ and the nodes $x_1, x_2, \dots, x_k$ are distinct from each other). Clearly, the so defined relation R is an equivalence relation on $E_2(D)$, and the 2-structure g that represents the cycle structure of G is usually quite different from the representing 2-structure $g(G)$ of G. $\square$

Each 2-structure g with n nodes has $n(n-1)$ edges, and thus *drawing* a larger g is usually inconvenient. However, drawing reversible 2-structures (defined below) is easier.

A 2-structure $g = (D, R)$ is *reversible*, if g satisfies

$$e_1 R e_2 \iff e_1^{-1} R e_2^{-1}$$

for all edges $e_1, e_2 \in E_2(D)$.

Equivalently, g is reversible, if for each edge class a there corresponds a unique edge class a^{-1}, called the *reverse class* of a, such that for each edge e,

$$e \in a \iff e^{-1} \in a^{-1}.$$

Notice that we may have $a = a^{-1}$, in which case the edge class a is symmetric.

We simplify the drawings of reversible 2-structures by

— omitting the reverse edge classes of the chosen edge classes,

— drawing a line segment between x and y, if the edge (x, y) is symmetric, and

— often omitting one symmetric edge class.

Example 2.4 The 2-structure g in Figure 3 with $\mathrm{dom}(g) = \{x_1, \dots, x_8\}$ has two symmetric edge classes c and d. The edges in the class d have not been drawn, *e.g.*, the edges (x_1, x_3) and (x_2, x_7) together with their reverse edges are in the edge class d among quite many other edges (and their reverse edges). The 2-structure g has four nonsymmetric edge classes a, a^{-1}, b and b^{-1}, of which we have drawn the classes a and b only. $\square$

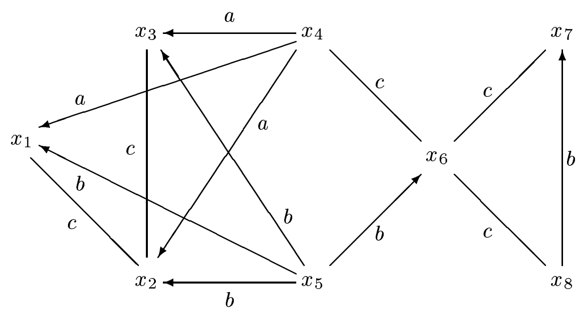


Figure 3: A simplified representation of a reversible 2-structure g

2.2 Clans

Let X be a nonempty subset of the domain D of a 2-structure $g = (D, R)$. Then the *substructure* induced by X is defined as the 2-structure

$$\mathrm{sub}_g(X) = (X, R \cap (E_2(X) \times E_2(X))),$$

which is obtained from g by restricting the relation R to the subset $E_2(X)$ of $E_2(D)$.

The edge classes of $h = \mathrm{sub}_g(X)$ are $eR_h = eR_g \cap E_2(X)$.

The most important technical notion of this paper is that of a clan. We call a subset $X \subseteq \mathrm{dom}(g)$ a *clan*, if every node $y \notin X$ sees the nodes of X in the same way,

$$(y, x_1)R(y, x_2) \quad \text{for all} \quad x_1, x_2 \in X,\ y \notin X,$$

and every two nodes $x_1, x_2 \in X$ see every node $y \notin X$ in the same way,

$$(x_1, y)R(x_2, y) \quad \text{for all} \quad x_1, x_2 \in X,\ y \notin X.$$

The set of all clans of g is denoted by $\mathcal{C}(g)$.

A clan $X \in \mathcal{C}(g)$ is said to be *proper*, if X is a proper subset of the domain of g.

It follows directly from the definition that the empty set $\emptyset$ and the domain $\mathrm{dom}(g)$ of g are clans of g. Also, it is equally clear that the singletons $\{x\}$, $x \in \mathrm{dom}(g)$, are clans. The sets $\emptyset$, $\mathrm{dom}(g)$ and the singletons are the *trivial clans* of g. A 2-structure g that has no other than the trivial clans is called *primitive*.

The substructure $\mathrm{sub}_g(X)$ induced by a nonempty clan $X \in \mathcal{C}(g)$ is called a *factor* of g.

Example 2.5 Consider the 2-structure $g = (D, R)$ of Example 2.4, see Figure 3. The two substructures $\mathrm{sub}_g(X)$ and $\mathrm{sub}_g(Y)$, that have been indicated by rectangles in Figure 4, are factors of g. $\square$

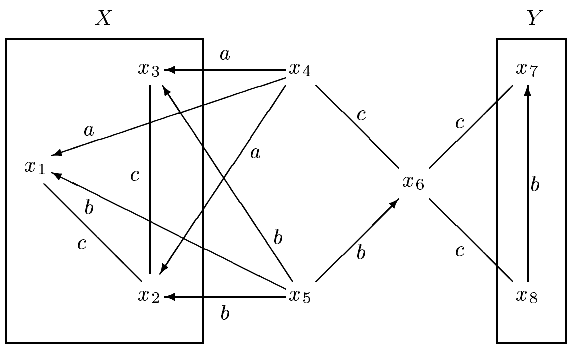


Figure 4: Two clans X and Y, and their factors in g

We show now that restricting ourselves to reversible 2-structures does not imply a loss of generality as far as the clans are concerned.

Consider a 2-structure $g = (D, R)$, and define a relation S on $E_2(D)$ by

$$e_1 S e_2 \iff e_1 R e_2 \text{ and } e_1^{-1} R e_2^{-1} \qquad (e_1, e_2 \in E_2(D)).$$

Clearly, S is an equivalence relation. We observe that the new 2-structure $h = (D, S)$ is reversible. Indeed, if $e_1 S e_2$, then by the definition of S, also $e_1^{-1} S e_2^{-1}$. On the other hand, if y, x_1 and $x_2 \in D$ are nodes such that either $(y, x_1)R \neq (y, x_2)R$ or $(x_1, y)R \neq (x_2, y)R$ in g, then by the definition of S, also $(y, x_1)S \neq (y, x_2)S$ in h. The converse holds equally well, and therefore we have the following 'normal form theorem' for 2-structures.

Theorem 2.6 *For each $g = (D, R)$ there exists a reversible 2-structure $h = (D, S)$ such that for all $X \subseteq D$, $\mathcal{C}(\mathrm{sub}_g(X)) = \mathcal{C}(\mathrm{sub}_h(X))$. In particular, $\mathcal{C}(g) = \mathcal{C}(h)$.* $\square$

Reversible 2-structures are simpler to handle than the general 2-structures. As an example, we have the following one-way condition for clans:

Lemma 2.7 *A subset $X \subseteq \mathrm{dom}(g)$ of a reversible g is a clan if and only if for all $y \notin X$ and $x_1, x_2 \in X$, $(y, x_1)R = (y, x_2)R$.*

We take the advantage of Theorem 2.6, and *consider only* reversible *2-structures in the rest of the paper.*

2.3 Basic properties of clans

Let $g = (D, R)$ be a (reversible) 2-structure. The following result gives the basic closure properties of clans.

Lemma 2.8 *Let* $X, Y \in \mathcal{C}(g)$. *Then*

(1) $X \cap Y \in \mathcal{C}(g)$;

(2) *if* $X \cap Y \neq \emptyset$, *then* $X \cup Y \in \mathcal{C}(g)$;

(3) *if* $Y \setminus X \neq \emptyset$, *then* $X \setminus Y \in \mathcal{C}(g)$.

Proof. The first two cases of the claim are rather easy to prove, and hence we prove only claim (3). Suppose that $y \in Y \setminus X$. Denote $Z = X \setminus Y$, and let $x \notin Z$, $z_1, z_2 \in Z$. If $x \notin X$, then $(x, z_1)R = (x, z_2)R$, because $z_1, z_2 \in X$ and X is a clan.

If, on the other hand, $x \in X$, then $x \in X \cap Y$. Because Y is a clan and $z_1, z_2 \notin Y$ and $x, y \in Y$, we have $(x, z_1)R = (y, z_1)R$ and $(x, z_2)R = (y, z_2)R$. Moreover, X is a clan with $y \notin X$ and $z_1, z_2 \in X$, and therefore $(y, z_1)R = (y, z_2)R$. Combining these we obtain

$$(x, z_1)R = (y, z_1)R = (y, z_2)R = (x, z_2)R,$$

and therefore $Z = X \setminus Y$ is a clan. □

We say that two subsets X and Y of a domain D *overlap*, if $X \cap Y \neq \emptyset$, $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$, *i.e.*, neither $X \subseteq Y$ nor $Y \subseteq X$ nor $X \cap Y = \emptyset$.

By Lemma 2.8, we have

Lemma 2.9 *If* $X, Y \in \mathcal{C}(g)$ *overlap, then* $X \cap Y$, $X \cup Y$, $X \setminus Y \in \mathcal{C}(g)$.

In fact, the above reasoning can be sharpened to show

Lemma 2.10 *If the clans* X *and* Y *overlap, then the edges in*

$$(X \setminus Y) \times (X \cap Y) \ \cup \ (X \setminus Y) \times (Y \setminus X) \ \cup \ (X \cap Y) \times (Y \setminus X)$$

belong to the same edge class.

On the other hand, the clans of a 2-structure g need not be closed under complement. Indeed, consider the primitive 2-structure g of four nodes and two symmetric edge classes from Figure 5. Since g has only the trivial clans, the complement $\mathrm{dom}(g) \setminus \{x\}$ of the singleton clan $\{x\}$, for $x \in \mathrm{dom}(g)$, is not a clan of g. Also, $\mathcal{C}(g)$ is not necessarily closed under symmetric difference. However, if g has only symmetric edge classes, then $\mathcal{C}(g)$ is also closed under symmetric difference.

The following rather obvious lemma is crucial for the definition of a quotient.

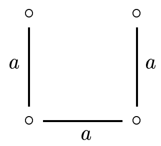


Figure 5: A primitive g

Lemma 2.11 *If $X, Y \in \mathcal{C}(g)$ and $X \cap Y = \emptyset$, then $(x_1, y_1)R = (x_2, y_2)R$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.*

By the following result the clans of the factors $\mathrm{sub}_g(X)$ are exactly those clans of g that are contained in X.

Theorem 2.12 *Let $h = \mathrm{sub}_g(X)$ be a factor of g. Then*

$$\mathcal{C}(h) = \{Y \mid Y \subseteq X,\ Y \in \mathcal{C}(g)\}.$$

Example 2.13 The 2-structure g from Example 2.4 (see Figure 3) has a factor $h = \mathrm{sub}_g(X)$ for $X = \{x_1, x_2, x_3\}$, and $Z = \{x_1, x_3\}$ is a clan of h. Hence Z is also a clan of g. $\square$

3 Decompositions of 2-Structures

3.1 Prime clans

A nonempty clan $P \in \mathcal{C}(g)$ is a *prime clan*, if it does not overlap with any clan of g. Clearly, the nonempty trivial clans are prime clans.

We denote by $\mathcal{P}(g)$ the set of all prime clans of g. For a prime clan P of g, the factor $\mathrm{sub}_g(P)$ is called a *prime factor* of g.

The next lemma follows immediately from the definition of a prime clan.

Lemma 3.1 *Let $g = (D, R)$. If P is a prime clan and $X \in \mathcal{C}(g)$ intersects with P, then either $P \subseteq X$ or $X \subseteq P$.*

By Theorem 2.12, we may easily deduce that if a prime clan $P \in \mathcal{P}(g)$ is contained in a clan $X \in \mathcal{C}(g)$, then P is a prime clan of the factor $\mathrm{sub}_g(X)$. Also the converse holds for proper prime clans of the factor $\mathrm{sub}_g(X)$ as can been seen using Lemma 3.1. Hence we have

Theorem 3.2 *If $h = \mathrm{sub}_g(X)$ is a factor of g, then*

$$\mathcal{P}(h) = \{P \mid P \in \mathcal{P}(g),\ P \subseteq X\} \cup \{X\}.$$

3.2 Quotients

Let g be a (reversible) 2-structure and let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be a partition of $\mathrm{dom}(g)$ into nonempty clans. We call $\mathcal{X}$ a *factorization* of g, and the substructures $\mathrm{sub}_g(X_i)$ are the *factors* of this factorization.

Each 2-structure g has at least the *trivial factorizations* $\mathcal{X} = \{\mathrm{dom}(g)\}$ and $\mathcal{X} = \{\{x\} \mid x \in \mathrm{dom}(g)\}$.

The *quotient* of a $g = (D, R)$ by a factorization $\mathcal{X}$ is defined to be the 2-structure $g/\mathcal{X}$ with the domain $\mathcal{X}$, and the equivalence relation $R_{\mathcal{X}}$, for which

$$(X_1, Y_1) R_{\mathcal{X}} (X_2, Y_2) \iff (x_1, y_1) R (x_2, y_2) \text{ for some } x_i \in X_i,\ y_i \in Y_i,$$

where each $X_i, Y_i \in \mathcal{X}$. Thus a quotient $g/\mathcal{X}$ is obtained from g by contracting each clan X in the partition $\mathcal{X}$ into a single node, and then inheriting the edge classes from g.

By Lemma 2.11, the quotient $g/\mathcal{X}$ is well-defined, *i.e.*, it is independent of the choice of the representatives: if $(x_1, y_1) R (x_2, y_2)$ for *some* $x_1 \in X_1$, $x_2 \in X_2$, and $y_1 \in Y_1$, $y_2 \in Y_2$, then $(x_1, y_1) R (x_2, y_2)$ for *all* $x_1 \in X_1$, $x_2 \in X_2$ and $y_1 \in Y_1$, $y_2 \in Y_2$ (where X_i, Y_i are in $\mathcal{X}$).

Example 3.3 The 2-structure g from Example 2.4 (see Figures 3 and 4) has a factorization

$$\mathcal{X} = \{X, Y, \{x_4\}, \{x_5\}, \{x_6\}\}.$$

We can redraw g with respect to $\mathcal{X}$ as in Figure 6. The quotient $g/\mathcal{X}$ is now obtained by contracting the factors into nodes as in Figure 7. $\square$

A *decomposition*

$$g/\mathcal{X}(\mathrm{sub}_g(X_1), \ldots, \mathrm{sub}_g(X_k))$$

of a 2-structure g consists of the factors $\mathrm{sub}_g(X_i)$ with respect to a factorization $\mathcal{X} = \{X_1, \ldots, X_k\}$ together with the quotient $g/\mathcal{X}$ that gives the relationships between the factors.

In the general case of 2-structures (that we have been studying now in contrast to the *labeled* 2-structures of Section 6) we may lose some information in the formation of a decomposition. Indeed, as the next example shows, two different 2-structures may possess a common decomposition.

Example 3.4 The 2-structures g_1 and g_2 of Figure 8 have a common factorization $\mathcal{X} = \{\{x_1, x_2\}, \{x_3, x_4\}\}$. Moreover,

$$\mathrm{sub}_{g_1}(\{x_1, x_2\}) = \mathrm{sub}_{g_2}(\{x_1, x_2\}) \quad \text{and} \quad \mathrm{sub}_{g_1}(\{x_3, x_4\}) = \mathrm{sub}_{g_2}(\{x_3, x_4\}).$$

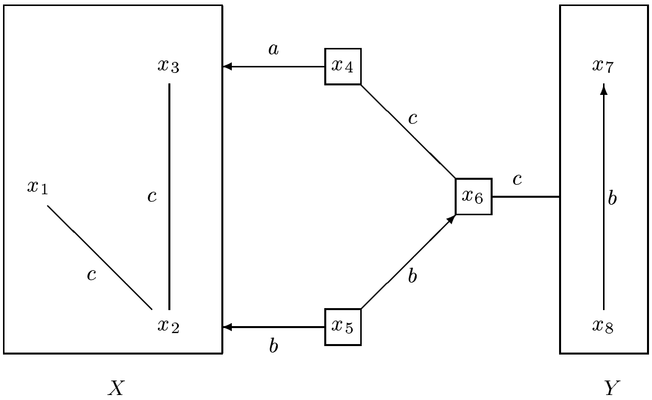


Figure 6: Disjoint factors of g

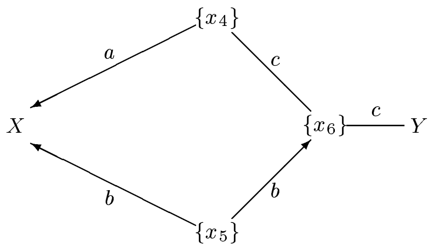


Figure 7: $g/\mathcal{X}$

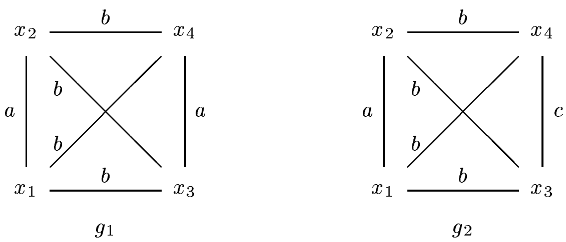


Figure 8: g_1 and g_2

Also, the quotients $g_1/\mathcal{X}$ and $g_2/\mathcal{X}$ are equal as shown in Figure 9. However, g_1 and g_2 are different 2-structures, since $(x_1, x_2)R_{g_1} = (x_3, x_4)R_{g_1}$, but $(x_1, x_2)R_{g_2} \neq (x_3, x_4)R_{g_2}$. $\square$

$$\{x_1, x_2\} \quad \overset{b}{\rule{2cm}{0.4pt}} \quad \{x_3, x_4\}$$

Figure 9: A common factorization $\mathcal{X}$

Next we shall study the clans of the quotients $g/\mathcal{X}$ for factorizations $\mathcal{X} \subseteq \mathcal{C}(g)$. For this we adopt the following notations. If $\mathcal{A}$ is a family of sets, then let

$$\cup\mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \text{and} \qquad \cap\,\mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

Let $g = (D, R)$ and let $\mathcal{X}$ be a factorization of g. For each subset $Y \subseteq D$, we let

$$K_{\mathcal{X}}(Y) = \{X \mid X \in \mathcal{X}, X \cap Y \neq \emptyset\}$$

be the *frame* of Y in $g/\mathcal{X}$; the situation is illustrated in Figure 10.

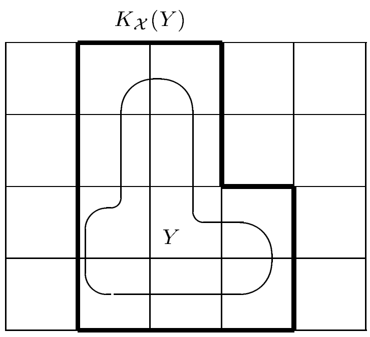


Figure 10: The frame of Y in $g/\mathcal{X}$

We begin with an immediate result.

Lemma 3.5 *Let $g = (D, R)$ and let $\mathcal{X}$ be a factorization of g. If $Y \in \mathcal{C}(g)$, then $K_{\mathcal{X}}(Y) \in \mathcal{C}(g/\mathcal{X})$ and $\cup K_{\mathcal{X}}(Y) \in \mathcal{C}(g)$.*

It can happen that for a prime clan $\mathcal{Z}$ of a quotient $g/\mathcal{X}$, the 'inverse image' $\cup\mathcal{Z}$ is not a prime clan of g. Indeed, consider a factorization $\mathcal{X}$ and a clan $X \in \mathcal{X}$. Now, X is a singleton clan of the quotient $g/\mathcal{X}$, and hence a prime clan in this quotient, but $\cup\{X\} \in \mathcal{P}(g)$ only if $X \in \mathcal{P}(g)$. The following lemma states that the singletons are the only examples of such a situation.

Theorem 3.6 *Let $g = (D, R)$ and let $\mathcal{X}$ be a factorization of g.*

(1) *If $\mathcal{Z} \in \mathcal{C}(g/\mathcal{X})$, then $\cup Z \in \mathcal{C}(g)$.*
(2) *If $\mathcal{Z} \in \mathcal{P}(g/\mathcal{X})$ is not a singleton, then $\cup Z \in \mathcal{P}(g)$.*
(3) *If $\mathcal{X} \subseteq \mathcal{P}(g)$, then for each prime clan $\mathcal{Z} \in \mathcal{P}(g/\mathcal{X})$, also $\cup\mathcal{Z} \in \mathcal{P}(g)$.*

Proof. Case (1) follows easily from the definition of clan, and Case (3) is a consequence of (2) and the discussion above. Hence we need to prove only (2).

Suppose $\mathcal{Z}$ is a proper prime clan of the quotient $g/\mathcal{X}$ (*i.e.*, $\mathcal{Z} \neq \mathcal{X}$). By (1), $\cup\mathcal{Z} \in \mathcal{C}(g)$. Assume now that there exists a clan $Y \in \mathcal{C}(g)$ that overlaps with $\cup\mathcal{Z}$. By Lemma 3.5, the frame $K_{\mathcal{X}}(Y)$ is a clan of $g/\mathcal{X}$, and $\cup K_{\mathcal{X}}(Y) \in \mathcal{C}(g)$.

Since $\mathcal{Z}$ is a prime clan in the quotient, $K_{\mathcal{X}}(Y)$ does not overlap with it. But clearly,

$$K_{\mathcal{X}}(Y) \cap \mathcal{Z} \neq \emptyset \qquad \text{and} \qquad K_{\mathcal{X}}(Y) \setminus \mathcal{Z} \neq \emptyset,$$

since $Y \cap \cup\mathcal{Z} \neq \emptyset$ and $Y \setminus \cup\mathcal{Z} \neq \emptyset$. Therefore $\mathcal{Z} \subset K_{\mathcal{X}}(Y)$, where the inclusion is proper.

By above, for each $z \in \cup\mathcal{Z}$ there exists an $X \in \mathcal{Z}$ such that $X \cap Y \neq \emptyset$ and $z \in X$.

Let $z \in \cup\mathcal{Z} \setminus Y$. Such a z exists, because $\cup\mathcal{Z}$ and Y overlap. Let $X \in \mathcal{Z}$ be such that $X \cap Y \neq \emptyset$ and $z \in X \setminus Y$. The situation is now as in Figure 11.

It follows that X and Y overlap with each other, and, in particular, $Y \setminus X$ is a clan of g by Lemma 2.9. By Lemma 3.5, $K_{\mathcal{X}}(Y \setminus X) \in \mathcal{C}(g/\mathcal{X})$. However, $K_{\mathcal{X}}(Y \setminus X) = K_{\mathcal{X}}(Y) \setminus \{X\}$, since $X \in \mathcal{X}$. Now, $K_{\mathcal{X}}(Y \setminus X)$ overlaps with $\mathcal{Z}$ unless $\mathcal{Z} = \{X\}$ is a singleton clan of $g/\mathcal{X}$. This proves the claim. □

3.3 Maximal prime clans

Let $g = (D, R)$. A clan $M \in \mathcal{C}(g)$ is called *maximal*, if it is a proper clan and for all proper clans X, $M \subseteq X$ implies that $X = M$.

A prime clan $P \in \mathcal{P}(g)$ is a *maximal prime clan*, if it is a proper clan, and maximal among the proper prime clans of g: for all proper prime clans Q, $P \subseteq Q$ implies that $Q = P$.

We denote by $\mathcal{P}_{\max}(g)$ the set of all maximal prime clans of g. For a maximal prime clan $P \in \mathcal{P}_{\max}(g)$, the factor $\mathrm{sub}_g(P)$ is called a *maximal prime factor* of g.

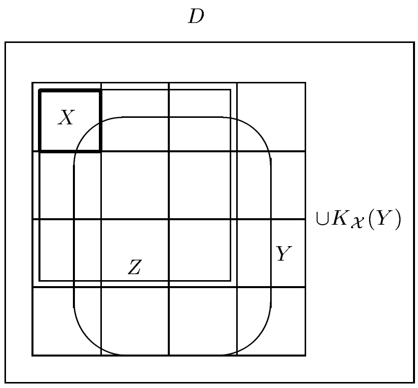


Figure 11: X and Y overlap

We observe that all 2-structures $g = (D, R)$ with $|D| \geq 2$ have maximal prime clans, because every singleton is a prime clan. By definition of maximality, this is not true in the trivial case, where $|D| = 1$. For this reason we set $\mathcal{P}_{\max}(g) = \{D\}$, if $|D| = 1$.

Lemma 3.7 *Let $g = (D, R)$. Every $x \in D$ belongs to a unique maximal prime clan P_x of g.*

Proof. Suppose $|D| \geq 2$. For each $x \in D$ there is a proper prime clan containing x, namely $\{x\}$. Let P_x be then a proper prime clan so that P_x is maximal with respect to the condition $x \in P_x$. Such a P_x exists because the domain D is finite. Now, if $P \neq P_x$ is a proper prime clan with $x \in P$, then $P \cap P_x \neq \emptyset$, and by the maximality of P_x, $P_x \subset P$ cannot hold. Hence since P is a prime clan, $P \subset P_x$, and therefore P_x is the unique maximal prime clan containing x. □

Theorem 3.8 *Let g be a 2-structure. The maximal prime clans form a partition $\mathcal{P}_{\max}(g)$ of $\mathrm{dom}(g)$, and therefore the quotient $g/\mathcal{P}_{\max}(g)$ is well-defined.*

Proof. By Lemma 3.7, each node x belongs to a unique maximal prime clan P_x, and thus $\mathrm{dom}(g)$ is a disjoint union of the maximal prime clans, because for each $x, y \in \mathrm{dom}(g)$ either $P_x = P_y$ or $P_x \cap P_y = \emptyset$. □

Lemma 3.9 *For each clan $Y \in \mathcal{C}(g)$ either $Y \subseteq P$ for a maximal prime clan $P \in \mathcal{P}_{\max}(g)$ or Y is a union of maximal prime clans.*

Proof. Assume Y is not a subset of any maximal prime clan of g. If $Y \cap P \neq \emptyset$ for some $P \in \mathcal{P}_{\max}(g)$, then Lemma 3.1 implies that $P \subset Y$. Since every $y \in Y$

belongs to a unique maximal prime clan P_y and $P_y \cap Y \neq \emptyset$, it follows that

$$Y = \bigcup_{y \in Y} P_y,$$

which proves the claim. □

Since each singleton is a prime clan, it follows that every clan $Y \in \mathcal{C}(g)$ is the union of prime clans $P \in \mathcal{P}(g)$ that are contained in Y. By Theorem 3.2, the prime clans P of the factor $\mathrm{sub}_g(Y)$ are exactly the prime clans of g that are contained in Y. Therefore we have the following result.

Theorem 3.10 *Each nonsingleton clan $Y \in \mathcal{C}(g)$ is the union of the prime clans $P \in \mathcal{P}(g)$ that are the maximal prime clans of the factor* $\mathrm{sub}_g(Y)$.

3.4 Special 2-structures

Recall that a 2-structure g is primitive if it has only the trivial clans. We generalize the notion of primitivity by defining a 2-structure g to be *special*, if all its prime clans are trivial. Hence if g is special, then $\mathcal{P}_{\max}(g)$ consists of the singletons of the domain.

The next result from [16] is crucial for the clan decomposition theorem.

Theorem 3.11 *The quotient $g/\mathcal{P}_{\max}(g)$ is special.*

Proof. Assume that $g/\mathcal{P}_{\max}(g)$ has a prime clan $\mathcal{X} = \{P_1, \ldots, P_k\}$ for $k \geq 2$, where each P_i is a maximal prime clan of g. By Theorem 3.6(2),

$$X = \bigcup_{i=1}^{k} P_i \in \mathcal{P}(g).$$

Hence, because each $P_i \subset X$ $(i = 1, 2, \ldots, k)$ is a maximal prime clan of g, we must have that $X = \mathrm{dom}(g)$. Thus $\mathcal{X}$ is the domain of the quotient $g/\mathcal{X}$, and hence not a proper prime clan. □

Lemma 3.12 *Let g be a special 2-structure. Then either*

(1) *the maximal clans M are all singletons, or*

(2) *all the complements $\mathrm{dom}(g) \setminus M$ of the maximal clans M are singletons.*

Proof. Suppose $M \in \mathcal{C}(g)$ is a maximal clan with at least two nodes. Since in a special g all prime clans are trivial, M is not a prime clan, and hence there exists a clan X that overlaps with M. By Lemma 2.9, $M \cup X \in \mathcal{C}(g)$ and $X \setminus M \in \mathcal{C}(g)$.

Now, since M is maximal and $M \subset M \cup X$, it follows that $M \cup X = \mathrm{dom}(g)$, and hence, in particular, $\mathrm{dom}(g) \setminus M = X \setminus M$. By above, the complement of M, $\overline{M} = \mathrm{dom}(g) \setminus M$, is a clan. If $\overline{M}$ is a prime clan, then it must be a singleton because g is special. On the other hand, if $\overline{M}$ overlaps with another clan Y, then Y overlaps also with M by maximality of M, and in this case $M \cup Y \in \mathcal{C}(g)$ would be a proper clan contradicting the maximality of M. We conclude that $\overline{M}$ is a singleton clan.

Finally, if N is any other maximal clan, then the above argument shows that either its complement $\overline{N}$ is a singleton or N itself is a singleton. The latter condition cannot hold, because $|N| = 1$ would imply $N = \overline{M}$ or $N \subseteq M$. The inclusion $N \subseteq M$ and the assumption $N \neq M$ imply $N \subset M$; this contradicts the maximality of N. Hence $N = \overline{M}$ holds. Since X overlaps with M, we get $N \subset X$ contradicting the maximality of N. $\square$

3.5 The clan decomposition theorem

We shall now give a new proof of the clan decomposition theorem of [16] using maximal clans.

A 2-structure $g = (D, R)$ is said to be *complete*, if it has only one edge class, *i.e.*, for all $e_1, e_2 \in E_2(D)$, $e_1 R e_2$. In particular, the only edge class is symmetric, *i.e.*, eRe^{-1} for all $e \in E_2(D)$. Clearly, in a complete 2-structure g each subset of the domain is a clan. It follows from this that a complete 2-structure is special.

A 2-structure g is said to be *linear*, if g has a nonsymmetric edge class a which linearly orders the domain D of g, *i.e.*, D has an ordering $x_1, x_2, \ldots, x_n$ such that $(x_i, x_j) \in a$ if and only if $i < j$. In this case the nonempty clans of g are exactly the segments $\{x_i, x_{i+1}, \ldots, x_{i+k}\}$ for $1 \leq i \leq i + k \leq n$. Consequently, a linear 2-structure is special.

Recall also that a g is *primitive*, if it has no nontrivial clans. Notice that a 2-structure g with at most two nodes is always primitive and at the same time either complete or linear. For this reason we say that g is *truly primitive*, if it is primitive and it has at least three nodes.

Figure 12: A complete and a linear 2-structure on four nodes

As stated in the following theorem (from [16]) there are only three types of special 2-structures.

Theorem 3.13 *A 2-structure g is special if and only if it is either linear, or complete, or truly primitive.*

Proof. It is easy to show that if g is complete, linear or primitive, then its prime clans are trivial.

Suppose then that $g = (D, R)$ is special. We show by induction on the number $|D|$ of nodes in the domain that g is of one of the types as claimed. The claim clearly holds if $|D| \leq 3$ (by simple case analysis).

Assume that $|D| > 3$, and let M be a maximal clan of g. If $|M| = 1$, then by Lemma 3.12 it follows that g is primitive, since now the maximal clans are singletons, and so all the proper clans are singletons.

We may thus assume that $|M| > 1$, and hence, again by Lemma 3.12, that the complement $\overline{M}$ of M is a singleton. Let $D \setminus M = \{v\}$. There exists a maximal clan N that contains v, and, as above, $|N| > 1$ and the complement $\overline{N}$ is a singleton, say $D \setminus N = \{w\}$, see Figure 13.

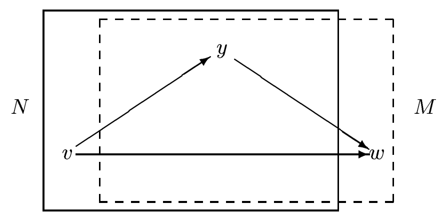


Figure 13: Maximal M and N

Consider the maximal factor $\text{sub}_g(M)$. If P is a proper prime clan of $\text{sub}_g(M)$, then P is also a proper prime clan of g by Theorem 3.2, and hence P is necessarily a singleton. Therefore $\text{sub}_g(M)$ is special, and by the induction hypothesis $\text{sub}_g(M)$ is complete, linear or primitive.

The factor $\text{sub}_g(M)$ cannot be primitive, since it has the nontrivial clan $N \cap M$ by Theorem 2.12 and by the fact that $|D| > 3$.

Assume that $\text{sub}_g(M)$ is linear or complete. Since $M \in \mathcal{C}(g)$ and $v \notin M$,

$$(v, y)R = (v, w)R \quad \text{for all} \quad y \in M.$$

Also, since $N \in \mathcal{C}(g)$ and $w \notin N$,

$$(v, w)R = (y, w)R \quad \text{for all} \quad y \in N.$$

From these it follows that $(v, y)R = (y, w)R$ for all $y \in M \cap N$, and thus g is linear if $\mathrm{sub}_g(M)$ is linear, and g is complete if $\mathrm{sub}_g(M)$ is complete. $\square$

We have thus proved the following *clan decomposition theorem* of [16].

Theorem 3.14 *For any 2-structure g, the quotient $g/\mathcal{P}_{\max}(g)$ is either linear, or complete, or truly primitive.*

There are also other proofs of the clan decomposition theorem, see Ehrenfeucht and Rozenberg [16], Engelfriet, Harju, Proskurowski and Rozenberg [23] and Harju and Rozenberg [31]. In a general setting the decomposition theorem is proved also in Möhring and Radermacher [37] and Möhring [36]. The decomposition theorem for submodular functions as presented by Cunningham [5] is closely related to the clan decomposition theorem. In [37] and [31] the proofs are given for infinite domains, for which the theorem has a minor modification due to the fact that maximal prime clans may not exist, if the domain of a 2-structure is infinite.

For a 2-structure g with the maximal prime clans $P_1, \dots, P_m$, the *clan decomposition* $g/\mathcal{P}_{\max}(g)(\mathrm{sub}_g(P_1), \dots, \mathrm{sub}_g(P_m))$ of g consists of the maximal prime factors $\mathrm{sub}_g(P_i)$ together with the special quotient $g/\mathcal{P}_{\max}(g)$.

Example 3.15 Consider the 2-structure g of Figure 14 with five vertices and five edge classes, a, a^{-1}, b, b^{-1}, c. In Figure 14 we have also extracted the maximal prime factors, and the linear quotient $g/\mathcal{P}_{\max}(g)$. $\square$

3.6 The shape of a 2-structure

By the clan decomposition theorem, each 2-structure g can be decomposed into its maximal prime factors so that the resulting quotient is special. Also, the maximal prime factors can be decomposed, if they are not special already. This allows us to decompose g completely into its prime factors, not only into its maximal prime factors. Indeed, by Theorem 3.2, we have the following immediate result.

Lemma 3.16 (1) *Let P be a nonsingleton prime clan of g. The maximal prime clans of the factor $\mathrm{sub}_g(P)$ are exactly the prime clans of g that are properly contained in P, and are maximal in this respect.*

(2) *Every proper prime clan X of g is a maximal prime clan of some prime factor $\mathrm{sub}_g(P)$.*

The partially ordered set $T(g) = (\mathcal{P}(g), \subseteq)$ of the prime clans forms a rooted tree, where the domain $\mathrm{dom}(g)$ is the root and the singletons are the leaves. We call the tree $T(g)$ the *prime tree* of g.

The *shape* of a 2-structure g is defined as a pair

$$\mathrm{shape}(g) = (T(g), \Psi_g),$$

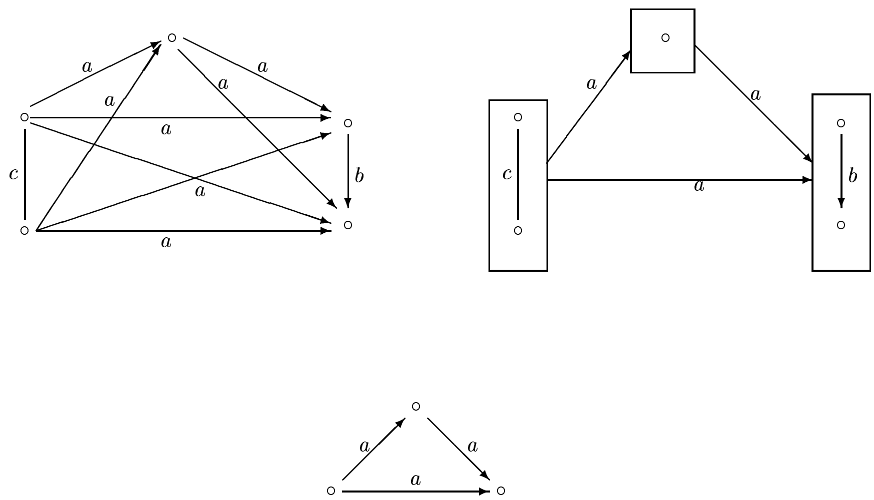


Figure 14: A clan decomposition

where $T(g)$ is the prime tree of g and

$$\Psi_g(P) = \mathrm{sub}_g(P)/\mathcal{P}_{\max}(\mathrm{sub}_g(P))$$

is a function, which associates the special quotient $\mathrm{sub}_g(P)/\mathcal{P}_{\max}(\mathrm{sub}_g(P))$ with each prime clan $P \in \mathcal{P}(g)$.

The shape of a 2-structure g can conveniently be represented as a tree-like object by contracting each prime factor $\mathrm{sub}_g(P)$ to the corresponding quotient $\Psi_g(P)$, and then drawing a line down from a node Q of $\Psi_g(P)$ to the corresponding quotient $\Psi_g(Q)$ of Q.

Example 3.17 The shape of the 2-structure g from Figure 16 is given in Figure 15. We have not identified the inner nodes (circles) of the quotients (rectangles). Indeed, each node $Q \in \Psi_g(P)$ is the prime clan consisting of the singletons x_i that are descendants of this node. Also, we have omitted the colours from this figure, since (in this example) each quotient $\Psi_g(P)$ has only one pair of edges, and hence it has either one symmetric edge class or two edge classes that are reverses of each other. $\square$

The observation of Example 3.4 is valid also for the shapes. We may lose some information on the 2-structure g while forming the shape of g, *i.e.*, two different 2-structures may have the same shape.

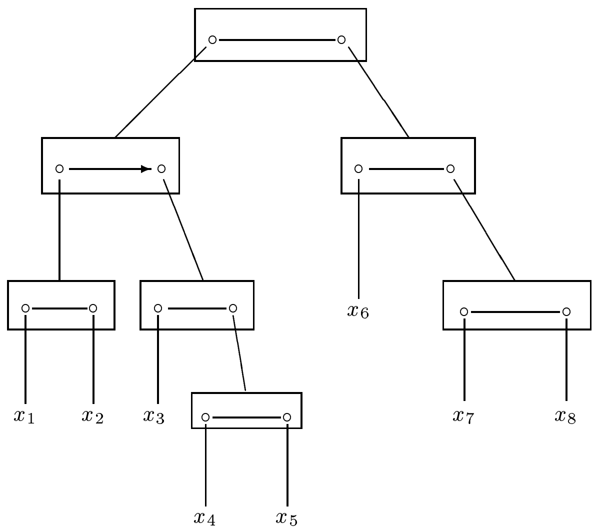


Figure 15: The shape of g

We can also draw the shape of a 2-structure g by nesting the special quotients $\Psi_g(P)$: In this representation a quotient (rectangle) $\Psi_g(Q)$ is drawn inside of quotient $\Psi_g(P)$ replacing the node Q of $\Psi_g(P)$.

Example 3.18 In Figure 17 there is a nested representation of the shape of the 2-structure g from Figure 16. The result is again rather economical compared to the full drawing of g. □

3.7 Constructions of clans and prime clans

The algorithmic complexity of forming the shape of a 2-structure was shown to be of order $O(n^2)$ in Ehrenfeucht, Gabow, McConnell and Sullivan [8]. Here n refers to the number of nodes in the 2-structures.

The algorithm given in [8] is not incremental. In an *incremental algorithm* the shape of a 2-structure g with a domain $\{x_1, x_2, \ldots, x_n\}$ is constructed starting from the shape of a single node substructure $\text{sub}_g(\{x_1\})$ and then for each $i = 1, 2, \ldots, n-1$ constructing the shape of the substructure $\text{sub}_g(\{x_1, x_2, \ldots, x_{i+1}\})$ by updating the shape of the substructure $\text{sub}_g(\{x_1, x_2, \ldots, x_i\})$.

An incremental algorithm of complexity $O(n^2)$ for the construction of the shape of a 2-structure is given in McConnell [35]. When restricted to undi-

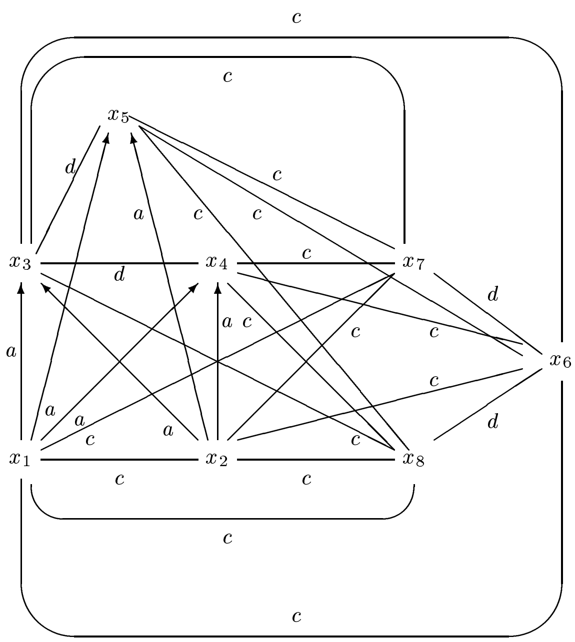


Figure 16: A 2-structure g

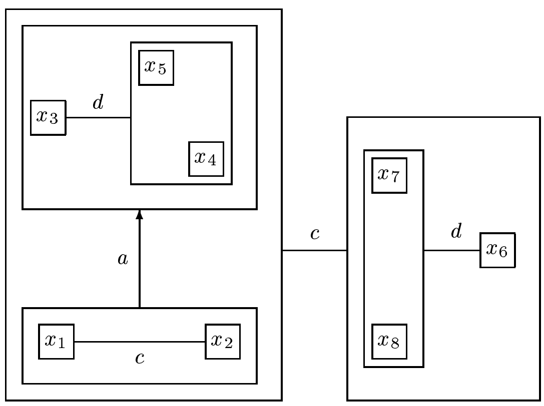


Figure 17: The nested representation of the shape of g

rected graphs this algorithm gives another method of constructing the modular decomposition of graphs in time $O(n^2)$, see also Muller and Spinrad [38].

We shall now give some constructive characterizations of clans and prime clans. These results were used in Engelfriet, Harju, Proskurowski and Rozenberg [23] in algorithmic constructions of shapes.

Let $g = (D, R)$ be a 2-structure, and let $X \subseteq D$. We shall denote by $C(X)$ the smallest clan of g that contains the subset X. Such a clan exists since $\mathcal{C}(g)$ is closed under intersection.

For each $x \in D$, define a relation R_x on D as follows,

$$R_x = \{(u, v) \mid (v, u)R \neq (v, x)R\}.$$

In words, if uR_xv, then v 'sees' u and x differently. A subset $X \subseteq D$ is said to be *closed* under R_x, if $u \in X$ and uR_xv imply $v \in X$. Further, let R_x^* be the reflexive and transitive closure of the relation R_x.

Lemma 3.19 *The following statements are equivalent for each nonempty subset $X \subseteq \mathrm{dom}(g)$:*

(1) *X is a clan of g,*
(2) *X is closed under R_x for all $x \in X$,*
(3) *X is closed under R_x for some $x \in X$.*

Proof. If $X \in \mathcal{C}(g)$, then clearly X is closed under R_x for all $x \in X$. Hence it suffices to show that (3) implies (1).

Assume that (3) holds, and let $x \in X$ be a node such that X is closed under R_x. Suppose to the contrary of the claim that X is not a clan. Hence there are nodes $u \notin X$ and $y, z \in X$ such that $(u, y)R \neq (u, z)R$. It follows that either $(u, y)R \neq (u, x)R$ or $(u, z)R \neq (u, x)R$. Hence yR_xu or zR_xu, which implies that $u \in X$; a contradiction. □

The following characterization of $C(X)$ was proved in [23].

Theorem 3.20 *For each subset $X \subseteq \mathrm{dom}(g)$ and node $x \in X$,*

$$C(X) = \{u \mid yR_x^*u \ \textit{ for some } y \in X\}.$$

Proof. Denote $W(X) = \{u | yR_x^*u \ \text{ for some } \ y \in X\}$. First of all, $X \subseteq W(X)$, because R_x^* is reflexive. Moreover, by definition, $W(X)$ is the smallest set containing X and closed under R_x, and hence, $W(X) = C(X)$ by Lemma 3.19. □

Let $x, y \in \mathrm{dom}(g)$. We write simply $C(x, y)$ instead of $C(\{x, y\})$. Not every clan is of the form $C(x, y)$, but as the next lemma shows a clan is a prime clan just in case it does not overlap with a clan of this form.

Lemma 3.21 *Let $X \in \mathcal{C}(g)$ for a 2-structure g. Then $X \in \mathcal{P}(g)$ if and only if X does not overlap with any $C(x,y)$, where $x \in X$ and $y \notin X$.*

Proof. In the one direction the claim follows by the definition of a prime clan. On the other hand, if a clan X is not a prime clan, then it overlaps with a clan $Y \in \mathcal{C}(g)$. Let $x \in X \cap Y$ and $y \in Y \setminus X$. Now, $C(x,y) \subseteq Y$, and hence $C(x,y)$ is a clan that overlaps with X. This proves the claim. □

Denote by $P(X)$ the smallest prime clan containing $X \subseteq \mathrm{dom}(g)$. Such a prime clan exists since the prime clans do not overlap, and $\mathrm{dom}(g) \in \mathcal{P}(g)$. Define for each $x \in \mathrm{dom}(g)$,

$$S_x = \{(u,v) \mid uR_xv \;\text{ or }\; u \notin C(x,v) \;\text{ or }\; x \notin C(u,v)\}.$$

Notice that the requirement $u \notin C(x,v)$ or $x \notin C(u,v)$ is equivalent with $C(x,v) \neq C(u,v)$. Also, denote by S_x^* the reflexive and transitive closure of S_x. Again, we write $P(x,y)$ instead of $P(\{x,y\})$.

In [23] the sets $P(X)$ were characterized as follows.

Theorem 3.22 *For each subset $X \subseteq \mathrm{dom}(g)$ and node $x \in X$,*

$$P(X) = \{u \mid yS_x^*u \;\textit{ for some }\; y \in X\}.$$

Proof. Let $W(X) = \{u \mid yS_x^*u \;\text{ for some }\; y \in X\}$. Clearly, $X \subseteq W(X)$. Moreover, by Lemma 3.19, $W(X)$ is a clan, since it is closed under the relation R_x.

We prove that $W(X)$ is a prime clan. For this let $u \notin W(X)$. Hence, for all $v \in W(X)$, vS_xu does not hold. By definition of S_x we have that $v \in C(x,u)$ and $x \in C(v,u)$. Hence $C(x,u) = C(v,u)$ for all $v \in W(X)$. Consequently, $W(X) \subseteq C(v,u)$ for all $u \notin W(X)$ and $v \in W(X)$, which proves that $W(X)$ is a prime clan by Lemma 3.21. Thus $P(X) \subseteq W(X)$.

Suppose then that $u \notin P(X)$, and let $v \in P(X)$. Clearly, vS_xu does not hold since $P(X)$ is a prime clan, and hence $u \notin W(X)$, which proves that $W(X) \subseteq P(X)$.

□

The following theorem (from [23]) shows that every prime clan is of the form $P(x,y)$.

Theorem 3.23 *Let g be a 2-structure. For each prime clan $P \in \mathcal{P}(g)$ there exist $x,y \in P$ such that $P = P(x,y)$.*

Proof. If $P = \{x\}$ is a singleton prime clan, then clearly $P = P(x,x)$. Suppose then that $|P| \geq 2$, and let P_1 and P_2 be two maximal prime clans of $\mathrm{sub}_g(P)$ with $P_1 \neq P_2$. Let $x \in P_1$ and $y \in P_2$. Now, $P = P(x,y)$ as required. □

3.8 Clans and sibas

We shall now describe how the clan decomposition theorem can be applied to systems which possess clan-like structures in the sense defined below.

Let $\mathcal{S}$ be a family of subsets of a finite set D. We say that $\mathcal{S}$ is a *siba* on D (or a *semi-independent partial boolean algebra*, see [15]), if it satisfies the following conditions:

(1) $\emptyset \in \mathcal{S}$, $D \in \mathcal{S}$ and $\{x\} \in \mathcal{S}$ for all $x \in D$;
(2) if $X, Y \in \mathcal{S}$, then $X \cap Y \in \mathcal{S}$;
(3) if $X, Y \in \mathcal{S}$ and $X \cap Y \neq \emptyset$, then $X \cup Y \in \mathcal{S}$;
(4) if $X, Y \in \mathcal{S}$ and $Y \setminus X \neq \emptyset$, then $X \setminus Y \in \mathcal{S}$.

The sets in (1) are called the *trivial elements* of $\mathcal{S}$.

Let $\mathcal{S}$ be a siba on D. We say that an element $X \in \mathcal{S}$ is a *prime element*, if X does not overlap with any element of $\mathcal{S}$, and that $X \in \mathcal{S}$ is a *maximal* (*prime*) *element*, if it is maximal among the proper (prime) elements of $\mathcal{S}$ with respect to inclusion.

Clearly, the maximal prime elements of $\mathcal{S}$ form a partition of the set D. Further, as in Lemma 3.9, we obtain

Lemma 3.24 *Let $\mathcal{S}$ be a siba on D. For each $Y \in \mathcal{S}$ either $Y \subseteq P$ for a maximal prime element P of $\mathcal{S}$ or Y is a union of maximal prime elements.*

A siba $\mathcal{S}$ is said to be *trivial*, if its elements are trivial, and $\mathcal{S}$ is *special*, if its prime elements are trivial. Further, $\mathcal{S}$ is *complete*, if it is the power set 2^D, and *linear*, if there exists a linear order $\{x_1, x_2, \ldots, x_n\}$ of D such that $\mathcal{S}$ consists of the emptyset and the segments $\{x_i, \ldots, x_j\}$ for $1 \leq i \leq j \leq n$.

We notice that the proof of Lemma 3.12 applies as such to sibas, and the proof of Theorem 3.13 can be rather easily modified to prove the following result, see also Möhring [36] and Deutz [7].

Lemma 3.25 *A siba is special if and only if it is either linear, or complete, or trivial.*

Let $\mathcal{S}$ be a siba on D. We denote

$$\mathcal{S}|X = \{X \cap Y \mid Y \in \mathcal{S}\}$$

for each $X \in \mathcal{S}$. It is easy to show that $\mathcal{S}|X$ is a siba on X.

If $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ is a partition of D into elements $X_i \in \mathcal{S}$, then we define the *quotient* $\mathcal{S}/\mathcal{X}$ as the family of subsets of $\mathcal{X}$ such that $\mathcal{Y} \in \mathcal{S}/\mathcal{X}$ if and only if $\cup\mathcal{Y} \in \mathcal{S}$. It is again easy to show that the quotient $\mathcal{S}/\mathcal{X}$ is a siba on $\mathcal{X}$.

By Lemma 2.8, $\mathcal{C}(g)$ is a siba for each 2-structure g. The following theorem shows that also the converse is true.

Theorem 3.26 *A family $\mathcal{S}$ of subsets of a finite set D is a siba if and only if there is a 2-structure g such that $\mathcal{S} = \mathcal{C}(g)$.*

Proof. Let $\mathcal{S}$ be a siba on D. We show by induction on the cardinality of D that there exists a 2-structure g with $\mathcal{C}(g) = \mathcal{S}$. If $|D| \leq 2$, then the claim is obvious. Suppose thus that $|D| \geq 3$.

Clearly, if $\mathcal{S}$ is special, then the claim holds by Lemma 3.25, since there exist linear, complete and primitive 2-structures on the domain D.

Assume now that $\mathcal{S}$ is not special, and let $\mathcal{P}$ be the partition of D into the maximal prime elements of $\mathcal{S}$. Denote $\mathcal{Q} = \mathcal{S}/\mathcal{P}$, for short. By induction hypothesis there are 2-structures $g_P = (P, R_P)$ for each $P \in \mathcal{P}$ such that $\mathcal{C}(g_P) = \mathcal{S}|P$, and a 2-structure $h = (\mathcal{P}, R_h)$ on the domain $\mathcal{P}$ such that $\mathcal{C}(h) = \mathcal{Q}$. We define a 2-structure $g = (D, R)$ on the domain D such that

$$(x_1, y_1)R(x_2, y_2) \iff \begin{cases} (x_1, y_1)R_P(x_2, y_2), & \text{if } (x_i, y_i) \in E_2(P) \text{ for} \\ & P \in \mathcal{P},\ i = 1, 2, \text{ or} \\ (P_1, Q_1)R_h(P_2, Q_2), & \text{if } x_i \in P_i, y_i \in Q_i \text{ for some} \\ & (P_i, Q_i) \in E_2(\mathcal{P}),\ i = 1, 2. \end{cases}$$

It is now straightforward to show that $\mathcal{P} = \mathcal{P}_{\max}(g)$, from which one can deduce that $\mathcal{C}(g) = \mathcal{S}$. $\square$

As shown by Möhring and Radermacher [37] the congruence classes of various systems such as set systems, relations and boolean functions form sibas.

Example 3.27 We consider here set systems as in Möhring and Radermacher [37]. A (*normal*) *set system* on a finite domain D is simply a family of subsets $\mathcal{L}$ of D such that $\cup\mathcal{L} = D$. Because of its generality, set systems are applicable in various contexts, see *e.g.* Shapley [40] for game theoretic applications.

Let $\mathcal{S}$ be a set system on A, and let $\mathcal{S}_x$ be a set system on A_x for each $x \in A$, where the domains A_x are nonempty and disjoint. The *substitution composition of* $\{\mathcal{S}_x\}_{x \in A}$ *for* $\mathcal{S}$ is defined as the set system $\mathcal{S}[\mathcal{S}_x; x \in A]$ consisting of the sets $\bigcup_{y \in Y} \tau(y)$, where Y runs through the sets in $\mathcal{S}$ and τ the functions such that $\tau(y) \in \mathcal{S}_y$ for each $y \in A$. The domain of $\mathcal{S}[\mathcal{S}_x; x \in A]$ is $\cup_{x \in A} A_x$.

For a set system $\mathcal{L}$ on D a subset $X \subseteq D$ is called an *autonomous set*, if it is empty or, in the above notations, $\mathcal{L} = \mathcal{S}[\mathcal{S}_x; x \in A]$ and $X = A_x$ for some $x \in A$. A partition of D into nonempty autonomous sets is called a *congruence partition* of $\mathcal{L}$.

It is now a straightforward task to verify that the autonomous sets of a set system $\mathcal{L}$ form a siba. Therefore each set system $\mathcal{L}$ can be represented as a 2-structure such that the autonomous sets of $\mathcal{L}$ are exactly the clans of g. This representation g of $\mathcal{L}$ is easily seen to be faithful in the sense that the congruence partitions of $\mathcal{L}$ are exactly the clan partitions of g, and hence, in a sense, the 'rough' algebraic structure of $\mathcal{L}$ is faithfully represented by the 2-structure g.

The autonomous sets of a set system $\mathcal{L}$ have further closure properties that are not necessarily true for arbitrary 2-structures. In fact, see [37], X is an autonomous set of $\mathcal{L}$ if and only if for all $S_1, S_2 \in \mathcal{L}$ with $S_i \cap X \neq \emptyset$, also $(S_1 \setminus X) \cup (S_2 \cap X) \in \mathcal{L}$.

□

4 Primitive 2-Structures

4.1 Hereditary properties

Of the three special 2-structures linear and complete are determined by (an ordering of) their domains, and therefore they need no further inspection. The only interesting special 2-structures are thus the primitive ones. Indeed, there are arbitrarily complex primitive 2-structures.

By definition, all 2-structures with two nodes are primitive, but these are also either complete or linear. We shall thus consider those primitive 2-structures that are truly primitive.

Figure 18: A path of length three, and a cycle of length three

In Figure 18 there are two simple truly primitive 2-structures. The first of these is a path of length three and the second is a cyclic triangle.

We shall state now without proofs some interesting properties of the primitive 2-structures. The first of these is from Ehrenfeucht and Rozenberg [17].

Theorem 4.1 (Downward hereditarity) *Let $g = (D, R)$ be a truly primitive 2-structure. There are nodes x and y (possibly $x = y$) such that* $\mathrm{sub}_g(D \setminus \{x, y\})$ *is primitive.*

As an example, from the path of length three (in Figure 18) one cannot remove a node without violating primitivity. Hence in this case two nodes have to be removed in order to obtain a primitive substructure.

The proof of Theorem 4.1 is based on the following lemma, which gives a simple condition ensuring that a primitive 2-structure $g = (D, R)$ has no primitive substructures of the form $\mathrm{sub}_g(D \setminus \{x\})$ with $x \in D$. We call such a 2-structure *critically primitive*.

Lemma 4.2 *Let $g = (D, R)$ be a truly primitive 2-structure, and suppose $\mathrm{sub}_g(X)$ is a truly primitive substructure such that $\mathrm{sub}_g(X \cup \{x\})$ is non-primitive for all $x \in D \setminus X$. Then for each $x \in D \setminus X$ the substructure $\mathrm{sub}_g(X \cup \{x\})$ has a unique nontrivial clan Z_x such that either $|Z_x| = 2$ with $x \in Z_x$ or $Z_x = X$.*

The critically primitive 2-structures were characterized in Schmerl and Trotter [41] and Bonizzoni [2]. In particular, the following result was proved in [41].

Theorem 4.3 *Let $g = (D, R)$ be a critically primitive 2-structure with $|D| = n$. If the substructure $h = \mathrm{sub}_g(X)$ is primitive with $|X| = m \geq 3$, then $n \equiv m \pmod 2$.*

This means that if a critically primitive g has an even (odd) number of nodes, then all its substructures with an odd (even) number of nodes are non-primitive. Moreover,

Corollary 4.4 *Let g be a critically primitive 2-structure, and $\mathrm{sub}_g(X)$ a primitive substructure of g with $|X| \geq 4$. Then $\mathrm{sub}_g(X)$ is critically primitive.*

Example 4.5 In Figure 19 we have a critically primitive 2-structure g. Notice that g has only two edge classes, a class together with its reverse class. In graph theoretical terms, g is a tournament. □

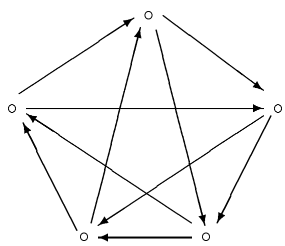


Figure 19: Critically primitive 2-structure g

Lemma 4.2 can be also used to prove the following hereditary result, see Harju and Rozenberg [31] or Schmerl and Trotter [41]. (Again, in [31] the result is proved for infinite 2-structures, for which Theorem 4.1 does not hold).

Theorem 4.6 (Upward hereditarity) *Let $g = (D, R)$ be primitive, and let $h = \mathrm{sub}_g(X)$ be a proper truly primitive substructure of g. Then there are nodes $x, y \in D \setminus X$ (possibly $x = y$) such that $\mathrm{sub}_g(X \cup \{x, y\})$ is primitive.*

Consider, for a while, a 2-structure g as a directed graph $g = (D, E, R)$ with an equivalence relation R on the edge set $E \subseteq E_2(D)$ as mentioned in the beginning of Section 2.

For an edge $e \in E$ of g we let $g - e = (D, E \setminus \{e, e^{-1}\}, R_e)$, where R_e is R restricted to $E \setminus \{e, e^{-1}\}$. (This means that e is classified as 'nonedge' by the equivalence relation R_e).

Theorem 4.1 gives a hereditary property of primitivity concerning removals of nodes from a primitive 2-structure. We ask now whether one can remove an edge e from a primitive 2-structure g so that the result $g - e$ is still primitive.

We call a primitive $g = (D, E, R)$ *unstable* (after Sumner [43]), if $g - e$ is non-primitive for all edges $e \in E$. Figure 20 gives two examples of unstable 2-structures.

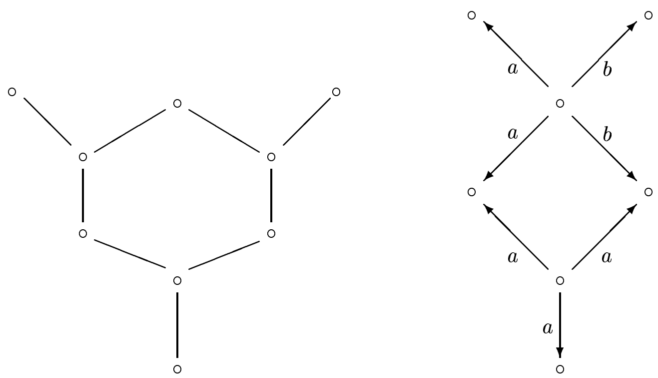


Figure 20: Two unstable 2-structures

For unstable 2-structures we have the following result from Harju and Rozenberg [30]. A *leaf* of a 2-structure $g = (D, E, R)$ is a node of degree one.

Theorem 4.7 *Let $g = (D, E, R)$ be a primitive 2-structure. Either*

(1) *g has an edge e such that $g - e$ is primitive, or*

(2) *g is unstable and it is either a path of length three or it has a leaf x such that* $\mathrm{sub}_g(D \setminus \{x\})$ *is primitive.*

4.2 Uniformly non-primitive 2-structures

We shall now consider those 2-structures that do not contain primitive components. We call a 2-structure g *uniformly non-primitive* if g has no truly primitive substructures.

A number of characterizations of the uniformly non-primitive 2-structures were proved by Engelfriet, Harju, Proskurowski and Rozenberg in [23]. In particular, the graphs and directed graphs that are representable by uniformly non-primitive 2-structures were characterized by forbidden subgraphs in [23]. In the following we shall state only one general characterization result (Theorem 4.11) concerning uniformly non-primitive 2-structures.

Theorem 4.8 *Let g be a uniformly primitive 2-structure. Then the substructures and the quotients of g are uniformly non-primitive.*

By Theorem 4.1, if g is a primitive 2-structure with n nodes, then g has a primitive substructure on $n-1$ or $n-2$ nodes. In particular,

Lemma 4.9 *Every truly primitive 2-structure g has a primitive substructure consisting of three or four nodes.*

Recall that $P(X)$ is the smallest prime clan of a 2-structure $g = (D, R)$ that contains the subset $X \subseteq D$. The following result was proved by Ehrenfeucht and Rozenberg [16].

Theorem 4.10 *Let $g = (D, R)$ be a 2-structure. If $\mathrm{sub}_g(X)$ is a truly primitive substructure of g, then the quotient $\Psi_g(P(X))$ is primitive.*

We say that a 2-structure g has the *2-block property*, if every substructure h of g with $|\mathrm{dom}(h)| \geq 2$ has a partition into two nonempty clans, *i.e.*, if every substructure h has a nonempty clan $X \in \mathcal{C}(h)$ the complement $\mathrm{dom}(h) \setminus X$ of which is also a nonempty clan of h. Also, we say that g has the *doubleton clan property*, if every substructure h with at least two nodes has a clan of size 2. The following theorem was proved in [23].

Theorem 4.11 *The following statements are equivalent for a 2-structure g:*

(1) *g is uniformly non-primitive,*

(2) *g has no primitive substructures of size 3 and 4,*

(3) *each node of* $\mathrm{shape}(g)$ *is linear or complete,*

(4) *g has the 2-block property,*

(5) *g has the doubleton clan property.*

In fact, the proof in [23] reveals that g is uniformly non-primitive if and only if each substructure $\mathrm{sub}_g(X)$ with $|X| \geq 2$ has a prime clan, the complement of which is a clan.

Example 4.12 Let g be the 2-structure from Figure 21. The shape of g is in Figure 22. By Theorem 4.11, g is uniformly non-primitive. $\square$

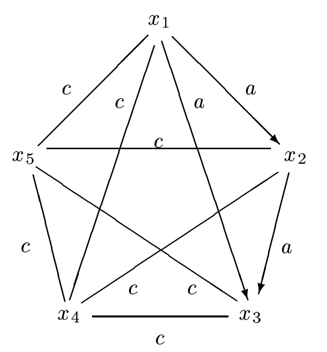


Figure 21: A uniformly non-primitive 2-structure g

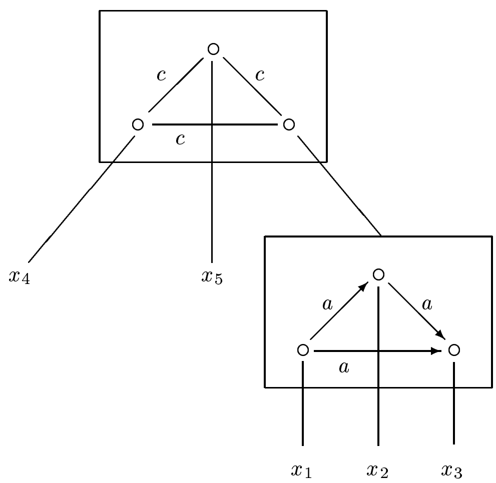


Figure 22: The shape of g

5 Angular 2-structures and T-structures

5.1 Angular 2-structures

By Lemma 4.9 a truly primitive 2-structure contains a primitive substructure with 3 or 4 nodes. We call a $g = (D, R)$ *angular*, if $\mathrm{sub}_g(X)$ is non-primitive for every subset X with $|X| = 3$.

Example 5.1 Clearly, if a primitive $g = (D, R)$ with $|D| \geq 4$ has only two edge classes, a and b, both of which are symmetric, then each $\mathrm{sub}_g(X)$ with $|X| = 3$ is non-primitive. Therefore g is angular, and, by Theorem 4.1, g has a primitive substructure on four nodes. It is straightforward to verify that then this primitive substructure is isomorphic to the path of length three. $\square$

We define the *feature* of an edge class a of g as the set $F_a = \{a, a^{-1}\}$. Note that a feature $\{a, a^{-1}\}$ is a singleton, if the edge class a is symmetric.

We start with some immediate observations.

Lemma 5.2 *Let g be an angular 2-structure.*

(1) *If g has only one feature, then it is either linear or complete.*

(2) *The substructures and the quotients of g are angular.*

By Theorem 5.5 below primitivity of angular 2-structures is preserved if the edge classes are made symmetric. For a 2-structure $g = (D, R)$ define its *symmetric version* as the 2-structure $g^s = (D, R^s)$, where

$$e_1 R^s e_2 \iff e_1 R e_2 \text{ or } e_1 R e_2^{-1}.$$

Clearly, g^s has only symmetric edge classes,

$$eR_{g^s} = eR_g \cup e^{-1}R_g = e^{-1}R_{g^s}.$$

Example 5.3 Consider the 2-structure g from the left part of Figure 23. Now, g is angular and primitive. The edge classes a and b of g are nonsymmetric, and the features $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$ are disjoint. The symmetric version g^s (the 2-structure from the right part of Figure 23) with two symmetric edge classes is also angular and primitive. $\square$

For an edge e of a 2-structure $g = (D, R)$, let $g_e = (D, eR_{g^s})$ be the (undirected) graph, which is obtained from g by setting an edge $\{x, y\}$ if and only if (x, y) belongs to the edge class eR_{g^s} of the symmetric version g^s.

We say that a 2-structure g is *connected*, if g_e is connected for all e. Further, a subset $X \subseteq D$ is an *a-connected component* for an edge class a, if X is a connected component of the graph g_e for some $e \in a$.

Lemma 5.4 *If g is an angular 2-structure, then the a-connected components are clans of g. In particular, if g is primitive, then it is connected.*

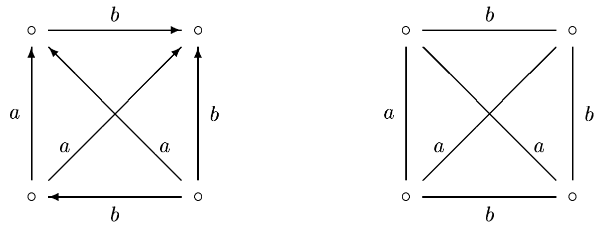


Figure 23: Primitive and angular g, and its symmetric version g^s.

Proof. Assume that X is an a-connected component of g for some a. If $X = D$ or $|X| = 1$, then the claim holds. Assume thus that $X \neq D$ and $|X| \geq 2$.

Let $y \notin X$ and $x_1, x_2 \in X$. Since X is an a-connected component, there are nodes $z_1, \ldots, z_k \in X$ such that $z_1 = x_1$, $z_k = x_2$, and $(z_i, z_{i+1})R \in F_a$ for all $i = 1, 2, \ldots, k-1$. Moreover, $(y, z_i)R \notin F_a$ for $i = 1, 2, \ldots, k$, since X is an a-connected component. By angularity, it follows that $(y, z_i)R = (y, z_{i+1})R$ for $i = 1, 2, \ldots, k-1$, and thus also $(y, x_1)R = (y, x_2)R$. We conclude that $X \in \mathcal{C}(g)$. $\square$

The next theorem reduces the primitivity of angular 2-structures to their symmetric versions.

Theorem 5.5 *Let $g = (D, R)$ be angular. Then g is truly primitive if and only if the symmetric version g^s is truly primitive.*

Finally, we have the following result, for the proof of which we refer again to [19].

Theorem 5.6 *Let $g = (D, R)$ be a truly primitive angular 2-structure. Then g has exactly two features.*

Theorem 5.6 generalizes to all angular 2-structures in the following sense, see Harju and Rozenberg [31].

Theorem 5.7 *If an angular g is connected, then it has at most two features.*

Proof. Let $g = (D, R)$ be angular with k features with $k \geq 2$. If it is primitive, then the claim holds by Theorem 5.6. Otherwise, consider the special quotient $g/\mathcal{P}_{\max}(g)$. By Lemma 5.2(2) $g/\mathcal{P}_{\max}(g)$ is also angular. Since g is connected, by the formation of a quotient also $g/\mathcal{P}_{\max}(g)$ is connected and therefore it has the same number of features as g. Now, since $k \geq 2$, $g/\mathcal{P}_{\max}(g)$ cannot be linear or complete, and hence it is primitive. Consequently, by Theorem 5.6, we have that $k = 2$. $\square$

Theorem 5.5 and Theorem 5.6 reduce essentially the study of primitive angular 2-structures to that of graphs. One can then use various methods of graph theory to decompose primitive angular 2-structures into smaller 2-structures. We mention here only the *split decomposition* of Cunningham and Edmonds [6] and the decomposition of graphs into P_4-*components* of Jamison and Olariu [33].

5.2 T-structures and texts

We say that a 2-structure g is a *partial order*, if it has two features one of which is symmetric and the other is nonsymmetric and such that its edge classes partially order the domain of g. It was shown in [19] that a truly primitive angular 2-structure g is either a graph (*i.e.*, g has two symmetric edge classes), or a partial order, or a T-structure. Here a 2-structure g is a *T-structure*, if g is angular and it has only nonsymmetric edge classes.

The following lemma is obvious.

Lemma 5.8 *Let g be a T-structure. Then the substructures and quotients of g are T-structures. Further, the special quotient $g/\mathcal{P}_{\max}(g)$ is either linear or truly primitive.*

Assume that g is a truly primitive T-structure. By Lemma 4.9 and the fact that g is angular, g has a primitive substructure $h = \mathrm{sub}_g(X)$ with $|X| = 4$. Moreover, by Theorem 5.6, g (and h) has exactly two (nonsymmetric) features. One can show that the 2-structure N from Figure 24 is the only primitive T-structure with four nodes (up to isomorphism).

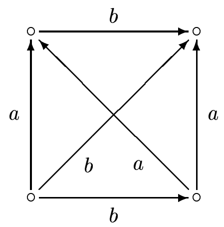


Figure 24: The primitive T-structure N

Lemma 5.9 *If g is a truly primitive T-structure, then g contains a substructure N.*

Consider then a 2-structure g without symmetric edge classes. A subset σ of edge classes is a *selection*, if for all features F, $|\sigma \cap F| = 1$. For a selection σ we denote by $\sigma(g)$ the directed graph $(\mathrm{dom}(g), \cup\sigma)$.

The next result was proved in Ehrenfeucht and Rozenberg [20].

Lemma 5.10 *Let g be a 2-structure having only nonsymmetric edge classes. Then g is a T-structure if and only if $\sigma(g)$ is a linear order of* $\mathrm{dom}(g)$ *for all selections σ.*

In general a T-structure may have arbitrarily many features, but by Theorem 5.6 a truly primitive T-structure has exactly two features. Therefore the nodes (the special quotients $\Psi_g(P)$ for prime clans P) in the shape of a T-structure g have at most two features. This fact was generalized in [20] to

Theorem 5.11 *For each T-structure g there exists a T-structure g' with at most two features such that* $\mathrm{shape}(g) = \mathrm{shape}(g')$.

Remark. Notice that in [20] Theorem 5.11 is stated using 'similarity' instead of equality of shapes. However, one can easily check that for (unlabeled) 2-structures the two statements are equivalent. Therefore, we shall now consider T-structures with at most two features. For a partial order ρ we let ρ^{-1} denote the dual partial order of ρ, *i.e.*, $\rho^{-1} = \{(x,y) \mid (y,x) \in \rho\}$.

For the proof of the following result see [20].

Theorem 5.12 *Let g be a 2-structure with at most two features. Then g is a T-structure if and only if there exist two linear orders ρ_1 and ρ_2 such that the edge classes of g are included in the set $\{\rho_1 \cap \rho_2,\ \rho_1 \cap \rho_2^{-1},\ \rho_1^{-1} \cap \rho_2$ and $\rho_1^{-1} \cap \rho_2^{-1}\}$.*

In Theorem 5.12 some of the intersections may be empty.

Notice that in the above each of the intersections of the linear orders ρ_1 and ρ_2 is a partial order of the domain, and moreover $(\rho_1 \cap \rho_2)^{-1} = \rho_1^{-1} \cap \rho_2^{-1}$ and $(\rho_1 \cap \rho_2^{-1})^{-1} = \rho_1^{-1} \cap \rho_2$.

From Theorem 5.12 it follows that each T-structure with at most two features is determined by two linear orders of its domain.

Texts were defined in [20] as triples $(\lambda, \rho_1, \rho_2)$, where $\lambda\colon D \to A$ is a function from the domain D into an alphabet A, and ρ_1, ρ_2 are two linear orders of D. Hence a text τ consists of a T-structure $g(\tau)$ (specified by Theorem 5.12) together with a labeling function λ.

A text $\tau = (\lambda, \rho_1, \rho_2)$ with $\lambda\colon D \to A$ can be considered as a 'structured word' over A. Indeed, let $D = \{x_1, x_2, \ldots, x_n\}$ and $\rho_1 = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, $\rho_2 = (x_{j_1}, x_{j_2}, \ldots, x_{j_n})$. The first linear order gives an ordinary word over the alphabet A,

$$\lambda(\rho_1) = (\lambda(x_{i_1}), \lambda(x_{i_2}), \ldots, \lambda(x_{i_n})),$$

and the linear orders ρ_1 and ρ_2 together assign to the word $\lambda(\rho_1)$ its structure as specified by the shape of the corresponding T-structure $g(\tau)$.

Example 5.13 Let $D = \{1, 2, 3, 4, 5\}$, and $A = \{x, y, z\}$. Define a text $\tau = (\lambda, \rho_1, \rho_2)$ as follows:

$$\begin{aligned} \lambda(1) &= x = \lambda(2),\ \lambda(3) = y = \lambda(4),\ \lambda(5) = z, \\ \rho_1 &= (2, 3, 5, 1, 4),\ \rho_2 = (3, 2, 4, 1, 5). \end{aligned}$$

The word defined by τ is $\lambda(\rho_1) = (x, y, z, x, y)$, and its structure is given by ρ_2. The T-structure $g(\tau)$ has the features $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$, where

$$\begin{aligned} a &= \rho_1 \cap \rho_2 = \{(2,1), (2,4), (2,5), (3,1), (3,4), (3,5)\}, \\ b &= \rho_1 \cap \rho_2^{-1} = \{(1,4), (2,3), (5,1), (5,4)\}, \end{aligned}$$

and hence $g(\tau)$ is as given in Figure 25.

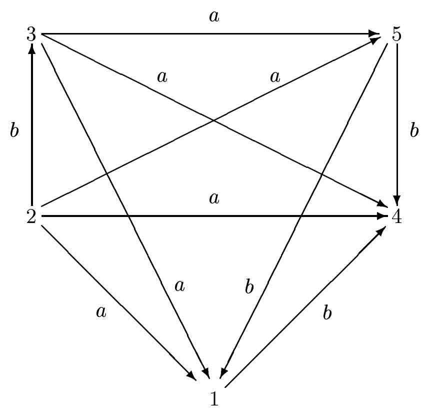


Figure 25: The T-structure $g(\tau)$

The maximal prime clans of $g(\tau)$ are $\{2, 3\}$ and $\{1, 4, 5\}$, and the shape of $g(\tau)$ is given in Figure 26.

□

The texts can be classified with respect to the properties of the shapes of the corresponding T-structures. We say that a text τ is *alternating*, if shape$(g(\tau))$ has only linear nodes, *i.e.*, the special quotients $\Psi_{g(\tau)}(P)$ are linear 2-structures for all prime clans P.

If a text $\tau = (\lambda, \rho_1, \rho_2)$ is not alternating, then the shape of $g(\tau)$ contains a primitive quotient, and therefore, by Lemma 5.9, $g(\tau)$ contains a substructure N. This means that the linear order ρ_1 of the text τ contains a suborder (i_1, i_2, i_3, i_4) such that either (i_2, i_4, i_1, i_3) or (i_3, i_1, i_4, i_2) is a suborder of ρ_2.

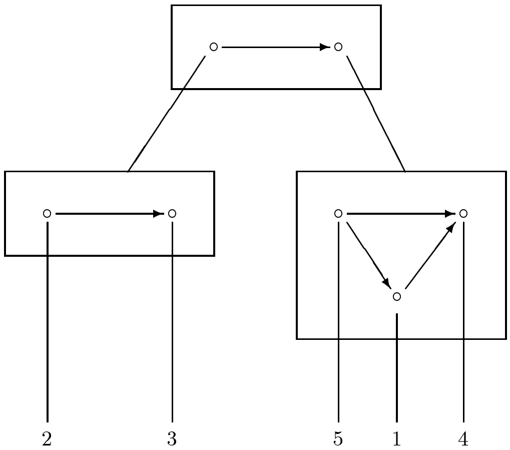


Figure 26: The shape of $g(\tau)$

Next we define two operations Vrev and $\oplus$ for texts. For a text $\tau = (\lambda, \rho_1, \rho_2)$ let

$$\mathrm{Vrev}(\tau) = (\lambda, \rho_1^{-1}, \rho_2).$$

For texts $\tau = (\lambda, \rho_1, \rho_2)$ and $\tau' = (\lambda', \rho_1', \rho_2')$ with disjoint domains let

$$\tau \oplus \tau' = (\lambda \cup \lambda', \rho_1 + \rho_1', \rho_2 + \rho_2'),$$

where $\rho_i + \rho_i' = \rho_1 \cup \rho_1' \cup \{(x,y) \mid x \in \mathrm{dom}(\tau),\, y \in \mathrm{dom}(\tau')\}$.

For the proof of the following theorem we refer to [20].

Theorem 5.14 *The family of alternating texts is the smallest class of texts containing the singleton texts and closed under the operations* Vrev *and* $\oplus$*.*

For further combinatorial properties and applications of texts we refer to [20], [14] and [13].

6 Labeled 2-Structures

6.1 Definition of a labeled 2-structure

As noted in Sections 2 and 3 the decompositions of 2-structures are not unique in the sense that two different 2-structures may possess the same decomposition. In *labeled* 2-structures this ambiguity does not occur.

A labeled 2-structure is obtained from a 2-structure g by setting a different label to each edge class of g.

Equivalently, we define a *labeled 2-structure* or an *ℓ2-structure*, for short, as a triple $g = (D, \alpha, \Delta)$, where D is the *domain*, Δ is the set of *labels* and $\alpha: E_2(D) \to \Delta$ is the *labeling function* on the edges. Again, we assume that our ℓ2-structures are *reversible*, *i.e.*, there is a permutation $\delta: \Delta \to \Delta$ of order two on the labels that satisfies the condition

$$\alpha(e^{-1}) = \delta(\alpha(e)) \qquad \text{for all } \; e \in E_2(D).$$

That the permutation δ is of order two means that $\delta^2(a) = a$ for all labels $a \in \Delta$. Again we may have $a = \delta(a)$ for a label $a \in \Delta$, in which case a is called *symmetric*.

In later sections the set Δ of labels will be a group in the algebraic sense, and the reversibility function $\delta: \Delta \to \Delta$ will be a group involution. For this reason we do not write a^{-1} for the label of the reverse edge.

We observe that an ℓ2-structure $g = (D, \alpha, \Delta)$ is completely determined by its labeling function α, and therefore *we shall identify g with α*, *i.e.*, we shall consider ℓ2-structures as functions

$$g: E_2(D) \to \Delta.$$

Notice, however, that the domain of the 2-structure g is D, not $E_2(D)$.

For each ℓ2-structure $g: E_2(D) \to \Delta$ there is the *underlying* unlabeled 2-structure $g' = (D, R)$, where R is defined by

$$e_1 R e_2 \iff g(e_1) = g(e_2).$$

For this reason the results of the previous sections can be easily modified for labeled 2-structures.

The converse of this does not hold, since the same 2-structure corresponds to several ℓ2-structures with different labeling functions.

Let $g_1: E_2(D_1) \to \Delta_1$ and $g_2: E_2(D_2) \to \Delta_2$ be two labeled 2-structures. We say that g_1 and g_2 are *isomorphic*, if there are bijections $\varphi: D_1 \to D_2$ and $\psi: \Delta_1 \to \Delta_2$ such that for all $(x, y) \in E_2(D_2)$,

$$g_2(\varphi(x), \varphi(y)) = \psi(g_1(x, y)).$$

Also, g_1 and g_2 are *strictly isomorphic*, if the bijection φ is the identity function on D_1 (and so $D_1 = D_2$). Hence two ℓ2-structures $g: E_2(D) \to \Delta_1$ and $h: E_2(D) \to \Delta_2$ are strictly isomorphic, if there is a bijection $\psi: \Delta_1 \to \Delta_2$ such that $h(e) = \psi(g(e))$ for all edges $e \in E_2(D)$.

The next result gives a connection between the unlabeled and the labeled 2-structures.

Theorem 6.1 *Two $\ell 2$-structures g and h are strictly isomorphic if and only if they have the same underlying (unlabeled) 2-structure.*

6.2 Substructures, clans and quotients

Let $g: E_2(D) \to \Delta$ be an $\ell 2$-structure. We shall now modify the basic results and definitions of the previous sections, and state them for labeled 2-structures.

The *substructure* $\mathrm{sub}_g(X)$ induced by a subset $X \subseteq D$ is defined as the restriction of g onto $E_2(X)$, *i.e.*, $\mathrm{sub}_g(X): E_2(X) \to \Delta$ satisfies $\mathrm{sub}_g(X)(e) = g(e)$ for all $e \in E_2(X)$. Clearly, a substructure $\mathrm{sub}_g(X)$ has the same reversibility function $\delta: \Delta \to \Delta$ as g.

A subset $X \subseteq D$ is a *clan* of $g: E_2(D) \to \Delta$, if for all $x, y \in X$ and all $z \notin X$, $g(z,x) = g(z,y)$ (and hence $g(x,z) = g(y,z)$ by the reversibility condition).

Again, let $\mathcal{C}(g)$ denote the family of clans of g. Now, the basic results of Sections 2 and 3 hold for $\mathcal{C}(g)$.

Lemma 6.2 *Let $g: E_2(D) \to \Delta$ and let $g' = (D,R)$ be the underlying 2-structure. Then*

(1) *for each $X \subseteq D$, $\mathrm{sub}_{g'}(X)$ is the underlying 2-structure of $\mathrm{sub}_g(X)$, and*

(2) $\mathcal{C}(g) = \mathcal{C}(g')$.

A partition $\mathcal{X} \subseteq \mathcal{C}(g)$ of the domain D of g into clans is called a *factorization* of g. The *quotient* of g by $\mathcal{X}$ is the $\ell 2$-structure $g/\mathcal{X}$ on the domain $\mathcal{X}$ such that for all $X, Y \in \mathcal{X}$ (with $X \neq Y$), $(g/\mathcal{X})(X,Y) = g(x,y)$, whenever $x \in X$ and $y \in Y$. We need a modified result to ensure that the quotient $g/\mathcal{X}$ is well-defined for all g.

Lemma 6.3 *Let δ be the reversibility function of Δ. Let $X, Y \in \mathcal{C}(g)$ be two disjoint clans of an $\ell 2$-structure $g: E_2(D) \to \Delta$. There exists a label $a \in \Delta$ such that $g(x,y) = a$ and $g(y,x) = \delta(a)$ for all $x \in X$ and $y \in Y$.*

We observe that, by Lemma 6.3, every quotient $g/\mathcal{X}$ is isomorphic with a substructure of g, *viz.* with $\mathrm{sub}_g(X)$, where X is a set of representatives of $\mathcal{X}$ (*i.e.*, $X \cap Y$ is a singleton for every $Y \in \mathcal{X}$); this is a rather unusual situation from the general algebraic point of view. In this isomorphism the bijection $\psi: \Delta \to \Delta$ is just the identity.

A *decomposition* $h(g_1, \ldots, g_m)$ of an $\ell 2$-structure g consists of the factors $g_i = \mathrm{sub}_g(X_i)$ and the quotient $h = g/\mathcal{X}$ by a factorization $\mathcal{X} = \{X_1, \ldots, X_m\}$. In contrast to the unlabeled 2-structures, see Example 3.4 the decompositions of a labeled 2-structure g are unique to g.

Theorem 6.4 *Two $\ell 2$-structures g_1 and g_2 have a common decomposition if and only if $g_1 = g_2$.*

The *prime clans* and the *maximal prime clans* of an $\ell 2$-structure g are defined as in Section 3. Again, the family of prime clans is denotes by $\mathcal{P}(g)$ and the family of the maximal prime clans is denoted by $\mathcal{P}_{\max}(g)$.

The following result modifies some of the results from Sections 2 and 3 for $\ell 2$-structures. It shows that the (prime) clans of an $\ell 2$-structure g are inherited by the substructures and by the quotients.

Lemma 6.5 *Let g be an $\ell 2$-structure.*

(1) *For each $X \in \mathcal{C}(g)$, $\mathcal{C}(\mathrm{sub}_g(X)) = \{Y \mid Y \in \mathcal{C}(g), Y \subseteq X\}$.*

(2) *For each $X \in \mathcal{P}(g)$, $\mathcal{P}(\mathrm{sub}_g(X)) = \{Y \mid Y \in \mathcal{P}(g), Y \subseteq X\}$.*

(3) *If $\mathcal{X} \subseteq \mathcal{P}(g)$ is a prime factorization, then $\mathcal{Q} \in \mathcal{C}(g/\mathcal{X})$ if and only if $\cup\mathcal{Q} \in \mathcal{C}(g)$.*

The clan decomposition theorem for 2-structures, Theorem 3.14, is restated in the labeled case as follows.

Theorem 6.6 *For any $\ell 2$-structure g, the quotient $g/\mathcal{P}_{\max}(g)$ is either linear, or complete, or truly primitive.*

We say that an $\ell 2$-structure g is *a-linear*, for a label $a \in \Delta$, if the relation $x <_a y$ linearly orders the domain of g, where $x <_a y$ if and only if $g(x, y) = a$. Also, g is *a-complete*, if $g(e) = a$ for all $e \in E_2(D)$.

As is the case for unlabeled 2-structures, the partially ordered set $T(g) = (\mathcal{P}(g), \subseteq)$ of prime clans forms a directed and rooted tree, the *prime tree* of g. The root of $T(g)$ is the domain $\mathrm{dom}(g)$, the leaves are the singleton sets, and there is an edge (X, Y) in $T(g)$, if X is the minimal element of $\mathcal{P}(g)$ such that $Y \subseteq X$ and $X \neq Y$.

Example 6.7 Consider the $\ell 2$-structure g from Figure 27 with nonsymmetric labels. In the figure we have omitted the edges having reverse labels $\delta(a)$, $\delta(b)$ and $\delta(c)$.

Here the nontrivial clans of g are $X_1 = \{x_1, x_4, x_5\}$ and $X_2 = \{x_4, x_5\}$; both clans are also prime clans of g, and X_1 is a maximal prime clan. We have $\mathcal{P}_{\max}(g) = \{X_1, \{x_2\}, \{x_3\}\}$, and hence the quotient $g/\mathcal{P}_{\max}(g)$ has three nodes, and it is primitive.

In Figure 28 of the prime tree $T(g)$ of g we have not identified the inner nodes (the prime clans), since each inner node consists of the leaves x_i that are descendants of this node.

The substructure $h_1 = \mathrm{sub}_g(X_1)$ has two maximal primes, X_2 and $\{x_1\}$, and the quotient $h_1/\mathcal{P}_{\max}(h_1)$ is linear. The substructure $h_2 = \mathrm{sub}_g(X_2)$ has maximal primes $\{x_4\}$ and $\{x_5\}$ and the quotient $h_2/\mathcal{P}_{\max}(h_2)$ is isomorphic to the linear $\ell 2$-structure h_2.

Notice that the partition $\mathcal{X} = \{\{x_1\}, \{x_2\}, \{x_3\}, X_2\}$ is also a clan partition of g into prime clans. The quotient $g/\mathcal{X}$ has four nodes, and since

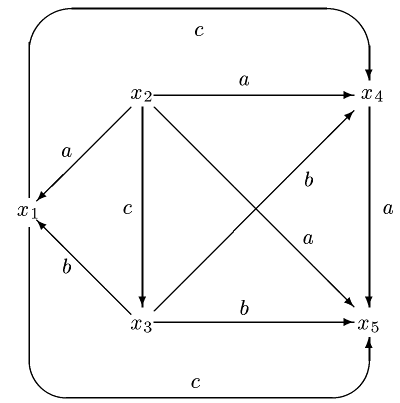


Figure 27: An $\ell 2$-structure g.

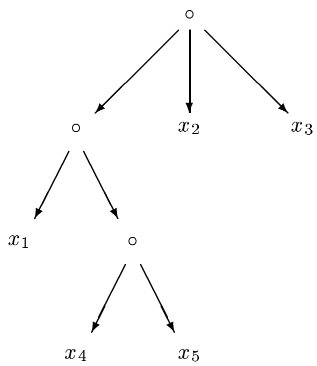


Figure 28: The prime tree of g.

$X_1 \in \mathcal{C}(g)$, also $\{\{x_1\}, X_2\}$ is a clan of $g/\mathcal{X}$. Therefore this quotient is not special. $\square$

The *shape* of an ℓ2-structure $g: E_2(D) \to \Delta$ is obtained in the same way as for unlabeled 2-structures. Hence the shape of g, $\mathrm{shape}(g) = (T(g), \Psi_g)$, consists of the prime tree of g together with the function

$$\Psi_g(P) = \mathrm{sub}_g(P)/\mathcal{P}_{\max}(\mathrm{sub}_g(P))$$

that maps each prime clan $P \in \mathcal{P}(g)$ to its special quotient.

Example 6.8 The shape of the ℓ2-structure g from Example 6.7, see Figure 27, can be drawn as in Figure 29. Here the labels of the edges in the special quotients $\Psi_g(P)$ have to be included, because now these quotients are labeled 2-structures. $\square$

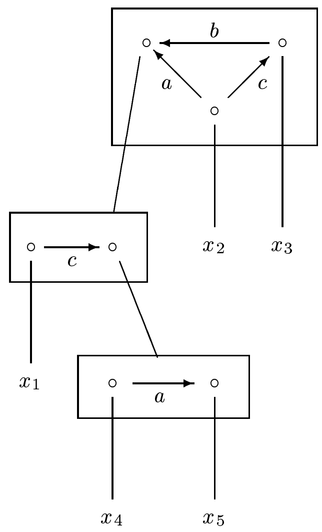


Figure 29: The shape of g.

7 Dynamic Labeled 2-Structures

7.1 Motivation

Usually in a valuation of objects of a mathematical structure the values come from a set that is structured itself. Frequently such a set is an algebraic structure, where the elements are bound together by one or more operations. As an example, a finite automaton may be considered as a directed graph, where the edges have values from a free monoid. Also, in graph theory the values of objects (the labels of edges or nodes) are often taken from an algebraic structure, *e.g.*, in many optimality problems such as the shortest path problem, the edges are labeled by elements of the field $(\mathbb{R}, +, \cdot)$ of real numbers.

In this and the following sections we shall study $\ell 2$-structures that have a group structure on their sets of labels. The group structure of the labels gives then a method of locally transforming $\ell 2$-structures into each other, and this leads to dynamic $\ell 2$-structures.

A group is a natural candidate for the set labels of a (reversible) $\ell 2$-structure. The dynamic $\ell 2$-structures were also motivated by Ehrenfeucht and Rozenberg [22] by evolution of networks and similar processes. Consider a network consisting of a set of processors D, where each $x \in D$ is connected by a channel to each $y \in D$ for $y \neq x$. A channel (x, y) may assume a certain value or state from the set Δ at a specific time, *e.g.*, if there are only two values $\Delta = \{0, 1\}$, then the channels may be interpreted as having a sleeping and an active state; on the other hand, if $\Delta = \mathbb{R}$, then the channels can be considered as weighted by reals.

The *concurrent* actions of the processors modify the states of the channels. The activity of each $x \in D$ will be described by two sets of actions, the *output actions* O_x and the *input actions* I_x, by which x changes the states of the channels from and to x, respectively. The actions of x are thus mappings of Δ into Δ.

Now, for each (x, y) there are two processors, x and y, that change the state of this channel concurrently (*i.e.*, independently of each other); x changes it by a mapping from O_x and y changes it by a mapping from I_y. In order to accommodate the concurrent behaviour of the processors, the mappings from O_x and I_y should commute, *i.e.*, for all $\varphi \in O_x$ and $\psi \in I_y$, $\varphi\psi = \psi\varphi$.

To avoid unnecessary sequencing of actions the composition of two actions from O_x (I_x, resp.) is again assumed to be an action, *i.e.*, the sets O_x and I_x form transformation semigroups on Δ under the operation of composition.

Further, to assure a minimal freedom of the actions for each x we assume that for each $a, b \in \Delta$ and $x \in D$ there exist a $\varphi \in O_x$ and a $\gamma \in I_x$ such that $\varphi(a) = b$ and $\gamma(a) = b$, *i.e.*, the semigroups O_x and I_x are transitive.

The above assumptions simplify the situation considerably. Indeed, it was shown in [22] (see, also [9]) that if $|D| \geq 3$, then there are two isomorphic groups O and I of permutations on Δ such that for each $x \in D$, $O_x = O$ and $I_x = I$. Thus the actions come from two groups O and I, which are independent of the processors. Moreover, we can define an operation on Δ

such that Δ becomes a group isomorphic to O and I. In fact, now the groups O and I become defined by left and (involutive) right multiplication of the group Δ.

These observations lead to the definition of a dynamic $\ell 2$-structure, given formally in Sections 7.2 and 7.3 in terms of *selectors*. A selector σ is a mapping, which captures the action of each processor $x \in D$ at a specific stage of the network. By the above observations, a selector will be simply a function $\sigma: D \to \Delta$.

A global state of a network is represented by an $\ell 2$-structure, for which Δ is the set of labels. An evolution of the network is presented as a set of $\ell 2$-structures that represent possible global states of the network. The transitions from one $\ell 2$-structure g to another h (and therefore from one global state to another) are the transformations $g \mapsto h$ induced by the selectors.

7.2 Group labeled 2-structures

We shall assume throughout the rest of the paper that the set Δ of labels forms a (possibly infinite) *group*. The identity element of Δ is usually denoted by 1_Δ.

A mapping $\delta: \Delta \to \Delta$ is an *involution* of the group Δ, if it is a permutation of order two such that $\delta(ab) = \delta(b)\delta(a)$ for all $a, b \in \Delta$. An involution δ satisfies $\delta(a^{-1}) = \delta(a)^{-1}$ for all $a \in \Delta$, and in particular, $\delta(1_\Delta) = 1_\Delta$.

Example 7.1 (1) For any group Δ the most obvious involution is the inverse function: $\delta(a) = a^{-1}$ for each $a \in \Delta$.

(2) If Δ is the group of nonsingular $n \times n$-matrices (over a field F), then transposition, $\delta(M) = M^T$, is an involution of Δ.

(3) If Δ is a finite group of even order, then it has an element a of order 2, and, as easily seen, the mapping $\delta(x) = ax^{-1}a$ ($x \in \Delta$) is an involution.

(4) If $\alpha: \Delta \to \Delta$ is an automorphism of order two, then δ defined by $\delta(a) = \alpha(a^{-1})$ is an involution. $\square$

As before we will be dealing only with reversible $\ell 2$-structures; now involutions will determine reversibility. Let δ be an involution of the group Δ. We say that $g: E_2(D) \to \Delta$ is a *δ-reversible $\ell 2$-structure on* Δ or a Δ^δ *2-structure*, for short, if the involution δ is its reversibility function: $g(e^{-1}) = \delta(g(e))$ for all edges $e \in E_2(D)$.

We reserve the symbol D for a domain, the symbol Δ for a group and the symbol δ for an involution of Δ.

Example 7.2 Let $\Delta = (\mathbb{R}^+, \cdot)$ be the multiplicative group of positive real numbers, and let δ be the inverse function of Δ. In Figure 30, we have an example of a $\Delta^\delta 2$-structure, where, as usual, the reverse edges are not shown.

Thus, *e.g.*, we have $g(x_2,x_1) = \delta(g(x_1,x_2)) = g(x_1,x_2)^{-1} = 1$, $g(x_2,x_3) = \frac{3}{2}$, and $g(x_3,x_2) = \frac{2}{3}$. □

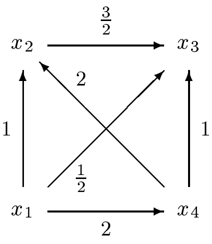


Figure 30: A $\Delta^\delta 2$-structure

7.3 Dynamic labeled 2-structures

The group Δ of labels of a $\Delta^\delta 2$-structure g becomes employed by the selectors, which, in essence, label the nodes $x \in D$ by the elements of the group Δ.

Let g be a $\Delta^\delta 2$-structure for a group Δ and its involution δ. A function $\sigma : D \to \Delta$ is called a *selector*. For a selector $\sigma : D \to \Delta$ define the $\ell 2$-structure g^σ by

$$g^\sigma(x,y) = \sigma(x) \cdot g(x,y) \cdot \delta(\sigma(y))$$

for all $(x,y) \in E_2(D)$. The family $[g] = \{g^\sigma \mid \sigma \text{ a selector}\}$ is a (*single axiom*) *dynamic* Δ^δ *2-structure* (*generated by* g).

Hence a selector $\sigma : D \to \Delta$ transforms each g into a g^σ by direct left and involutive right multiplication. The new value of an edge depends on the (values of the) nodes and on the label of the edge.

Example 7.3 Let g be as in Example 7.2, see Figure 30. Define a selector $\sigma : D \to \mathbb{R}$ by $\sigma(x_i) = 5 - i$. Then, *e.g.*, we have in the image g^σ, $g^\sigma(x_1,x_2) = 4 \cdot 1 \cdot \frac{1}{3} = \frac{4}{3}$. The image g^σ is drawn in Figure 31, where we have labeled the nodes by the values of σ.

□

In the pictorial representations of $\Delta^\delta 2$-structures, *we shall usually omit the edges labeled by the identity element* 1_Δ *of the group* Δ. Recall that $\delta(1_\Delta) = 1_\Delta$. This convention reflects a natural interpretation of 1_Δ as the 'no edge' symbol.

The dynamic $\Delta^\delta 2$-structures on the group $\mathbb{Z}_2$ of labels are closely connected to *Seidel switching*, which was defined in connection with a problem of finding equilateral n-tuples of points in elliptic geometry (in this respect, see Van Lint and Seidel [34]). This problem has the following subproblem

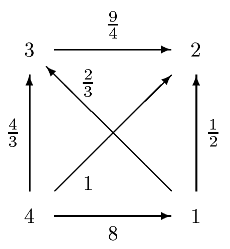


Figure 31: The image g^σ

for undirected graphs. Determine the equivalence classes of undirected graphs with n nodes with respect to the following operation: Let $G \to G'$, if there is a node x such that $G' = (D, E')$ is obtained from $G = (D, E)$ by removing all edges (x, y) and (y, x) incident with x, and adding all pairs (x, y) and (y, x) not in E. Hence $G \to G'$, if for some node x,

$$E' = (E \setminus \{(x,y),(y,x) \mid y \neq x\}) \cup \{(x,y),(y,x) \mid (x,y) \notin E\}.$$

Let then $\sim$ be the equivalence relation determined by $\to$, *i.e.*, $G \sim G'$ if and only if $G = G'$ or there exists a finite sequence $G_0 \to G_1 \to \ldots \to G_k$ with $\{G, G'\} = \{G_0, G_k\}$. One now asks how many equivalence of $\sim$ classes are there for a set D of nodes?

We can reformulate the above problem in terms of dynamic Δ^δ2-structures as follows. Let $\Delta = (\mathbb{Z}_2, +)$ be the cyclic group of order two. This group has only one involution δ, the inverse mapping, which in this simple case is the identity mapping of Δ: $\delta(0) = 0$ and $\delta(1) = 1$. We may consider a Δ^δ2-structure $g\colon E_2(D) \to \Delta$ as an undirected graph, where $g(e) = 1$ ($g(e) = 0$) means that e is (not, respectively) an edge of g. Consider a node $x \in D$ and a selector σ, for which $\sigma(x) = 1$ and $\sigma(y) = 0$ for all other nodes $y \neq x$. Clearly, the image g^σ represents a graph, where the existing connections from x are removed and the nonexistent connections from x are created. Therefore $G \to G'$ holds if and only if for the corresponding Δ^δ2-structures g and g', $g' = g^\sigma$ for such a selector σ. From this we obtain that $g \sim g'$ if and only if $g' = g^\sigma$ for a selector $\sigma\colon D \to \mathbb{Z}_2$.

For connections of Seidel switching to *signed graphs* and *two-graphs* we refer to Harary [29], Seidel and Taylor [42] and Zaslavsky [45], [46].

Example 7.4 Let $\Delta = (\mathbb{Z}_2, +)$, and let δ be the identity mapping of Δ. Let $D = \{x_1, x_2, x_3, x_4\}$ and consider a Δ^δ2-structure g as in Figure 32, where a line denotes value 1.

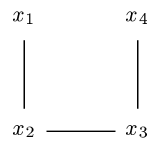


Figure 32: g with labels in $\mathbb{Z}_2$

There are $2^{|D|} = 16$ different selectors $\sigma: D \to \Delta$, but some of them have the same image g^σ. In fact, there are only 8 different images g^σ as depicted in Figure 33, where again the nodes are labeled by the values of a selector σ which applied to g yields g^σ.

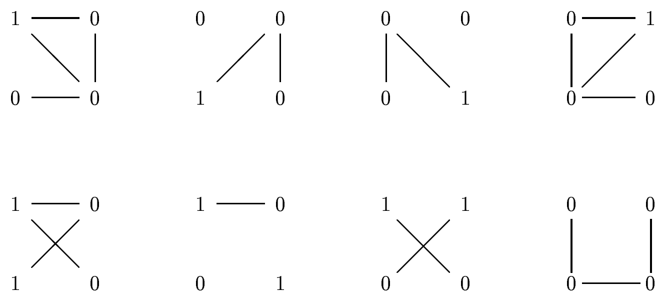


Figure 33: The images g^σ

As can be observed from this example, it can happen that $g(e_1) = g(e_2)$, but, nevertheless, $g^\sigma(e_1) \neq g^\sigma(e_2)$ for some edges $e_1, e_2 \in E_2(D)$. Notice also that g is primitive, but in $[g]$ there are nonprimitive structures. $\square$

The notion of a substructure carries over to dynamic Δ^δ2-structures as follows.

Lemma 7.5 *Let $[g]$ be a dynamic Δ^δ2-structure, and let $X \subseteq \mathrm{dom}(g)$. Then*

$$[\mathrm{sub}_g(X)] = \{\mathrm{sub}_{g^\sigma}(X) \mid \sigma \quad \textit{selector}\}.$$

We denote by $\mathrm{sub}_{[g]}(X) = [\mathrm{sub}_g(X)]$ for each dynamic Δ^δ2-structure g and subset $X \subseteq \mathrm{dom}(g)$.

The following lemma shows that the image g^σ under a selector σ of a Δ^δ2-structure g is also δ-reversible.

Lemma 7.6 *For a Δ^δ 2-structure g and a selector $\sigma: D \to \Delta$, the image g^σ is also a Δ^δ 2-structure.*

Proof. Let σ be a selector and g a Δ^δ2-structure. For each edge $e = (x, y) \in E_2(D)$ we have $g(e^{-1}) = \delta(g(e))$, and hence

$$\begin{aligned} g^\sigma(e^{-1}) &= \sigma(y) \cdot g(e^{-1}) \cdot \delta(\sigma(x)) = \sigma(y) \cdot \delta(g(e)) \cdot \delta(\sigma(x)) \\ &= \delta^2(\sigma(y)) \cdot \delta(g(e)) \cdot \delta(\sigma(x)) = \delta(\sigma(x) \cdot g(e) \cdot \delta(\sigma(y))) = \delta(g^\sigma(e)). \end{aligned}$$

Therefore g^σ is also δ-reversible. □

The *product* of two selectors $\sigma_1, \sigma_2: D \to \Delta$ is defined by

$$\sigma_2\sigma_1(x) = \sigma_2(x)\sigma_1(x) \qquad (x \in D).$$

Clearly, $\sigma_2\sigma_1$ is a selector.

Lemma 7.7 *For any selectors $\sigma_1, \sigma_2: D \to \Delta$ and Δ^δ 2-structure g, $g^{\sigma_2\sigma_1} = (g^{\sigma_1})^{\sigma_2}$.*

It is now immediate that *the selectors form a group* under this product. The inverse σ^{-1} of a selector σ becomes defined by $\sigma^{-1}(x) = \sigma(x)^{-1}$ for $x \in D$. In group theoretic terms the group of selectors *acts on* the Δ^δ2-structures, see Rotman [39].

The following lemma is now easy to prove.

Lemma 7.8 *Let g be a Δ^δ 2-structure. Then $[g^\sigma] = [g]$ for all $\sigma: D \to \Delta$.*

7.4 Clans of a dynamic Δ^δ 2-structure

Let $[g]$ be a dynamic Δ^δ2-structure. A clan of each individual $h \in [g]$ is called a *clan* of $[g]$. We shall write

$$\mathcal{C}[g] = \bigcup_{h \in [g]} \mathcal{C}(h)$$

for the family of clans of $[g]$.

The closure properties of the clans of a dynamic Δ^δ2-structure $[g]$ are quite different from those of ordinary ℓ2-structures.

Example 7.9 Recall that the intersection of two clans $X, Y \in \mathcal{C}(g)$ of an ℓ2-structure g is always a clan. This does not hold in general for the dynamic Δ^δ2-structures. Indeed, consider again $\Delta = \mathbb{Z}_2$, and the following ℓ2-structure g together with its image g^σ:

Here $X = \{x_2, x_3, x_4\}$ is a clan of g, and $Y = \{x_1, x_2, x_3\}$ is a clan of g^σ, and hence both of these are in $\mathcal{C}[g]$. However, the intersection $X \cap Y = \{x_2, x_3\}$ is not in $\mathcal{C}[g]$ as can be easily verified. □

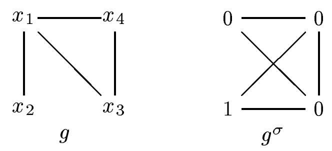


Figure 34: A Δ^δ2-structure g and its image g^σ.

On the other hand, as shown in Lemma 7.10, the clans of $[g]$ are closed under complement. As observed in Section 2.3 this is in general not true for ℓ2-structures.

We let $\overline{X}$ denote the complement $D \setminus X$ of $X \subseteq D$.

A set $X \in \mathcal{C}(g)$ is said to be *isolated*, if $g(x, y) = 1_\Delta$ for all $x \in X$ and $y \in \overline{X}$. Hence, if X is an isolated clan, then its complement $\overline{X}$ is also a clan of g.

Lemma 7.10 *Let g be a Δ^δ2-structure, and let $X \in \mathcal{C}(g)$ be a clan. There exists a selector σ such that X is an isolated clan in g^σ.*

Proof. Let $X \in \mathcal{C}(g)$, and define σ as follows $\sigma(x) = 1_\Delta$ for all $x \in X$, and $\sigma(y) = \delta(g(x, y)^{-1})$ for all $y \notin X$. Now, for all $e = (x, y)$ with $x \in X$ and $y \notin X$,

$$g^\sigma(e) = 1_\Delta \cdot g(e) \cdot g(e)^{-1} = 1_\Delta$$

as required. □

In particular, Lemma 7.10 implies the following closure result for dynamic Δ^δ2-structures, see [22].

Theorem 7.11 *The set $\mathcal{C}[g]$ of clans of a dynamic Δ^δ2-structure $[g]$ is closed under complements.*

7.5 Horizons

Let $g: E_2(D) \to \Delta$ be a Δ^δ2-structure. By Lemma 7.10, for each clan $X \in \mathcal{C}[g]$ there exists an $h \in [g]$ such that X and $\overline{X}$ are both clans of h. In particular, this holds for the singleton clans $\{x\}$ of $[g]$, and therefore for each $x \in D$, $D \setminus \{x\} \in \mathcal{C}[g]$.

We shall call a node $x \in D$ a *horizon* of $g: E_2(D) \to \Delta$, if $g(x, y) = 1_\Delta$ for all $y \neq x$. By the proof of Lemma 7.10, for each g and each $x \in \mathrm{dom}(g)$ x is a horizon of g^{σ_x}, where

$$\sigma_x(x) = 1_\Delta \quad \text{and} \quad \sigma_x(y) = \delta(g(x, y)^{-1}) \text{ for } y \neq x$$

is the *horizontal selector for x in g*.

We have the following uniqueness result for the horizontal selectors, which follows immediately from the definition of a horizon.

Lemma 7.12 *Let g be a Δ^δ 2-structure, and $x \in \mathrm{dom}(g)$. If for a selector σ, $\sigma(x) = 1_\Delta$ and x is a horizon in g^σ, then $\sigma = \sigma_x$.*

A selector $\sigma: D \to \Delta$ is said to be *constant* on a subset $X \subseteq D$, if $\sigma(x_1) = \sigma(x_2)$ for all $x_1, x_2 \in X$.

Lemma 7.13 *Let $g: E_2(D) \to \Delta$ be a Δ^δ 2-structure and $h = g^\sigma$ for a selector σ. If for a subset $Z \subseteq D$ and a node $x \notin Z$, $g(x, z_1) = g(x, z_2)$ and $h(x, z_1) = h(x, z_2)$ for all $z_1, z_2 \in Z$, then σ is constant on Z.*

Proof. Suppose $Z \subseteq D$ is as stated in the lemma, and let $z_1, z_2 \in Z$. We have

$$g^\sigma(x, z_i) = \sigma(x) \cdot g(x, z_i) \cdot \delta(\sigma(z_i)) \qquad (i = 1, 2),$$

and hence $\sigma(z_1) = \sigma(z_2)$, because $g^\sigma(x, z_1) = g^\sigma(x, z_2)$ and $g(x, z_1) = g(x, z_2)$. This proves the claim. □

When Lemma 7.13 is applied to clans, we obtain the next lemma which states that the derivation of a clan $X \in \mathcal{C}[g]$ in a dynamic $[g]$ depends only on X.

Lemma 7.14 *Let X be a proper clan of a Δ^δ 2-structure g and let σ be a selector. Then $X \in \mathcal{C}(g^\sigma)$ if and only if σ is constant on X.*

As a corollary of the above lemma we obtain that for each clan X of a dynamic Δ^δ2-structure $[g]$ either X or its complement is a clan of any $h \in [g]$ which has a horizon.

Theorem 7.15 *Let $[g]$ be a dynamic Δ^δ 2-structure, and let $h \in [g]$ be such that it has a horizon. Then*

$$\mathcal{C}[g] = \{X \mid X \in \mathcal{C}(h) \ \ \text{or} \ \ D \setminus X \in \mathcal{C}(h)\}.$$

Proof. Let $x \in D$ be a horizon of h. Let $X \in \mathcal{C}[g]$, and let $Z \in \{X, \overline{X}\}$ be such that $x \notin Z$. By Lemma 7.10, there exists a $h' \in [g]$ such that $Z \in \mathcal{C}(h')$ and $\overline{Z} \in \mathcal{C}(h')$. Moreover, since $[g] = [h]$, there exists a selector σ such that $h' = h^\sigma$. Because $Z \in \mathcal{C}(h')$ and x is a horizon in h, Lemma 7.13 implies that the selector σ is constant on Z. By Lemma 7.14, $Z \in \mathcal{C}(h)$, and thus $\mathcal{C}[g] \subseteq \{X \mid X \in \mathcal{C}(h) \ \text{ or } \ D \setminus X \in \mathcal{C}(h)\}$.

In the other direction the proof is obvious by Theorem 7.11. □

8 Dynamic ℓ2-structures with Variable Domains

8.1 Disjoint union of ℓ2-structures

In some situations it is desirable that the domain of an ℓ2-structure can be changed; one may, *e.g.*, want to add or delete nodes of the ℓ2-structure. Such change of a domain is standard in the theory of graph grammars. We shall now consider this problem in the framework of dynamic ℓ2-structures as presented by Ehrenfeucht and Rozenberg [?]. The basic operation that we use to combine two ℓ2-structures is the operation of disjoint union.

Let Δ be a group and δ its involution, and assume that $g\colon E_2(D_1) \to \Delta$ and $h\colon E_2(D_2) \to \Delta$ are disjoint Δ^δ2-structures, *i.e.*, $D_1 \cap D_2 = \emptyset$. We define the *disjoint union* of g and h as the ℓ2-structure $g + h\colon E_2(\mathrm{dom}(g) \cup \mathrm{dom}(h)) \to \Delta$ such that

$$(g+h)(e) = \begin{cases} g(e), & \text{if } e \in E_2(D_1), \\ h(e), & \text{if } e \in E_2(D_2), \\ 1_\Delta, & \text{otherwise.} \end{cases}$$

Clearly, $g + h$ is a Δ^δ2-structure, see Figure 35.

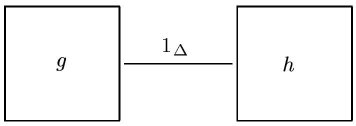


Figure 35: Disjoint union $g + h$

The following lemma is immediate from the definition.

Lemma 8.1 *Let g and h be disjoint Δ^δ 2-structures. Then* $\mathrm{dom}(g) \in \mathcal{C}([g+h])$ *and* $\mathrm{dom}(h) \in \mathcal{C}([g+h])$.

By Lemma 7.10, for any ℓ2-structure $g\colon E_2(D) \to \Delta$ and for any clan $X \in \mathcal{C}([g])$, there exists a selector σ such that

$$g = (\mathrm{sub}_g(D \setminus X) + \mathrm{sub}_g(X))^\sigma.$$

In particular, when we apply this result to the singleton clans, we obtain a construction of g from the singleton ℓ2-structures as follows: Let $D = \{x_1, x_2, \ldots, x_n\}$, and denote for simplicity by x the singleton ℓ2-structure with domain $\{x\}$. There are then Δ^δ2-structures g_i and selectors σ_i for $i = 1, 2, \ldots, n-1$ such that $\mathrm{dom}(g_i) = \{x_1, \ldots, x_i\}$, $g_{i+1} = (g_i + x_{i+1})^{\sigma_i}$ and $g = g_n$.

8.2 Comparison with grammatical substitution

In the theory of graph grammars the derivations of graphs are based on the operation of grammatical substitution. We shall now consider this operation in the context of dynamic Δ^δ2-structures.

Let g and h be Δ^δ2-structures with Δ as their group of labels. Assume that $x \in \mathrm{dom}(g)$ and that $(\mathrm{dom}(g) \setminus \{x\}) \cap \mathrm{dom}(h) = \emptyset$. The *grammatical substitution of h for x in g* is the Δ^δ2-structure $g(x \leftarrow h)\colon (\mathrm{dom}(g) \setminus \{x\}) \cup \mathrm{dom}(h) \to \Delta$ such that

$$g(x \leftarrow h)(y,z) = \begin{cases} g(y,z), & \text{if } (y,z) \in E_2(\mathrm{dom}(g) \setminus \{x\}), \\ h(y,z), & \text{if } (y,z) \in E_2(\mathrm{dom}(h)), \\ g(y,x), & \text{if } y \in \mathrm{dom}(g) \setminus \{x\} \text{ and } z \in \mathrm{dom}(h), \\ g(x,z), & \text{if } z \in \mathrm{dom}(g) \setminus \{x\} \text{ and } y \in \mathrm{dom}(h). \end{cases}$$

The operation of grammatical substitution as defined above corresponds to the case of node label controlled (eNLC) graph grammars (see [25]) where the connection relation is hereditary in the sense that the labels of the edges adjacent to the 'mother node' carry over without change to the edges adjacent to the daughter graph.

In the following if $f = g+h$, then we denote $h = f-g$. For a Δ^δ2-structure g and a node x, let σ_x be the horizontal selector for x. The following result from [21] states that the operation of substitution can be expressed by the operations of disjoint union and difference, and by (horizontal) selectors.

Theorem 8.2 *Let g and h be Δ^δ2-structures, and let $x \in \mathrm{dom}(g)$ be such that $(\mathrm{dom}(g) \setminus \{x\}) \cap \mathrm{dom}(h) = \emptyset$. Then*

$$g(x \leftarrow h) = \sigma_x^{-1}((\sigma_x(g) - x) + h).$$

Let g and h be disjoint Δ^δ2-structures. We say that a Δ^δ2-structure f *decomposes directly into g, h*, denoted $f \Rightarrow \{g,h\}$, if $f \in [g+h]$, *i.e.*, if there exists a selector σ such that $f = (g+h)^\sigma$. Further, a dynamic Δ^δ2-structure F *decomposes directly into* dynamic Δ^δ2-structures G and H, denoted $F \Rightarrow \{G,H\}$, if there exist an $f \in F$, $g \in G$ and $h \in H$ such that $f \Rightarrow \{g,h\}$, *i.e.*, if $F = [g+h]$ for some $g \in G$ and $h \in H$.

The next theorem shows that a direct decomposition of F amounts to decomposing F into clans.

Theorem 8.3 *Let F, G and H be dynamic Δ^δ2-structures, and let the domain of F be D. $F \Rightarrow \{G,H\}$ if and only if there exists a nonempty proper clan $X \in \mathcal{C}(F)$ such that $G = \mathrm{sub}_F(X)$ and $H = \mathrm{sub}_F(D \setminus X)$.*

8.3 Amalgamated union

We relax now the disjointness condition of the domains and consider the situation, where two Δ^δ2-structures g and h share a common substructure.

Let g and h be two Δ^δ2-structures with $A = \mathrm{dom}(g) \cap \mathrm{dom}(h)$ such that $\mathrm{sub}_g(A) = \mathrm{sub}_h(A)$, $\mathrm{dom}(g) \setminus A \in \mathcal{C}(g)$ and $\mathrm{dom}(h) \setminus A \in \mathcal{C}(h)$. We define the *amalgamated union* $g \amalg h$ of g and h as the Δ^δ2-structure

$$(g \amalg h)(x,y) = \begin{cases} g(x,y), & \text{if } (x,y) \in E_2(\mathrm{dom}(g)), \\ h(x,y), & \text{if } (x,y) \in E_2(\mathrm{dom}(h)), \\ 1_\Delta, & \text{otherwise.} \end{cases}$$

Clearly, $g \amalg h$ is a Δ^δ2-structure, see Figure 36.

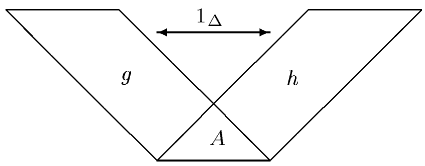


Figure 36: Amalgamated union $g \amalg h$

Lemma 8.4 *Let $f = g \amalg h$. Then* $\mathrm{dom}(g) \in \mathcal{C}([f])$ *and* $\mathrm{dom}(h) \in \mathcal{C}([f])$.

The following decomposition result for dynamic Δ^δ2-structure using amalgamated unions was proved in [21].

Theorem 8.5 *Let F be a dynamic Δ^δ2-structure on a domain D. If $X, Y \in \mathcal{C}(F)$ are nonempty and $X \cup Y = D$, then there exist $g \in \mathrm{sub}_F(X)$ and $h \in \mathrm{sub}_F(Y)$ such that $F = [g \amalg h]$.*

One can now define the direct decomposition relation based on amalgamated unions analogously to the direct decomposition relation based on disjoint unions. For this let g and h be Δ^δ2-structures with $A = \mathrm{dom}(g) \cap \mathrm{dom}(h)$ such that $\mathrm{sub}_g(A) = \mathrm{sub}_h(A)$, $\mathrm{dom}(g) \setminus A \in \mathcal{C}(g)$ and $\mathrm{dom}(h) \setminus A \in \mathcal{C}(h)$. We say that a Δ^δ2-structure f *directly amalgam decomposes into* g, h, denoted $f \Rightarrow^{\amalg} \{g, h\}$, if $f \in [g \amalg h]$. Further, we write $F \Rightarrow^{\amalg} \{G, H\}$ for dynamic Δ^δ2-structures F, G and H, if $F = [g \amalg h]$ for some $g \in G$ and $h \in H$.

We obtain for amalgamated unions an analogous result to Theorem 8.3:

Theorem 8.6 *Let $F = [f]$ be a dynamic Δ^δ2-structure. Then $F \Rightarrow^{\amalg} \{G, H\}$ if and only if there exist nonempty $X, Y \in \mathcal{C}(F)$ such that $X \cup Y = \mathrm{dom}(f)$, $G = \mathrm{sub}_F(X)$ and $H = \mathrm{sub}_F(Y)$.*

9 Quotients and Plane Trees

9.1 Quotients of dynamic $\Delta^\delta 2$-structures

Let $[g]$ be a dynamic $\Delta^\delta 2$-structure. We define now the factorizations and quotients of $[g]$. These constructions are not quite so straightforward as for 2-structures, because the clans $\mathcal{X} \subseteq \mathcal{C}[g]$ forming a partition of the domain D of $[g]$ may come from different images of g.

A partition $\mathcal{X}$ of the domain D into clans of $[g]$ is called a *factorization* of $[g]$. The *quotient* of $[g]$ by $\mathcal{X}$ is the family

$$[g]/\mathcal{X} = \{h/\mathcal{X} \mid h \in [g], \mathcal{X} \text{ a factorization of } h\}.$$

The family $[g]/\mathcal{X}$ is well-defined in the sense that each $h/\mathcal{X}$ is a $\Delta^\delta 2$-structure. Below we show that for each factorization $\mathcal{X} \subseteq \mathcal{C}[g]$, the quotient $[g]/\mathcal{X}$ is nonempty and, indeed, it is the dynamic $\Delta^\delta 2$-structure $[h/\mathcal{X}]$ for some $h \in [g]$.

If $\sigma(x) = a$ is a constant selector on the whole domain D, then for all $e \in E_2(D)$, $g^\sigma(e) = a \cdot g(e) \cdot \delta(a)$. Clearly, in this case, g^σ is strictly isomorphic with g. Hence the following lemma holds.

Lemma 9.1 *Let g be a $\Delta^\delta 2$-structure. If $h = g^\sigma$ for a selector σ, which is constant on D, then h and g are strictly isomorphic. In this case, the substructures $\mathrm{sub}_g(X)$ and $\mathrm{sub}_h(X)$ and the quotients $g/\mathcal{X}$ and $h/\mathcal{X}$ are strictly isomorphic for all $X \subseteq D$ and all factorizations $\mathcal{X}$ of g.*

The following theorem was proved in Ehrenfeucht, Harju and Rozenberg [12].

Theorem 9.2 *If $\mathcal{X}$ is a factorization of a dynamic $[g]$, then there exists a selector σ such that $\mathcal{X}$ is a factorization of g^σ, and*

$$[g]/\mathcal{X} = [g^\sigma/\mathcal{X}].$$

Proof. Let $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ and $[g]$ be as stated in the claim. For each $X \in \mathcal{X}$, we have $X \in \mathcal{C}(g^{\sigma_X})$ for some selector σ_X. By Lemma 7.14, we may assume that $\sigma_X(y) = 1_\Delta$ for all $y \notin X$. Define $\sigma = \sigma_{X_m}\sigma_{X_{m-1}} \dots \sigma_{X_1}$. Hence $\sigma(x) = \sigma_X(x)$, if $x \in X$, since $\mathcal{X}$ is a factorization of $[g]$. Moreover, by the choice of the selectors σ_{X_i}, $\sigma = \sigma_{X_{\pi(m)}}\sigma_{X_{\pi(m-1)}} \dots \sigma_{X_{\pi(1)}}$ for any permutation π of $\{1, 2, \dots, m\}$. Now, $g^\sigma = (g^{\sigma_X})^{\sigma \cdot \sigma_X^{-1}}$, where $\sigma \cdot \sigma_X^{-1}$ is constant on X. Hence by Lemma 7.14, $X \in \mathcal{C}(g^\sigma)$ for each $X \in \mathcal{X}$, and thus $\mathcal{X}$ is a factorization of $h = g^\sigma$.

Now, if $\mathcal{X}$ is a factorization of some $g_1 \in [g]$, then from Lemma 7.8 we obtain that $g_1 = h^{\sigma_1}$ for a selector σ_1. Consequently, Lemma 7.14 yields that σ_1 is constant on each $X \in \mathcal{X}$, and, therefore $g_1/\mathcal{X} = (h/\mathcal{X})^{\sigma_2}$ for the selector $\sigma_2\colon \mathcal{X} \to \Delta$ defined by $\sigma_2(X) = \sigma_1(x)$ $(x \in X)$ for all $X \in \mathcal{X}$.

On the other hand, if $\sigma_2\colon \mathcal{X} \to \Delta$ is a selector, then it can be extended to a selector $\sigma_1\colon D \to \Delta$ by setting $\sigma_1(x) = \sigma_2(X)$, if $x \in X$ with $X \in \mathcal{X}$. Evidently, $h^{\sigma_1}/\mathcal{X} = (h/\mathcal{X})^{\sigma_2}$, and thus $[g]/\mathcal{X} = \{(h/\mathcal{X})^{\sigma_2} \mid \sigma_2 \text{ a selector}\}$ as required. $\square$

9.2 Plane trees

In general, two given g_1 and g_2 in $[g]$ can be quite different from each other. Indeed, the clan structures of g_1 and g_2 may be drastically different as shown in the following example.

Example 9.3 Let $D = \{x_1, x_2, x_3\}$, $\Delta = \mathbb{Z}_5$ and let δ be the inverse function of Δ. Let g be the complete $\Delta^\delta 2$-structure corresponding to the label 0, *i.e.*, $g(e) = 0$ for all $e \in E_2(D)$. Hence all subsets of D are clans of g. Let then σ be a selector defined by $\sigma(x_1) = 0$, $\sigma(x_2) = 1$ and $\sigma(x_3) = 3$. Then $h = g^\sigma$ is specified by $h(x_1, x_2) = 4$, $h(x_2, x_3) = 3$ and $h(x_3, x_1) = 3$. Now, h is primitive – has only the trivial clans. $\square$

Let $G = [g]$ be a dynamic $\Delta^\delta 2$-structure, and assume that x is a horizon of g. The substructure

$$\pi_x(g) = \mathrm{sub}_g(D \setminus \{x\})$$

is called the *x-plane* of g (or an *x-plane* of G). We shall denote by G_π^x the family of x-planes of G.

Since for each node x of a $G = [g]$, $\{x\} \in \mathcal{C}[g]$, Lemma 7.10 implies that there exists an $h \in G$ such that x is a horizon of h. Hence G_π^x is always nonempty.

The following result from [12] shows that the planes in G_π^x are quite similar to each other.

Theorem 9.4 *Let G be a dynamic $\Delta^\delta 2$-structure. The planes in G_π^x are strictly isomorphic.*

Proof. Let $g_1, g_2 \in G$ be such that x is a horizon in both of them, and let $h_i = \pi_x(g_i)$ for $i = 1, 2$. By Lemma 7.8, there exists a selector σ such that $g_2 = g_1^\sigma$. Clearly, $h_2 = h_1^{\sigma_1}$, where $\sigma_1\colon D \setminus \{x\} \to \Delta$ is the restriction of σ. Since $D \setminus \{x\}$ is a common clan of g_1 and g_2, Lemma 7.14 implies that σ is constant on $D \setminus \{x\}$. By Lemma 9.1, h_1 and h_2 are strictly isomorphic, since h_1 and h_2 have the same domain $D \setminus \{x\}$. $\square$

In particular, it follows from Lemma 9.1 that for any two planes $h_1, h_2 \in G_\pi^x$, $\mathcal{C}(h_1) = \mathcal{C}(h_2)$ and $\mathcal{P}(h_1) = \mathcal{P}(h_2)$. It follows also that for all $X \in \mathcal{P}(h_1)$ the special quotients $\Psi_{h_1}(X)$ and $\Psi_{h_2}(X)$ are of the same type, *i.e.* they are both primitive or both complete or both linear. This justifies the following definition, where an element $X \in \mathcal{P}(h_1)$ obtains a label p, c or ℓ according to the type of $\Psi_{h_1}(X)$.

Let $x \in D$ for a dynamic Δ^δ2-structure G.

(1) Denote $\mathcal{C}(x) = \mathcal{C}(h_x)$, $\mathcal{P}(x) = \mathcal{P}(h_x)$ and $\mathcal{N}(x) = \mathcal{P}(h_x) \cup \{x\}$, where h_x is an arbitrary element of G_π^x.

(2) Define $\Phi_x(\{x\}) = c$ and for all $X \in \mathcal{P}(h_x)$, let $q_X = \Psi_{h_x}(X)$ and

$$\Phi_x(X) = \begin{cases} p, & \text{if } q_X \text{ is truly primitive,} \\ c, & \text{if } q_X \text{ is complete,} \\ \ell, & \text{if } q_X \text{ is linear and } |q_X| > 1. \end{cases}$$

In the above definition we have the requirement $|q_X| > 1$ in the case of linear q_X in order to have Φ_x well-defined, because a singleton quotient is, by definition, both complete and linear.

(3) The *plane tree* $L(x)$ of the node x is the node-labeled directed rooted tree with the set of nodes $\mathcal{N}(x)$ labeled by Φ_x, which is obtained from the prime tree $T(h_x)$ by adding $\{x\}$ as a new root and $(x, D \setminus \{x\})$ as a new edge.

Example 9.5 Let $\Delta = \mathbb{Z}_3$ and let δ be the inverse function of Δ. Consider the Δ^δ2-structure g from Figure 37.

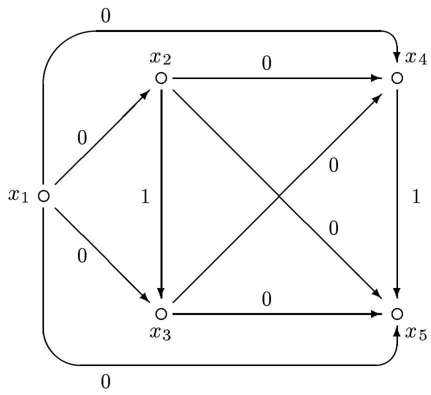


Figure 37: Δ^δ2-structure g

We note that g has a horizon x_1. The x_1-plane h_1 of g is given in Figure 38. The nontrivial prime clans of h_1 are $\{x_2, x_3\}$ and $\{x_4, x_5\}$, and the plane tree

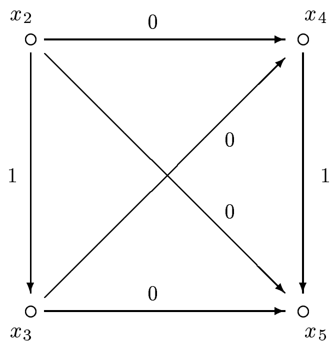


Figure 38: x_1-plane of g

of x_1 is given in Figure 39, where a nonsingleton node X has been labeled by d, if $\Phi_x(X) = d$. Again, each nonsingleton node consists of the leaves that are its descendants in the directed tree, and so we have omitted the identities of these nodes in Figure 39. The singleton nodes x_i have been identified, but without the label c.

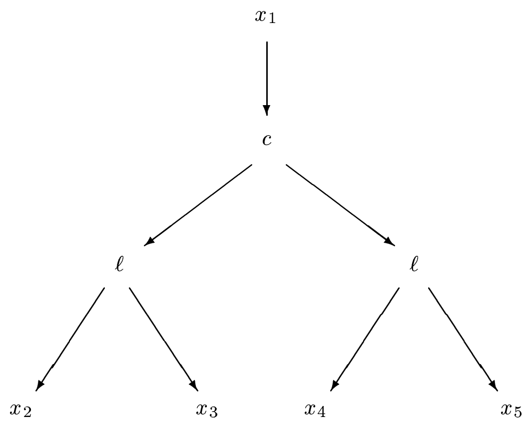


Figure 39: Plane tree $L(x_1)$

Let us then make x_4 a horizon. This is done by the horizonal selector σ as follows: $\sigma(x_4) = 0$ and $\sigma(x_i) = g(x_4, x_i)$ for $i \neq 4$. Now, the x_4-plane h_4 of g^σ is given in Figure 40.

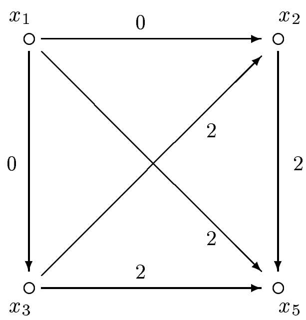


Figure 40: Plane h_4

The nontrivial primes of h_4 are $\{x_1, x_2, x_3\}$ and $\{x_2, x_3\}$, and the plane tree of x_4 is given in Figure 41.

The plane trees $L(h_1)$ and $L(h_4)$ are not isomorphic, but their underlying unrooted and undirected trees are node-isomorphic, and indeed, as we shall see, the quotients of the corresponding nodes are of the same size and type.

□

By Theorem 7.15 we have the following lemma.

Lemma 9.6 *Let x be a node of a dynamic Δ^δ2-structure G. Then for each clan X of G such that $X \notin \{\emptyset, D\}$, $X \in \mathcal{C}(x)$, if $x \notin X$, and $\overline{X} \in \mathcal{C}(x)$, if $x \in X$.*

Proof. Let x be a horizon of $g \in G$ and let $\pi_x(g) = h$. Assume X is a clan of G.

Assume $Y \in \{X, \overline{X}\}$ is such that $Y \in \mathcal{C}(g)$. If $x \notin Y$, then since the domain of h, $D \setminus \{x\}$, is a clan of g, Lemma 6.5 implies that $Y \in \mathcal{C}(h)$.

On the other hand, if $x \in Y$, then $\overline{Y} \in \mathcal{C}(g)$. Indeed, for all $y \in Y$ $(y \neq x)$ and $z \notin Y$ we have that $g(y,x) = 1_\Delta = g(z,x)$, since x is a horizon of g. Moreover, since $Y \in \mathcal{C}(g)$, $g(x,z) = g(y,z)$. Hence $g(y,z) = 1_\Delta$ for all $y \in Y$ and $z \notin Y$, which shows that $\overline{Y} \in \mathcal{C}(g)$. In this case $\overline{Y} \in \mathcal{C}(h)$ by Lemma 6.5. Now the claim follows from Theorem 7.15. □

Our purpose is to compare the planes of a dynamic Δ^δ2-structure G. For this we first study the relationships between the prime clans.

Lemma 9.7 *Let G be a dynamic Δ^δ2-structure, and let x, y be two nodes of its domain. Assume that $X \in \mathcal{P}(x)$.*

(1) *If $y \notin X$, then $X \in \mathcal{P}(y)$.*

(2) *If $y \in X$, then $\overline{X} \in \mathcal{P}(y)$. In particular, $\mathcal{P}(x)$ and $\mathcal{P}(y)$ have the same number of prime clans.*

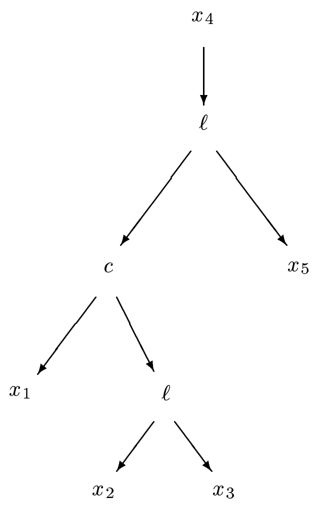


Figure 41: Plane tree $L(x_4)$

Proof. The claim follows directly from Lemma 9.6, when we observe that a clan Y of G overlaps with a subset $Z \subseteq D$ if and only if Y overlaps with $\overline{Z}$. $\square$

Example 9.8 Consider the plane trees $L(x_1)$ and $L(x_4)$ of Example 9.5 (see, Figures 39 and 41). Denote $A = \{x_2, x_3\}$ and $B = \{x_4, x_5\}$. We have redrawn these plane trees in Figure 42 in order to illustrate the 1-1 correspondence between them. Hence $L(x_4)$ is obtained from $L(x_1)$ by reversing the direction of the path from x_1 to x_4, and each inner node of this reversed path is the complement of its successor in the original path in $L(x_1)$; hence, *e.g.*, the node B becomes $\overline{x_4}$. The singleton node x_4 that begins the reversed path does not change, because it does not have a successor in $L(x_1)$. Note that the corresponding nodes in these plane trees are of the same type. $\square$

Following Lemma 9.7 (and the previous example) we define for all $x, y \in D$ the mapping $\eta_{xy}: \mathcal{N}(x) \to \mathcal{N}(y)$ as follows:

$$\eta_{xy}(X) = \begin{cases} X, & \text{if } y \notin X \text{ or } X = \{y\}, \\ \overline{Y}, & \text{if } y \in X,\ y \in Y \text{ and } (X,Y) \text{ is an edge in } L(x). \end{cases}$$

Lemma 9.9 *Let G be a dynamic Δ^δ2-structure, and let x, y be two of its nodes. Then η_{xy} is a bijective correspondence between $\mathcal{N}(x)$ and $\mathcal{N}(y)$.*

Proof. First of all η_{xy} is well-defined. Indeed, for each $X \in \mathcal{N}(x)$ with $y \in X$ and $X \neq \{y\}$ there is a unique edge (X, Y) in $L(x)$ such that $y \in Y$ and $Y \subset X$.

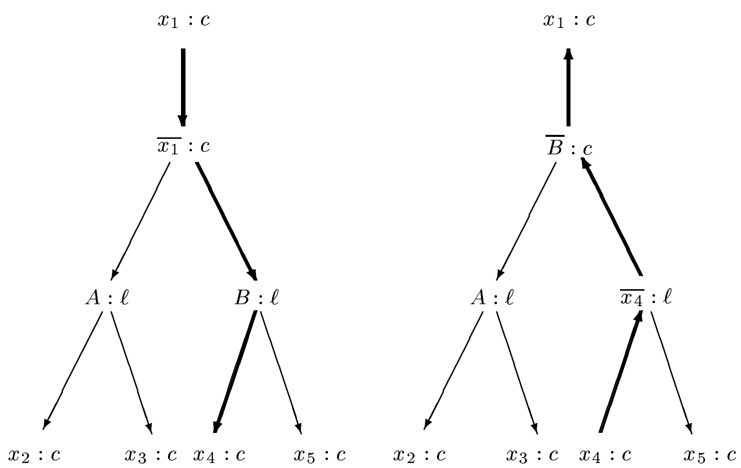


Figure 42: Comparison of two plane trees

Now, by Lemma 9.7, $\overline{Y} \in \mathcal{N}(y)$ (with $y \notin \overline{Y}$), and, by the definition of η_{xy}, $\eta_{xy}(X) = \overline{Y}$.

The injectiveness claim follows from the above stated uniqueness of the edge (X, Y), since if $y \in X$ for some $X \in \mathcal{N}(x)$ with $X \neq \{y\}$, then for the unique $Y \in \mathcal{N}(x)$ with $\eta_{xy}(X) = \overline{Y}$ we have that $\overline{Y} \notin \mathcal{N}(x)$ (because $x \in \overline{Y}$). It follows that the mapping η_{xy} is surjective, because the sets $\mathcal{N}(x)$ and $\mathcal{N}(y)$ are finite and have the same number of elements. $\square$

The following result from [12], which we state without proof, gives a simple construction that allows one to obtain the plain tree $L(y)$ from $L(x)$. Indeed, *$L(y)$ is obtained from $L(x)$ by first changing the nodes by the mapping η_{xy}, and then reversing the direction of the path from x to y.*

Theorem 9.10 *Let G be a dynamic Δ^δ2-structure and let x, y be two of its nodes. If (X, Y) is an edge of $L(x)$, then*

(1) *$(\eta_{xy}(X), \eta_{xy}(Y))$ is in $L(y)$, if $y \notin Y$,*

(2) *$(\eta_{xy}(Y), \eta_{xy}(X))$ is in $L(y)$, if $y \in Y$.*

The correspondence η_{xy} also preserves the types of the quotients:

Theorem 9.11 *Let G be a dynamic Δ^δ2-structure and let x, y be two of its nodes. Then $\Phi_x(X) = \Phi_y(\eta_{xy}(X))$ for all $X \in \mathcal{N}(x)$.*

Now, let x be a node of a dynamic Δ^δ2-structure G. Define $F(x) = (\mathcal{N}(x), \Phi_x)$ as the *form* of x, that is, $F(x)$ is undirected and unrooted tree such that $\{X, Y\}$ is an edge in $F(x)$ if and only if either (X, Y) or (Y, X) is an edge of $L(x)$. The labeling $\Phi_x: \mathcal{N}(x) \to \{p, c, \ell\}$ of the nodes remains the same as in $L(x)$.

Theorem 9.12 *Let x, y be two nodes in a dynamic Δ^δ2-structure G. The forms $F(x)$ and $F(y)$ are isomorphic. Indeed, $\eta_{xy}: \mathcal{N}(x) \to \mathcal{N}(y)$ is a label preserving isomorphism from $F(x)$ onto $F(y)$.*

Proof. By Theorem 9.10, if $\{X, Y\}$ is an edge of $F(x)$, then $\{\eta_{xy}(X), \eta_{xy}(Y)\}$ is an edge of $F(y)$. Moreover, η_{xy} is a bijection from $\mathcal{N}(x)$ onto $\mathcal{N}(y)$. This shows that η_{xy} is an isomorphism between $F(x)$ and $F(y)$.

By Theorem 9.11, $\Phi_x(X) = \Phi_y(\eta_{xy}(X))$ for all $X \in \mathcal{N}(x)$, and hence η_{xy} preserves the labels of the nodes. This proves the claim. $\square$

Theorem 9.12 allows us to construct a tree $F(G)$, the *form* of G, for each dynamic Δ^δ2-structure G as follows. Consider the tree $F(x)$ for a node x and leave the nodes unidentified, but labeled by Φ_x. Hence a tree $F(G)$ becomes an 'abstract tree' in the sense that the identities of the nodes are omitted. To be more precise, $F(G)$ is the isomorphism class of the forms $F(x)$ with x in the domain of G.

Example 9.13 The form $F(G)$ of the the dynamic Δ^δ2-structure G from Example 9.5 (see, Figure 37), is given in Figure 42. $\square$

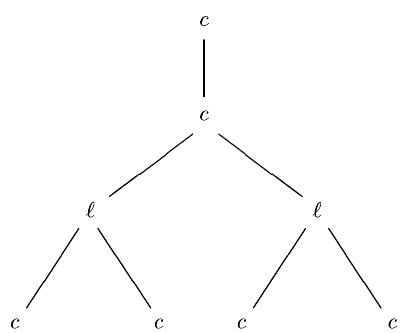


Figure 43: The form $F(G)$

10 Invariants

10.1 Introduction to invariants

In this section we shall consider the problem how to recognize that an ℓ2-structure g is obtainable from another ℓ2-structure h by an application of a selector. In order to answer this question invariant properties of $\ell 2$-structures were studied by Ehrenfeucht, Harju and Rozenberg [11], where the invariants were shown to be closely connected to the verbal identities of the group Δ of labels. In this paper we shall not go into these group theoretical considerations, and furthermore, we state a detailed characterization only for the abelian case (in Section 104). For the proofs and further results we refer to [11].

In this section *the involution δ of the given group Δ will be the inverse function of* Δ,

$$\delta(a) = a^{-1} \qquad (a \in \Delta).$$

For this reason we call a $\Delta^{\delta}2$-structure more conveniently an *inversive* $\ell 2$-structure.

Denote by $\mathcal{R}(D, \Delta)$ the family of all inversive $\ell 2$-structures $g: E_2(D) \to \Delta$ for a given domain D and for a given group Δ of labels.

The inversive $\ell 2$-structures in $\mathcal{R}(D, \Delta)$ are naturally divided into dynamic $\ell 2$-structures, *i.e.*, the family $\{[g] \mid g \in \mathcal{R}(D, \Delta)\}$ forms a partition of $\mathcal{R}(D, \Delta)$. A function $\eta: \mathcal{R}(D, \Delta) \to \Delta$ is an *invariant*, if it maps the elements of each dynamic $[g]$ into the same element,

$$\eta(g_1) = \eta(g_2) \qquad \text{if} \quad [g_1] = [g_2].$$

Hence an invariant is immune to the selectors, and the study of invariants is the study of those properties of a network that remain unchanged during its evolution.

A constant mapping, $\eta(g) = a$ ($a \in \Delta$, $g \in \mathcal{R}(D, \Delta)$) is a simple example of an invariant.

We shall also say that two $\ell 2$-structures g_1 and g_2 are *equivalent*, if $[g_1] = [g_2]$. Hence a function $\eta: \mathcal{R}(D, \Delta) \to \Delta$ is an invariant, if it maps equivalent $\ell 2$-structures to the same element of the group Δ.

10.2 Free invariants

In general an invariant, as defined above, is a function that can be independent of the specific properties of the $\ell 2$-structures. In order to reflect these properties of the $\ell 2$-structures, we shall restrict ourselves to those invariants that are more faithful to the $\ell 2$-structures in the sense that they are defined by variables corresponding to the edges.

Consider the free monoid $M(D) = E_2(D)^*$ generated by the edges. The elements of $M(D)$ are the words $e_1 e_2 \dots e_k$ for $k \geq 0$ and $e_i \in E_2(D)$. The empty word, denoted by 1, is the identity element of the monoid $M(D)$. For a word $w = e_1 e_2 \dots e_k \in M(D)$ we let $w^{-1} = e_k^{-1} e_{k-1}^{-1} \dots e_1^{-1}$. In this way $M(D)$ becomes a *monoid with involution*.

In order to clarify the distinction between the edges in $\ell 2$-structures and the generators of the free monoid $M(D)$, the generators $e \in M(D)$ will be called *variables*.

Each word $w = e_1 e_2 \dots e_k \in M(D)$ defines a function ψ_w from $\mathcal{R}(D,\Delta)$ into Δ as follows,

$$\psi_w(g) = g(e_1)g(e_2)\dots g(e_k) \qquad (g \in \mathcal{R}(D,\Delta)).$$

We call the function ψ_w the *variable function represented by* w, and we let $\mathrm{Var}(D,\Delta)$ denote the set of all variable functions represented by the words in $M(D)$.

Two words w_1 and w_2 from $M(D)$ are said to be *equivalent* (*over* Δ), denoted $w_1 \equiv w_2$, if they represent the same variable function, $\psi_{w_1} = \psi_{w_2}$.

Further, if a variable function $\psi_w \in \mathrm{Var}(D,\Delta)$ is an invariant, then it is a *free invariant* of $\mathcal{R}(D,\Delta)$. Denote by $\mathrm{Inv}(D,\Delta)$ the set of all free invariants ψ_w of $\mathcal{R}(D,\Delta)$.

Example 10.1 Let $\Delta = \mathbb{Z}_3 = \{0,1,2\}$ be the cyclic group on three elements. Assume further that $D = \{x_1,x_2,x_3\}$, and write e_{ij} for (x_i,x_j). The word $w_1 = e_{12}e_{23}e_{31}$ represents an invariant, *i.e.*, $\psi_{w_1} \in \mathrm{Inv}(D,\Delta)$, since for all $g \in \mathcal{R}(D,\Delta)$ and all selectors $\sigma\colon D \to \Delta$ with $\sigma(x_i) = a_i$, $\psi_{w_1}(g) = g(e_{12}) + g(e_{23}) + g(e_{31})$ and

$$\begin{aligned}
\psi_{w_1}(g^\sigma) &= g^\sigma(e_{12}) + g^\sigma(e_{23}) + g^\sigma(e_{31}) \\
&= (a_1 + g(e_{12}) - a_2) + (a_2 + g(e_{23}) - a_3) + (a_3 + g(e_{31}) - a_1) \\
&= g(e_{12}) + g(e_{23}) + g(e_{31}) = \psi_{w_1}(g)\ ,
\end{aligned}$$

because Δ is abelian.

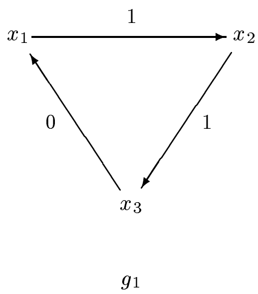
For the two structures g_1 and g_2 from Figure 44, we have $\psi_{w_1}(g_1) = 2 \neq 1 = \psi_{w_1}(g_2)$, which implies that for all selectors σ, $g_2 \neq g_1^\sigma$, since $\psi_{w_1}(g_1^\sigma) = 2$ but $\psi_{w_1}(g_2) = 1$ for each σ. Hence in this case $[g_1] \neq [g_2]$.

On the other hand, let $w_2 = e_{12}e_{32}e_{31}$. Now, $\psi_{w_2}(g) = g(e_{12}) + g(e_{32}) + g(e_{31})$ and

$$\psi_{w_2}(g^\sigma) = g^\sigma(e_{12}) + g^\sigma(e_{32}) + g^\sigma(e_{31}) = 2a_3 - 2a_2 + \psi_{w_2}(g)\ ,$$

and thus $\psi_{w_2}(g^\sigma) = \psi_{w_2}(g)$ if and only if $2a_3 - 2a_2 = 0$ in $\mathbb{Z}_3$. It follows that ψ_{w_2} is not an invariant of $\mathcal{R}(D,\Delta)$, since, *e.g.*, the selector may have chosen $a_2 = 1$ and $a_3 = 2$. □

The constant functions $\eta\colon \mathcal{R}(D,\Delta) \to \Delta$ are invariants, but as can be easily shown only the trivial constant function is a free invariant.



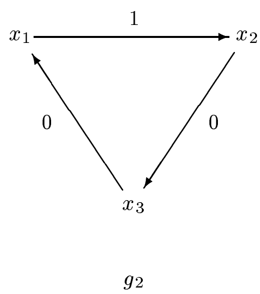


Figure 44: Nonequivalent inversive $\ell 2$-structures

10.3 Basic properties of free invariants

From the definition of $\mathrm{Var}(D, \Delta)$ we obtain that for all words $w, w_1, w_2 \in M(D)$

$$\psi_{w_1 w_2}(g) = \psi_{w_1}(g) \cdot \psi_{w_2}(g) \ , \quad \psi_{w^{-1}}(g) = \psi_w(g)^{-1} \quad \text{ and } \quad \psi_1(g) = 1_\Delta,$$

whenever g is an inversive $\ell 2$-structure. (Here 1 is the empty word). Therefore $\mathrm{Var}(D, \Delta)$ is a group with the operation defined by

$$\psi_{w_1} \cdot \psi_{w_2} = \psi_{w_1 w_2}.$$

The identity of the group $\mathrm{Var}(D, \Delta)$ is the variable function ψ_1 represented by the empty word. Further, we have $\psi_{w^k}(g) = \psi_w(g)^k$ for all $w \in M(D)$, $k \in \mathbb{Z}$ and $g \in \mathcal{R}(D, \Delta)$. Therefore we can write

$$\psi_{w^k} = \psi_w^k \qquad \text{and} \qquad \psi_{w^{-1}} = \psi_w^{-1}.$$

Moreover, $\mathrm{Var}(D, \Delta)$ is generated by the variable functions ψ_e represented by the variables, $e \in E_2(D)$.

The following result from [11] shows that a free invariant ψ_w is, in fact, a mapping $\psi_w : \mathcal{R}(D, \Delta) \to Z(\Delta)$ into the center of the group Δ. This implies that the group $\mathrm{Inv}(D, \Delta)$ is always abelian. Recall that the *center* of a group Δ is defined by

$$Z(\Delta) = \{a \in \Delta \mid ax = xa \ \text{ for all } \ x \in \Delta\}.$$

Theorem 10.2 (1) *For all free invariants* $\psi_w \in \mathrm{Inv}(D, \Delta)$ *and inversive* g, $\psi_w(g) \in Z(\Delta)$.

(2) $\mathrm{Inv}(D, \Delta)$ *is a subgroup of* $Z(\mathrm{Var}(D, \Delta))$. *In particular,* $\mathrm{Inv}(D, \Delta)$ *is an abelian group.*

By Theorem 10.2, if the center $Z(\Delta)$ is trivial, then there are no nontrivial free invariants for the inversive $\ell 2$-structures g with the group Δ of labels, see Example 10.4 below.

A word $w = (x_0, x_1)(x_1, x_2) \dots (x_{k-1}, x_0)$ from $M(D)$ is called a *closed walk of length* k. A closed walk

$$t(x_0, x_1, x_2) = (x_0, x_1)(x_1, x_2)(x_2, x_0)$$

is called a *triangle* (*at* x_0), if $x_i \neq x_j$ for each $i \neq j$. The empty word $1 \in M(D)$ is a *trivial walk*.

The next result is rather easy to prove.

Theorem 10.3 *Let $w \in M(D)$ be a closed walk. Then $\psi_w \in \mathrm{Inv}(D, \Delta)$ if and only if $\psi_w(g) \in Z(\Delta)$ for all inversive $\ell 2$-structures g. In particular, if Δ is abelian, then ψ_w is a free invariant of $\mathcal{R}(D, \Delta)$.*

Example 10.4 Consider the dihedral group D_{2n} of $2n$ elements ($n \geq 3$),

$$D_{2n} = \{1, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\},$$

where 1 is the identity element of D_{2n}. The group D_{2n} is the symmetry group of a regular n-gon of the Euclidean plane, and it is generated by a rotation a (with an angle of $2\pi/n$) together with a reflection b of the n-gon. These generators satisfy the following defining relations

$$a^n = 1, \qquad b^2 = 1, \qquad ab = ba^{-1}.$$

It is rather easy to show that if n is odd, then $Z(D_{2n})$ is trivial, and hence in this case the group $\mathrm{Inv}(D, \Delta)$ of free invariants is also trivial by Theorem 10.2.

On the other hand, if n is even, then $Z(D_{2n})$ contains two elements, *i.e.*, it is isomorphic to the cyclic group $\mathbb{Z}_2$. Let us consider the case $n = 4$. In this case $Z(D_8) = \{1, a^2\}$. Assume that the domain D has at least three nodes, and let w be any closed walk. We claim that the variable function ψ_{w^2} is a free invariant. Here w^2 is the closed walk that traverses w twice around.

Indeed, let g be an inversive $\ell 2$-structure. Then

$$\psi_{w^2}(g) = (\psi_w(g))^2,$$

and it is easy to check that for all $c \in D_8$, $c^2 = 1$ or a^2. Therefore $\psi_{w^2}(g) \in Z(D_8)$ for all $g \in \mathcal{R}(D, \Delta)$, and thus ψ_{w^2} is a free invariant by Theorem 10.3. $\square$

10.4 Invariants on abelian groups

We shall show that in the abelian case the invariants represented by the triangles at x_0 form a 'complete' set of invariants.

Clearly, in the abelian case, for $w_1, w_2 \in M(D)$ we have $w_1 w_2 \equiv w_2 w_1$, and hence the occurrences of the variables in $w \in M(D)$ can be freely permuted without violating invariant properties.

For an abelian group Δ the group $\mathrm{Inv}(D, \Delta)$ has properties that are independent of the 2-structures in the following sense. Let $w = (x_1, y_1) \dots (x_n, y_n)$ be a word and $\sigma: D \to \Delta$ a selector. Then we have for all $g \in \mathcal{R}(D, \Delta)$,

$$\begin{aligned}\psi_w(g^\sigma) &= \sigma(x_1) g(x_1, y_1) \sigma(y_1)^{-1} \dots \sigma(x_n) g(x_n, y_n) \sigma(y_n)^{-1} \\ &= \sigma(x_1)\sigma(y_1)^{-1} \dots \sigma(x_n)\sigma(y_n)^{-1} \cdot \psi_w(g) \ ,\end{aligned}$$

and thus ψ_w is a free invariant if and only if $\sigma(x_1)\sigma(y_1)^{-1} \dots \sigma(x_n)\sigma(y_n)^{-1} = 1_\Delta$ for all selectors σ. Notice that the latter condition does not depend on g, and hence we have the following characterization result from [11].

Theorem 10.5 *Let Δ be an abelian group. The following conditions are equivalent for a variable function ψ_w.*

(1) *ψ_w is a free invariant of $\mathcal{R}(D, \Delta)$.*

(2) *For each $g \in \mathcal{R}(D, \Delta)$ and for each selector σ, $\psi_w(g^\sigma) = \psi_w(g)$.*

(3) *There exists a $g \in \mathcal{R}(D, \Delta)$ such that for each selector σ, $\psi_w(g^\sigma) = \psi_w(g)$.*

For a fixed node $x_0 \in D$, the set

$$T_{x_0} = \{ t(x_0, y, z) \mid x_0, y, z \in D \text{ are distinct } \}$$

is the *bucket of triangles at x_0*.

Each triangle $t \in T_x$ is a closed walk, and so it represents a free invariant $\psi_t \in \mathrm{Inv}(D, \Delta)$ for abelian Δ. The next theorem (from [11]) states that these invariants generate $\mathrm{Inv}(D, \Delta)$. The proof of this is based on the following simple identity, which allows us to use triangles. Let x_0 be a fixed node. For each $e = (y, z)$, where $y \neq x_0$ and $z \neq x_0$, we have

$$e \equiv (x_0, y)^{-1} \cdot (x_0, y)(y, z)(z, x_0) \cdot (z, x_0)^{-1} = (x_0, y)^{-1} \cdot t(x_0, y, z) \cdot (x_0, z) \ .$$

Theorem 10.6 *Let Δ be an abelian group and x_0 an element of the domain D. Then* $\mathrm{Inv}(D, \Delta)$ *is generated by the free invariants represented by the triangles at x_0.*

The triangles (at a fixed node x_0) not only represent generators of $\mathrm{Inv}(D, \Delta)$ but form a large enough set to characterize the relation $[g] = [h]$ between the inversive labeled 2-structures on an abelian Δ.

A set W of invariants for $\mathcal{R}(D, \Delta)$ is said to be a *complete*, if W satisfies for all g_1, g_2 the condition: $[g_1] = [g_2]$ if and only if $\eta(g_1) = \eta(g_2)$ for all $\eta \in W$.

We notice that in the above definition the converse implication is always valid, that is, if $g_1^\sigma = g_2$ for a selector σ, then for every invariant η, $\eta(g_2) = \eta(g_1^\sigma) = \eta(g_1)$. On the other hand, if a set of invariants W is complete and $[g_1] \neq [g_2]$, then there exists an invariant $\eta \in W$ such that $\eta(g_1) \neq \eta(g_2)$.

Theorem 10.7 *Let Δ be an abelian group. For each $x_0 \in D$ the bucket of triangles T_{x_0} forms a complete set of invariants.*

The above theorem allows the use of 'local' triangles for checking whether or not $[g_1] = [g_2]$, *i.e.*, if $[g_1] \neq [g_2]$, then we can find a (common) triangle in g_1 and g_2 that induces non-equivalent substructures in g_1 and g_2.

Example 10.8 Let $\Delta = \mathbb{Z}_2$ and let g_1, g_2, g_3 be the structures from Figure 45. Consider the triangle $t = t(x_1, x_2, x_3)$. We have $\psi_t(g_1) = 1 \neq 0 = \psi_t(g_2)$, and hence by the above theorem there does not exist a selector σ for which $g_2 = g_1^\sigma$.

On the other hand, we observe that for the triangles $t = t(x_1, x_2, x_3)$, $t = t(x_1, x_3, x_4)$ and $t = t(x_1, x_2, x_4)$, $\psi_t(g_1) = \psi_t(g_3)$, and hence $\psi_t(g_1) = \psi_t(g_3)$ for all triangles $t \in T_{x_1}$. Hence, by Theorem 10.7, $[g_1] = [g_3]$. $\square$

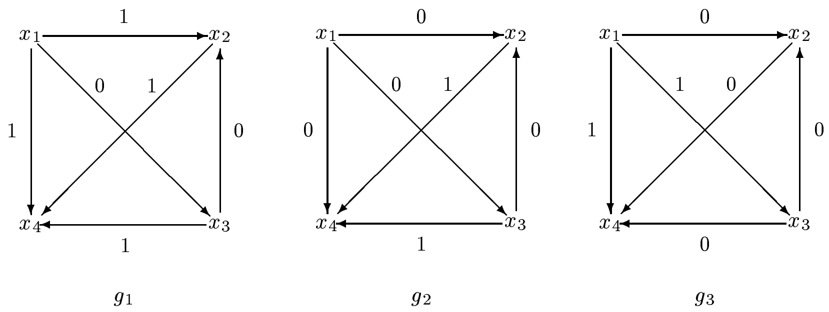


Figure 45: $[g_1] \neq [g_2]$, but $[g_1] = [g_3]$

10.5 Clans and invariants

Now, we do not assume that the group Δ is abelian. We shall state an invariant property of inversive $\ell 2$-structures connected to the clans of $\ell 2$-structures.

Let a closed walk

$$s(x_0, x_1, x_2, x_3) = (x_0, x_1)(x_1, x_2)(x_2, x_3)(x_3, x_0) \in M(D)$$

with four nodes be called a *square*. The following rather straightforward theorem (from [11] states that being a clan is an invariant property of the dynamic inversive $\ell 2$-structure $[g]$.

Theorem 10.9 *Let $g: E_2(D) \to \Delta$ be an inversive $\ell 2$-structure on a group Δ of labels. A subset $X \subseteq D$ is a clan of $[g]$, if $\psi_s(g) = 1_\Delta$ for all squares $s = s(x_1, y_1, x_2, y_2)$ with $y_1, y_2 \in D \setminus X$ and $x_1, x_2 \in X$.*

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