Graph Theory: Problem Set 1

January 18: 12 – 14. With an appendix on Games. (2018)

1 Determine the least n for which there are two non-isomorphic graphs G and H of order n such that $d_G(v) = d_H(v)$ for all $v \in V_G = V_H$.

2 For each available n and m, let

$$A(n,m) = \min\Big\{ \sum_{v \in G} \frac{1}{d_G(v)+1} \mid \nu_G = n \text{ and } \varepsilon_G = m \Big\}.$$

Let G be a graph of order n and size m obtaining the (minimum) value $A(n,m)$. Show that G is "almost regular": $\Delta(G) - \delta(G) \le 1$.

3 Generalize the Handshaking Lemma as follows: Let $f : V \to \mathbb{R}$ be any function on the vertex set of $G = (V,E)$. For an edge e, write $f(e) = f(u) + f(v)$ if $e = uv$. Show that

$$\sum_{e \in E_G} f(e) = \sum_{v \in V_G} f(v)\, d_G(v).$$

4 Let $G = (V,E)$ be a graph. Show that there exists a binary colouring $\alpha \colon V \to \{0,1\}$ such that, for each $v \in V$, at least half of the neighbours $u \in N_G(v)$ have a different colour than v: $\alpha(u) \neq \alpha(v)$.

Hint. Consider the number $B(\alpha)$ of edges with different values on its ends.

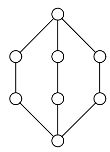
DEFINITION. (**Cops and Robbers**)[1] Let G be a connected graph, where cops $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_k$ (white hats) and a robber $\hat{r}$ (black hat) move along the edges.

- Initially the cops and the robber are placed on some vertices.
- In a **move** each cop changes first to a neighbouring vertex, or stays in her current vertex, after which the robber changes to his neighbouring vertex, or stays put, *unless* a cop already caught him by occupying the same vertex.
- The game ends if a cop occupies the same vertex as the robber. The **cop number** $\operatorname{cop}(G)$ is the minimum number of cops needed to catch the robber starting from *any* initial configuration.

5 **(a)** Show that $\operatorname{cop}(C_n) = 2$ for all cycles C_n with $n \ge 4$.

(b) What is $\operatorname{cop}(G)$ for the graph on the right?

(c) Determine $\operatorname{cop}(T)$ for trees T.



6 Let $G = (V,E)$ be a fixed graph. Consider all nonnegative weight functions $\alpha \colon V \to \mathbb{R}$ that are not identically zero. Define $\mathcal{N}_\alpha \colon V \to \mathbb{R}$ by

$$\mathcal{N}_\alpha(v) = \sum_{u \in N_G(v)} \alpha(u).$$

We write $\alpha \to \alpha'$, if there are vertices u and v with $\mathcal{N}_\alpha(u) \le \mathcal{N}_\alpha(v)$ such that

$$\alpha'(v) = \alpha(v) + \alpha(u), \ \ \alpha'(u) = 0 \ \text{ and otherwise } \alpha'(x) = \alpha(x).$$

Show that there is a complete subgraph H of G together with a sequence $\alpha \to \alpha' \to \ldots \to \alpha^{(k)} = \beta$ for some k such that $\beta(x) > 0$ if $x \in H$ and $\beta(x) = 0$ if $x \notin H$.

Hint. Consider $\mathcal{N}_\alpha = \sum_{v \in V} \alpha(v)\mathcal{N}_\alpha(v)$.

[1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Applied Math.* 8 (1984), 1-12.

Games and riddles*

Ginsberg's theorem:[2]

> *You can't win.*
> *You can't break even.*
> *You can't even get out of the game.*

Consequences of the laws of thermodynamics?

Alcuin's riddle.[3] A shepherd wants to cross a river by boat. He has with him a wolf, a lettuce and a goat. The boat can carry only the shepherd and one of the three. As is well known, and unfortunate for the shepard, the wolf will eat the goat and the goat will eat the lettuce if left unguarded on the same bank of the river. How does the shepherd bring all three over the river – using a graph?

This is a typical instance where one can apply graphs on a puzzle. The vertices will be the possible instances (i.e., where the three are), and an edge is drawn between safe choices. In general, a finitely described game can be presented by a (directed, sometimes infinite) graph, where the vertices are the configurations (i.e., possible positions of the game) and edges tell which configurations are legal after a given configuration.

The graph is simple for the Alcuin's problem as well as, say, for the Tic-Tac-Toe. Sometimes it gets complicated. It is not known if the first player has a winning strategy for the 3-dimensional $5 \times 5 \times 5$ cube of Tic-Tac-Toe, where the winner is the player selecting 5 cubes in a row.[4]

> Beck: "For example, the $4 \times 4 \times 4 = 4^3$ Tic-Tac-Toe is a first player's win, but the winning strategy is extremely complicated: it is the size of a phone-book (computer-assisted task due to O. Patashnik)."

The graph for the Rubik's Cube has 43 252 003 274 489 856 000 vertices. Chess has nearly 10^{47} correct positions (vertices). The number of games is something else.

Sudoku. Consider a graph $G = (V, E)$, where $V = \{1, 2, \ldots, 81\}$ and $ij \in E$ if the corresponding Sudoku cells are in the same column or row or a 3×3 box. Then colour each vertex having a number by the corresponding number in $\{1, 2, \ldots, 9\}$. Clearly solving a Sudoku puzzle is equivalent to finding an extension of the colouring to a *proper colouring*, where no adjacent vertices have the same colour.

There is a method for calculating the number of extended proper colourings of a given partially coloured graph.[5]

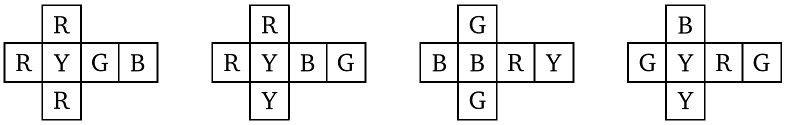
Instant Insanity. In this puzzle we consider four cubes with faces coloured by R(ed), B(lue), G(reen) and Y(ellow) as in the below (where the cubes have been cut out on the table). The problem of the puzzle is to pile the cubes in such a way that each of the four sides of the pile show the four colours.

[2] Allen Ginsberg in the Coevolution Quarterly (1975)

[3] Alcuin of York (c.735 – 804)

[4] J. Beck: Combinatorial Games. Tic-Tac-Toe theory, Cambridge Univ. Press, 2008.

[5] A.M. Herzberg and M. Ram Murty: Sudoku squares and chromatic polynomials. *Notices of the American Mathematical Society* 54 (6) 2007, 708-717.



Solution. Construct a graph for each of the cubes such that two colours are adjacent if they are on opposite faces of the cube. Form an edge coloured graph H by merging the four graphs together: an edge colour in $\{1, 2, 3, 4\}$ indicates from which cube the edge is from. Next find subgraphs H_1 and H_2 of H such that they have

(1) no edges in common,
(2) exactly one edge from each cube, and
(3) only vertices of degree two.

These subgraphs are cycles of length 4.

Colouring games: There are two players R and B, who colour alternatingly the edges of a complete graph K_n by red and blue colours, respectively. Let G be a *test graph*.

- In the game, R looses, if he draws a red subgraph (isomorphic to) G, i.e., the edges of which are all red.
- B looses if he draws a blue G.

Can a game end in a draw? Of course it can, at least for small values of n. What else?

It turns out that, for any G, there exists a bound m such that for all $n \geq m$, there is a looser on K_n. For $G = C_4$, the smallest bound is $m = 6$. So if you colour the edges of a K_6 by two colours, there will be a monochromatic square. For $G = K_4$, the smallest bound is already reasonably large, $m = 18$. For $G = K_5$, a bound is $m = 49$, but nobody knows if this is the smallest bound. Such colouring games need Ramsey theory.

Mazes. To solve a general maze, maybe n-dimensional, using graphs one sets a vertex at each crossing, and an edge if there is a clear view between the crossings.