Combinatorial Enumeration (2017)

Problem Set 1 (Jan. 19)

1 **(a)** How many elements are there in $2^{[1,12]}$?

(b) How many $X \subseteq [1,12]$ contain at least one odd integer?

(c) How many $X \subseteq [1,12]$ contain exactly one odd integer?

Solution. (a) The set $[1,12]$ has $2^{12} = 4096$ subsets.

(b) There are 6 even integers in $[1,12]$, and thus $2^6 = 64$ subsets of $[1,12]$ have only even integers (including the empty set $\emptyset$). Now $4096 - 64 = 4032$.

(c) $64 \cdot 6 = 384$ (add one odd integer to each of the 64 sets).

2 (Finite symmetry). Let $k \in \mathbb{N}$. Show that there exists a bound $n \in N$ such that if $|X| \geq n$ and $f : X \to 2^X$ is any assignment with $|f(x)| = k$ for all $x \in X$, then there are $x, y \in X$ such that $x \notin f(y)$ and $y \notin f(x)$.

Hint. $n = (k+1)k + 1$.

Solution. (Hamkins) Choose $n = (k+1)k+1$. Let $x_0, x_1, \ldots, x_k \in X$. Now

$$\left|\bigcup_{i=0}^{k} f(x_i)\right| \leq (k+1)k \text{ and hence there exists } y \notin \bigcup_{i=0}^{k} f(x_i).$$

For this y (and, indeed, for every element of X), there exists a j such that $x_j \notin f(y)$ (since $|f(y)| = k$). This proves the claim.

3 (Pigeohole) Let $f : A \to B$ be a function between finite nonempty sets A and B. Show that there exists an element $b \in B$ such that $|f^{-1}(b)| \geq |A|/|B|$, where $f^{-1}(b) = \{x \mid f(x) = b\}$.

Solution. If $|f^{-1}(b)| < |A|/|B|$ to all $b \in B$, then

$$|A| = \sum_{b \in B} |f^{-1}(b)| < \frac{|A|}{|B|} \cdot |B| = |A|;$$

a contradiction. (Here the sets $f^{-1}(b)$ for a partition of A.)

4 Let $f : A \to B$ be a function between finite nonempty sets. Prove

$$|f(A)| = \sum_{x \in A} \frac{1}{|f^{-1}(f(A))|}.$$

Solution. We count the sum in a different way by summing over $y \in f(A)$:

$$\sum_{x \in A} \frac{1}{|f^{-1}(f(x))|} = \sum_{y \in f(A)} \sum_{x \in f^{-1}(y)} \frac{1}{|f^{-1}(f(x))|}$$
$$= \sum_{y \in f(A)} \sum_{x \in f^{-1}(y)} \frac{1}{|f^{-1}(y)|} = \sum_{y \in f(A)} 1 = |f(A)|.$$

5 Let X be an n-set. How many pairs (A, B) are there for which $A \subseteq B \subseteq X$?

Solution. There are 3^n such pairs, because the solutions (A,B) are in 1-1 correspondence with the triples $(A, B \setminus A, X \setminus B)$ (of disjoint subsets that form a partition of X). The elements of X can be placed in 3^n different ways to these three sets.

6 Let $\pi\colon X \to X$ be a permutation of prime order p on a finite set X, i.e., $\pi^p = \pi\pi\cdots\pi$ (p times) is the identity function on X. Shown that $|X| \equiv k \pmod p$, where k is the number of the fixed points of π: $\pi(x) = x$.

Solution. For each $x \in X$, $\pi^i(x) = x$ if and only if i divides the order p of π. Hence $i = 1$ or p, since p is a prime number. Hence each orbit $\mathrm{Orb}(x) = \{x, \pi(x), \dots\}$ has size 1 or p. The orbits form a partition of X. Those of size 1 consist of the fixed points.

7 Let A and B be finite sets and $\alpha\colon A \times B \to \mathbb{R}$ a (weight) function. Show that

$$\sum_{f:A\to B} \prod_{a\in A} \alpha(a, f(a)) = \prod_{a\in A} \sum_{b\in B} \alpha(a, b).$$

Solution. Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_k\}$. Each ordered n-tuple $(x_1, x_2, \dots, x_n) \in B^n$ corresponds to a unique function $f : A \to B$ specified by $f(a_i) = x_i$. Therefore

$$\begin{aligned}
\sum_{f:A\to B} \prod_{a\in A} \alpha(a, f(a)) &= \sum_{(x_1,x_2,\dots,x_n)\in B^n} \alpha(a_1,x_1)\alpha(a_2,x_2)\cdots\alpha(a_n,x_n) \\
&= \sum_{x_1\in B}\sum_{x_2\in B}\cdots\sum_{x_n\in B} \alpha(a_1,x_1)\alpha(a_2,x_2)\cdots\alpha(a_n,x_n) \\
&= \sum_{x_1\in B} \alpha(a_1,x_1)\cdot\left(\sum_{x_2\in B}\cdots\sum_{x_n\in B} \alpha(a_2,x_2)\cdots\alpha(a_n,x_n)\right) \\
&= \cdots \\
&= \sum_{x_1\in B} \alpha(a_1,x_1) \sum_{x_2\in B} \alpha(a_2,x_2) \cdots \sum_{x_n\in B} \alpha(a_n,x_n) \\
&= \sum_{x\in B} \alpha(a_1,x) \sum_{x\in B} \alpha(a_2,x) \cdots \sum_{x\in B} \alpha(a_n,x) \\
&= \prod_{a\in A} \sum_{b\in B} \alpha(a, b).
\end{aligned}$$

Remark. The field $\mathbb{R}$ can obviously be replaced by any commutative ring.