Combinatorial Enumeration (2017)

Problem Set 3 (Feb. 2)

1 How many lattice paths $w \in \{u,r\}^*: (0,0) \to (n,k)$ are there that visit the point (x,y), where $0 \le x \le n$ and $0 \le y \le k$?

Solution. There are $\binom{x+y}{x}$ paths $(0,0) \to (x,y)$ and $\binom{n-x+k-y}{n-x}$ paths $(x,y) \to (n,k)$. Hence the answer is

$$\binom{x+y}{x} \cdot \binom{n-x+k-y}{n-x}.$$

2 Let n and k be fixed positive integers. How many *positive* integer solutions are there for the (in)equality

(a) $x_1 + x_2 + \cdots + x_k = n$? (b) $x_1 + x_2 + \cdots + x_k < n$?

Solution. (a) Reduce the equality to $(x_0+1)1 + (x_1+1) + \cdots + (x_{k-1}+1) = n$. This gives $\binom{n-1}{k-1}$.

Second solution. Let

$$A = \{(x_1, x_2, \ldots, x_k) \mid \sum_{i=1}^{k} x_i = n,\ x_i \ge 1\},$$

and consider the mapping $\alpha : A \to 2^{[1,n-1]}$ defined by

$$\alpha(x_1, x_2, \ldots, x_k) = \{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_{k-1}\}.$$

Clearly, α is bijective onto the $(k-1)$-subsets of $[1, n-1]$ the number of which is $\binom{n-1}{k-1}$.

(b) The number of solutions is the same as for $x_1 + x_2 + \cdots + x_{k+1} = n$, i.e., $\binom{n-1}{k}$.

3 We say that $w = a_1 a_2 \cdots a_n$ with $n \ge 1$ and $a_i \in [0, k-1]$ is a **step word**, if for each index i, we have $a_{i+1} = a_i$ or $a_{i+1} = a_i + 1 \pmod{k}$ (where the index $i+1$ is modulo n). For instance, $w = 2234001$ is a step word for $k = 5$, but 112 is not, since while closing the sequences, $2+1 \not\equiv 1 \pmod{5}$. Show that the number $G(n)$ of step words of length n is

$$G(n) = k \sum_{i \ge 0} \binom{n}{ki}.$$

(Recall that $\binom{n}{m} = 0$ if $m > n$.)

Solution. The step words are in $1-1$ correspondence with the sequences

$$(a_1; r_1, r_2, \ldots, r_n), \text{ where } a_{i+1} = a_i + r_i \text{ with } r_i \in \{0, 1\}.$$

E.g. 2234001 is represented by $(2; 0, 1, 1, 1, 0, 1, 1)$. A sequence is a step word if and only if the number of ones is divisible by k, i.e., $G(n) = k \sum_{i \ge 0} \binom{n}{ki}$, where the coefficient k counts the first elements a_1 of the words.

Additional note. For $k \ge 3$, further evaluation of the sum is bit more complicated. By Guichard's theorem (1995), $k \sum_{i \ge 0} \binom{n}{ki} = \sum_{i=1}^{k} (1 + \zeta_k^i)^n$, where $\zeta_k = e^{2\pi i/k}$ is a complex primitive kth root of unity.

4 Let $f(n)$ be the number of integer sequences $1 \le a_1 \le a_2 \le \ldots \le a_n \le n$ such that $a_i \ge i$ for each $i \in [1, n]$. Show that $f(n)$ is a Catalan number.

Solution. Note first that $a_n = n$ by the conditions. There is a bijection onto the lattice paths $(0,0) \to (n,n)$. Each a_i corresponds to the word $u^{a_i - a_{i-1}} r$. E.g., the sequence $2,3,3,4$ with $n = 4$ corresponds to $uururrur$.

5 A finite set $S \subseteq [1,n]$ is said to be **self-centred**, if $|S| \in S$. Denote by M_n the family of the **minimal** self-centred sets of $[1,n]$, i.e., those self-centred sets that do not have proper self-centred subsets. Find $|M_n|$ for all $n \geq 1$.

Solution. The answer is the nth Fibonacci number.

First, S is minimal self-centred if and only if $|S| = \min(S)$ (the smallest element of S). Indeed, if $k \in S$ with $k < |S|$, then the k smallest elements of S form a self-centred subset.

We prove the claim *by induction on* n. For $n \leq 2$, the set $\{1\}$ is the unique set in M_n. These give Fibonacci numbers. Let then $n \geq 3$. We construct $|M_{n-1}| + |M_n|$ minimal self-centred subsets of $[1, n+1]$.

(A) Clearly $M_n \subseteq M_{n+1}$. This gives $|M_n|$ sets in M_{n+1} that are exactly those $S \in M_{n+1}$ with $n+1 \notin S$.

(B) On the other hand, let $S \in M_{n-1}$ be any. Define

$$A(S) = \{i + 1 \mid i \in S\} \cup \{n+1\}.$$

Its smallest element is $|S| + 1 = |A(S)|$, and so $A(S) \in M_{n+1}$. Also, $A(S) \notin M_n$, since $n+1 \in A(S)$, and hence $A(S)$ is not counted in (A). There are $|M_{n-1}|$ minimal self-centred subsets $A(S)$. Finally, each $A \in M_{n+1}$ with $n+1 \in A$ is of the form $A = A(S)$. Indeed, necessarily $1 \notin A$, and for $S = \{x - 1 \mid x \in A, x \neq n+1\}$, $|S| = |A| - 1$ is the smallest element of S, and hence $S \in M_{n-1}$, and then $A = A(S)$.

Remark. As a consequence, we have the following binomial formula for the nth Fibonacci number a_n:

$$a_n = \sum_{k=1}^{n} \binom{n-k}{k-1}.$$

Indeed, this the number of pairs (k, A), where $k \in [1,n]$ and $A \subseteq [k+1, n]$ with $|A| = k - 1$.

6 Count the number of permutations in S_n, for $n \geq 3$, that fix at least one element of $\{1,2,3\}$.

Solution. Let F_i denote the set of all α such that $\alpha(i) = i$. By PIE,

$$\begin{aligned}
|F_1 \cup F_2 \cup F_3| &= |F_1| + |F_2| + |F_3| - |F_1 \cap F_2| - |F_1 \cap F_3| - |F_2 \cap F_3| + |F_1 \cap F_2 \cap F_3| \\
&= (n-1)! + (n-1)! + (n-1)! - (n-2)! - (n-2)! - (n-2)! + (n-3)! \\
&= 3(n-1)! - 3(n-2)! + (n-3)! \\
&= (n-3)!\left(3n^2 - 12n + 13\right).
\end{aligned}$$

7 Show that the number a_n of permutations $\alpha \in S_n$, where $\alpha(i) \neq i+1$, for $i = 1, 2, \dots, n-1$, is

$$a_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!\,.$$

Solution. Let $A_i = \{\alpha \mid \alpha(i) = i+1\}$. Then $|A_i| = (n-1)!$, and

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)! \quad \text{for } i_1 < i_2 < \ldots < i_k\,.$$

Now, $a_n = \left|\overline{A}_1 \cap \cdots \cap \overline{A}_{n-1}\right|$. By PIE,

$$\begin{aligned} a_n &= |S_n| + \sum_{k=1}^{n-1} (-1)^k \sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \\ &= \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!\,. \end{aligned}$$