Graph Theory: Problem Set 4[1]

February 8 (2018)

1 Let G be a 3-regular graph. Show that $\kappa(G) = \kappa'(G)$.

2 Suppose G is 2-connected.

(a) Let $uv \in G$ be an edge. Show that $G{-}uv$ is 2-connected if and only if u and v are on a common cycle of $G{-}uv$.

(b) Show that G is **minimally** 2**-connected**, i.e., none of its proper spanning subgraphs are 2-connected, if and only if all cycles are induced subgraphs (i.e., they have no chords).

The next exercise will be used in the new proof of Brooks' theorem.

3 Let G be a 2-connected graph with $\delta(G) \geq 3$. Show that either G is complete or it has an induced path $z_1 - x - z_2$ such that $G{-}\{z_1, z_2\}$ is connected.

4 Let G_1 and G_2 be two different maximal k-connected subgraphs of the graph G. Show that $|V_{G_1} \cap V_{G_2}| < k$.

5 Prove the part of Theorem 2.10 needed in Theorem 2.11:

(a) Let $d_G(v) \geq k$ for a vertex $v \in G$. Show that if $G{-}v$ is k-connected, then G is k-connected.

(b) Let G be k-connected with $v \in G$ and $S \subseteq V_G \setminus \{v\}$ such that $|S| \geq k$. Then there exists a (v, S)-fan of k paths.

6 Let G be a connected graph. Show that G is eulerian if and only if there are cycles $C_1, C_2, \ldots, C_k$ in G for some $k \geq 1$ such that each edge of G belongs to exactly one of these.

7 Let G be a connected graph where each edge belongs to a triangle (i.e., in a C_3). Show that G has a spanning Eulerian subgraph.

Remark. The complete bipartite graph $K_{n,m}$, where n and m are both odd, is not a spanning subgraph of any eulerian graph G. However, *A all other connected graphs are spanning subgraphs of eulerian graphs.*[2]

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[1] Appendix for Menger's theorem

[2] Boesch, Suffel, Tindell: The spanning subgraphs of Eulerian graphs, J. Graph Theory 1 (1977) 79 – 84.

Menger's theorem for $\kappa = 2$

Here is a short proof of Menger's theorem for $\kappa = 2$: *Let $\nu_G \geq 3$. A graph G is 2-connected if and only if every two vertices are connected by at least two independent paths.*

Proof. Suppose first that G has a cut vertex w. Then there are vertices u, v such that every path $u \xrightarrow{\star} v$ goes through w. Hence, in this case there is only one independent path $u \xrightarrow{\star} v$.
Assume then that G is 2-connected. We prove the claim by induction on the distance $d_G(u, v)$. First let $d_G(u, v) = 1$. The edge uv is not a bridge, and hence it belongs to a cycle. Hence there are two independent paths $u \xrightarrow{\star} v$.
Suppose then that $d_G(u, v) = k \geq 2$, and let $P: u \xrightarrow{\star} v$ be a path of length k. Assume that $P: u \xrightarrow{\star} w \longrightarrow v$. By the induction hypothesis, there are independent paths $P_1, P_2: u \xrightarrow{\star} w$, since now $d_G(u, w) = k - 1$. Moreover, since G is 2-connected, $G{-}w$ is connected. Thus there exists a path $Q: u \xrightarrow{\star} v$ in $G{-}w$. Let x be the last vertex of Q that is in P_1 or in P_2, say P_1. (Such an x exists since, at least, u is in Q and in both P_1 and P_2.) Let $P_1 = P_{11}P_{12}: u \xrightarrow{\star} x \xrightarrow{\star} w$, and $Q = Q_1Q_2: u \xrightarrow{\star} x \xrightarrow{\star} v$. The required independent paths are now: $P_{11}Q_2$ and $P_2(uv)$. □