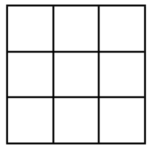
Combinatorial Enumeration (2017)

Problem Set 6 (Feb. 23)

1 Apply Cauchy-Frobenius to the following 3×3 board, where each square can be coloured by red or blue. The plane isometries (rotations and reflections) determine which boards are similar.



(a) What is the answer if there are no restrictions?

(b) What is the answer if exactly two of the squares are red?

Solution. (a) The set X od all colourings has $2^9 = 512$ elements. The isometries and the number of their fixed points are given in the following table.

α	ε	σ_1	σ_2	σ_3	ρ_1	ρ_2	ρ_3	ρ_4
$\lvert\mathrm{Fix}(\alpha)\rvert$	2^9	2^3	2^5	2^3	2^6	2^6	2^6	2^6

Here σ_i's are rotations of 90^o, 180^o and 270^o, and ρ_i's are the reflections.
We have $|G| = 8$ and so $\mathrm{Orb}_G(X)| = \frac{816}{8} = 102$.

(b) Assume exactly two squares have colour 1. Now, $|X| = \binom{9}{2} = 36$.

α	ε	σ_1	σ_2	σ_3	ρ_1	ρ_2	ρ_3	ρ_4
$\lvert\mathrm{Fix}(\alpha)\rvert$	36	0	4	0	6	6	6	6

So that $|\mathrm{Orb}_G(X)| = \frac{64}{8} = 8$.

2 How many open necklaces are there of n beads in k colours? In other words, let A be an alphabet of k letters, and define the **mirror image** of a word $w = a_1 a_2 \cdots a_n$ by $w^R = a_n a_{n-1} \cdots a_1$. Two words as similar, if they are mirror images of each other. How many dissimilar words are there of length n?

Solution. Let X be the set of all words of length n. Then $|X| = k^n$. Now, $G = \{\varepsilon, \rho\}$ and so $|G| = 2$.

α	$\lvert\mathrm{Fix}(\alpha)\rvert$
ε	k^n
ρ	$k^{[n+1/2]}$
	$k^n + k^{[n+1/2]}$

$$|\mathrm{Orb}_G(X)| = \frac{k^n + k^{[n+1/2]}}{2}.$$

3 Consider a cube with a colouring of its vertices. Find the cycle index polynomial P_G. Apply it to the case of 2 colours. Symmetries are the rotations of the 3-dimensional space; see Example 7.8.

Solution. See Example 7.8: There are 24 rotations.
Rotations and vertex colours:

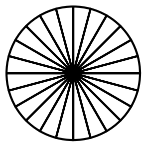
$$\begin{aligned}
z_1^8 &\text{ (identity)},\\
8z_1^2z_3^2 &\text{ (rotations } 120^o),\\
6z_2^4 &\text{ (rotations } 180^o),\\
3z_2^4 &\text{ (rotations } 180^o),\\
6z_4^2 &\text{(rotations } 90^o),\\
P_G &= \frac{1}{24}(z_1^8 + 9z_2^4 + 6z_4^2 + 8z_1^2z_3^2)\,.
\end{aligned}$$

When $|K| = m$, then

$$P_G(m, \ldots m) = \frac{1}{24}(m^8 + 9m^4 + 6m^2 + 8m^4) = \frac{1}{24}(m^8 + 17m^4 + 6m^2)\,.$$

and when $m = 2$, we have $P_G(2, \ldots, 2) = 23$.

4 Consider a wheel of 24 sectors that rotates around its centre. Count by Pólya-Redfield the number of ways the sectors can be coloured with two colours.



Solution. The group is $G = C_{24}$ for which the cycle index polynomial is

$$P_G(z_1, z_2, \ldots, z_{24}) = \frac{1}{24}\left(z_1^{24} + z_2^{12} + 2z_3^8 + 2z_4^6 + 2z_6^4 + 4z_8^3 + 4z_{12}^2 + 8z_{24}\right),$$

and here $P_G(2, \ldots, 2) = \frac{16\,782\,048}{24} = 699252$.

5 Consider an 8×8 checker board. In a **configuration** of the board each square is either left empty or a piece is placed on it. Find the polynomial P_G when similarities are defined by the rotations of the board. Find the total number of configurations of the board.

Solution. Let $X = [1, 64]$ be the set of squares enumerated from the upper left corner to right and down. The rotations with their polynomials are:

$$\begin{aligned}
\varepsilon &= (1)(2)\ldots(64) && z_1^{64}\\
\sigma_1 &= (1\ 8\ 64\ 57)(2\ 16\ 63\ 49)\ldots && z_4^{16}\\
\sigma_2 &= (1\ 64)(2\ 63) && z_2^{32}\\
\sigma_3 &= (1\ 57\ 64\ 8)(2\ 49\ 63\ 16)\ldots && z_4^{16}
\end{aligned}$$

Hence

$$P_G(z_1, z_2, z_3, z_4) = \frac{1}{4}(z_1^{64} + z_2^{32} + 2z_4^{16})\,.$$

The colours are $K = \{0,1\}$ (empty square/checker). The total number of patterns without restrictions is

$$P_G(2,2,2,2) = \frac{1}{4}(2^{64} + 2^{32} + 2^{17}).$$

6 Continue the previous exercise. How many different configurations are there on a 8×8 board, when 16 checkers are placed on its squares?

Solution. We give a general type of solution here; $K = \{0,1\}$, $X = [1,64]$, and again $\omega\colon K \to \mathbb{Q}[x,y]$ with $\omega(0) = y$ and $\omega(1) = x$. Then $\sum_{a\in K} \omega(a) = x + y$. By Theorem 8.6,

$$\sum_P \omega(P) = P_G((x+y),(x^2+y^2),(x^3+y^3),(x^4+y^4)). \tag{0.1}$$

Let $f : X \to K$ be a way to put the checkers on the board, and let $P = \mathrm{Orb}_G(f)$. Thus P consists of the similar configurations. Then $\omega(P) = \omega(f) = \prod_{x\in X} \omega(f(x)) = x^i y^{64-i}$. We have

$$P_G((x+y),(x^2+y^2),(x^3+y^3),(x^4+y^4)) =$$
$$\frac{1}{4}((x+y)^{64} + (x^2+y^2)^{32} + 2(x^4+y^4)^{16}).$$

The coefficient $x^{16}y^{48}$ (put 16 checkers) is

$$\frac{1}{4}\left(\binom{64}{16} + \binom{32}{8} + 2\binom{16}{4}\right).$$

7 Show that the cycle index polynomial of the cyclic group C_n is

$$P_{C_n}(z_1, z_2, \ldots, z_n) = \frac{1}{n}\sum_{d|n} \phi(d) z_d^{n/d}.$$

Solution. The rotations of the necklace form C_n, where the generator angle $360/n$. If a rotation α has order d, then α is a composition of n/d d-cycles, and thus it has cycle type $z_d^{n/d}$.