Graph Theory: Problem Set 2[1]

January 25 (2018)

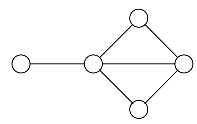
1 Apply Havel-Hakimi to the following sequences. If the sequence is graphical find a graph having the degree sequence.

(a) $3,\ 3,\ 3,\ 3,\ 3,\ 1$ **(b)** $3, 3, \overbrace{2, \dots, 2}^{k}\ (k \geq 0)$

2 Let G be a graph with $\delta(G) \geq 2$. Show that there is a connected graph H having the same degree sequence as G.

3 Let T be a tree with $\nu_T \geq 3$. Let L denote the set of leaves of T. Show that T has a vertex $v \notin L$ that is adjacent to at most one vertex of $V_T \setminus L$.

4 Find a 4-regular graph G that has the following graph as an induced subgraph.



5 Let G be a graph, and $r \geq \Delta(G)$ an integer. Show that there exists an r-regular graph H with G as an induced subgraph.

6 Given a graph G define a relation of the edges $e_1, e_2 \in E_G$ by

$$e_1 \sim e_2 \iff e_1 = e_2 \text{ or there is a cycle } C \text{ in } G \text{ such that } e_1, e_2 \in E_C.$$

Show that $\sim$ is an equivalence relation on E_G.

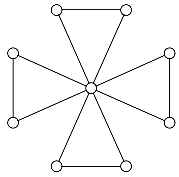
The following exercise gives an alternative algorithm for the minimum weighted spanning trees.

7 Let G^α be a connected graph with $\alpha \colon E_G \to \mathbb{R}$, and let F be any **greedy subgraph** of G, i.e., F is a subgraph of a minimum weight spanning subgraph T of G. Let $e \in E_G \setminus E_F$ be such that $\alpha(e)$ is the minimum among the edges that have at least one end in F. Show that also $F + e$ is greedy.

[1] Appendix on Windmills.

Windmills of friendship*

A graph G is a **friendship graph** if every two vertices $u, v \in G$, $u \neq v$, have a unique common neighbour. A graph G is a **windmill** if it consists of triangles that have a unique common vertex (known as the **politician**). Clearly, a windmill is a friendship graph.



Four-leaf clover

This is the **Friendship Theorem** by ERDÖS, RÉNYI AND SÓS (1966), which hopefully has no political implications.

Theorem. *If G is a friendship graph, then it is a windmill.*

The Friendship Theorem is a bit difficult to prove. It does have a reasonably short proof about five pages by LONGYEAR and PARSONS (1972), but a truly elegant short proof was given by TVERBERG using eigenvalues of matrices.